Lecture 8

We are working on understanding Borinsky's theorem about $\chi(\mathrm{Out}(F_n))$

We've written $T(z) = c_1 z + c_2 z^2 + \cdots$
$$= \sum_{n=1}^{\infty} c_n z^n$$

$\left(c_n = \chi(\mathrm{Out}\, F_{n+1}) \right)$

and $$\exp(T(z)) = \sum_{n=0}^{\infty} b_n z^n$$

Borinsky: Let $f(n) = \sqrt{2\pi} \left(\frac{n}{e}\right)^n$

Then $\sum_{k=0}^{\infty} b_k \varphi_k(n)$ is an asymptotic expansion for $f(n)$ as $n \to \infty$, where $\varphi_k(n) = \Gamma\left(n + \frac{1}{2} - k\right)$

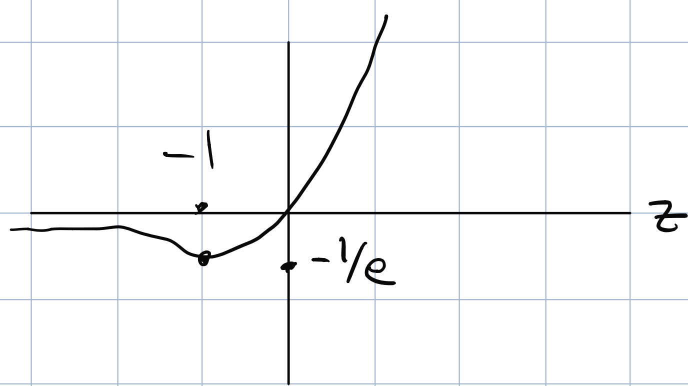
This is useful because:

(1) Asymptotic expansions (with respect to a given $\{\varphi_k\}$ and limit (here $n \to \infty$)) are unique

(2) There already exist expansions for $f(n)$ and facts about their coefficients have been proved.

One existing asymptotic expansion for $\sqrt{2\pi}\left(\frac{n}{e}\right)^n$ is derived from the Lambert W-function:

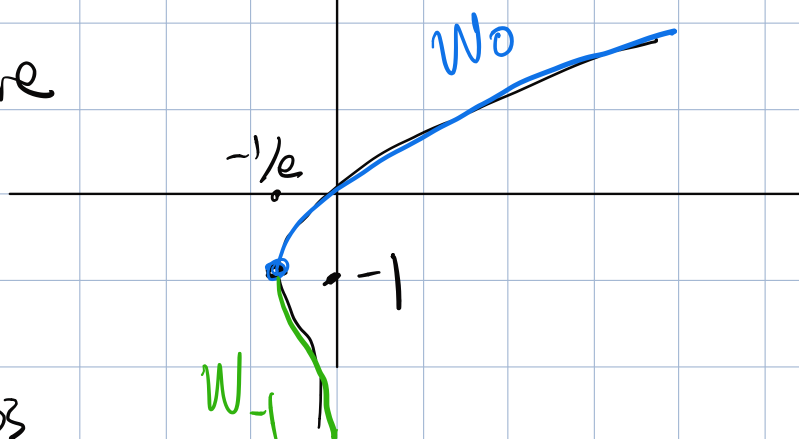
Let $f(z) = ze^z$. For real z the graph is:



The Lambert W-function is the inverse (multi-valued) function, whose real part is:

W_0 and W_{-1} are well-defined on $[-1/e, \infty)$

They can be defined on $\mathbb{C} - \{0\}$



and are analytic on $\mathbb{C} - (-\infty, 0]$, with a square-root-type singularity at $-\frac{1}{e}$

There are Puiseux expansions for $W_0(z)$ and $W_{-1}(z)$ at their common branch point $-1/e$ in powers of $\sqrt{ez+1}$:

$$(*)\begin{cases} W_0(z) = \sum_{p=0}^{\infty} (-1)^{p+1} w_p (ez+1)^{p/2} \\ W_{-1}(z) = -\sum_{p=0}^{\infty} w_p (ez+1)^{p/2} \end{cases}$$

Thm (Volkmer)

$$w_p = -\frac{1}{2\pi}\int_0^\infty (1-x)^{-p/2-1} \underbrace{\Im}_{\text{imaginary part}}(W_{-1}(e^{-1}x))\,dx$$

from $(*)$ you can get an expansion for $zW_0'(z)$:

$$(**)\begin{cases} zW_0'(z) = \sum_{k=-1}^{\infty} (-1)^{k+1} v_k (ez+1)^{k/2} \\ zW_{-1}'(z) = -\sum_{k=0}^{\infty} v_k (ez+1)^{k/2} \end{cases}$$

where $\begin{cases} v_k := \frac{1}{2} k w_k - \frac{1}{2}(k+2) w_{k+2} \\ w_{-1} := 0 \end{cases}$

Thm (Volkmer) $v_p > 0$ for $p \geq 1$

and $\sum_{p=0}^{\infty} v_{2p} = \sum_{p=0}^{\infty} v_{2p-1} = -\frac{1}{2}$.

Here's another series for $zW_0'(z)$, convergent for $|z| < \frac{1}{e}$:

$W_0(z)$ has a Taylor expansion at 0:

Differentiating it and multiplying by z gives

$$(***) \quad zW_0'(z) = \sum (-1)^{n+1} \frac{n^n}{n!} z^n$$

Borinsky now says: "Comparing $(**)$ and $(***)$ and using the reflection property of the Γ-function this becomes

$$\sqrt{2\pi}\left(\frac{n}{e}\right)^n = -\sqrt{\frac{2}{\pi}} \sum_k \Gamma\left(k+\tfrac{1}{2}-n\right)(-1)^k \Gamma\left(k+\tfrac{1}{2}\right) v_{2k-1}"$$

By uniqueness of asymptotic expansions we get

$$-\sqrt{\frac{2}{\pi}}\,\Gamma\left(k+\tfrac{1}{2}\right) v_{2k-1} = b_k$$

so $v_k > 0 \Rightarrow b_k < 0 \Rightarrow c_k < 0$ for $k \geq 1$

and $|c_k|$ grows more than exponentially in k.

<u>Now</u>: How do you get from

$$\exp(T(z)) = \sum b_n z^n \in \mathbb{Q}[[z]]$$

to an asymptotic expansion for $\sqrt{2\pi}\left(\frac{n}{e}\right)^n$?

Start with an <u>equation</u> in $\mathbb{Q}[[z]]$. Since b_n is calculated using τ, this equation should involve τ.

Recap of notation:

Γ a graph

$E(\Gamma) = \{\text{edges of } \Gamma\}$

$e(\Gamma) = \# E(\Gamma)$

$V(\Gamma) = \{\text{vertices of } \Gamma\}$

$v(\Gamma) = \# V(\Gamma)$

$|v|_\Gamma$ = valence of $v \in V(\Gamma)$ (or just $|v|$ if Γ is clear)

$|\Gamma| = e(\Gamma) - v(\Gamma)$

$$\tau(\Gamma) = \sum_{\varphi \subset \Gamma \text{ forest}} (-1)^{e(\varphi)} \quad (\text{including } \varphi = \emptyset)$$

$\mathcal{G}$ = graphs with no bivalent or univalent vertices

$\mathcal{G}^c$ = connected graphs in $\mathcal{G}$

$$c_n = \sum_{\substack{\Gamma \in \mathcal{G}^c \\ |\Gamma| = n}}' \frac{\tau(\Gamma)}{|\mathrm{Aut}(\Gamma)|} = \chi(\mathrm{Out} F_{n+1})$$

$$T(z) = c_1 z + c_2 z^2 + \cdots = \sum_{n=1}^{\infty} c_n z^n$$

$$\exp(T(z)) = 1 + b_1 z + b_2 z^2 + \cdots = \sum_{n=0}^{\infty} b_n z^n$$

so $c_1 = b_1$, $c_2 = b_2 - c_1 b_1$, etc.

<u>ie</u>: $c_k = b_k - \sum_{i=1}^{k-1} c_i b_{k-i}$

More graph terminology:

Γ a graph. A subgraph $\gamma \subset \Gamma$ is (the closure of) a subset of $E(\Gamma)$.

$\gamma \subset \Gamma$ a subgraph. Γ/γ is the graph obtained by collapsing all edges of γ to points.

Definition Γ is a core graph if it has no separating edges or isolated vertices. (this includes $\Gamma = \emptyset$)

Exercise Γ a core graph, $\gamma \subset \Gamma$ a subgraph $\Rightarrow$ Γ/γ is a core graph.

Def Γ a graph. $\mathrm{core}(\Gamma) \subset \Gamma$ is the subgraph obtained by deleting all separating edges.

Lemma $\Gamma \in \mathcal{G}$, $\Theta \subset \Gamma$ a core subgraph. Then there is a bijection.

$$\{\text{subgraphs of } \Gamma \text{ with core} = \Theta\} \longleftrightarrow \{\text{Forests in } \Gamma/\Theta\}$$

Pf Exercise

Define $\sigma(\Gamma) = (-1)^{e(\Gamma)}$ ("sign of Γ")

Lemma $\Gamma \in \mathcal{G}$ then

① $$\sum_{\substack{\theta \subset \Gamma \\ \text{core}}} \sigma(\theta)\, \tau(\Gamma/\theta) = 0$$

② $$\sum_{\substack{\theta \subset \Gamma \\ \text{core}}} \tau(\theta)\, \sigma(\Gamma/\theta) = 0$$

Pf

① The set of all subgraphs of Γ is just the set of subsets of $E(\Gamma)$

So $$\sum_{\gamma \subset \Gamma} (-1)^{e(\gamma)} = \sum_{k=0}^{e(\Gamma)} (-1)^k \binom{e(\Gamma)}{k} = 0$$

By the previous Lemma every subgraph γ is uniquely determined by its core $\theta = \text{core}(\gamma)$ and a forest $\varphi \in \Gamma/\theta$,

so

$$0 = \sum_{\gamma \subset \Gamma} (-1)^{e(\gamma)} = \sum_{\substack{\theta \subset \Gamma \\ \text{core}}} \sum_{\substack{\varphi \subset \Gamma/\theta \\ \text{forest}}} (-1)^{e(\theta)+e(\varphi)}$$

$$= \sum_{\substack{\theta \subset \Gamma \\ \text{core}}} (-1)^{e(\theta)} \tau(\Gamma/\theta)$$

$$= \sum_{\substack{\theta \subset \Gamma \\ \text{core}}} \sigma(\theta) \tau(\Gamma/\theta)$$

② (Thanks to Sam Shepherd!)

Recall that $\tau(\gamma) = 0$ if γ has a separating edge

so $$\sum_{\substack{\theta \subset \Gamma \\ \text{core}}} \tau(\theta) \sigma(\Gamma/\theta)$$

$$= \sum_{\gamma \subset \Gamma} \tau(\gamma) \sigma(\Gamma/\gamma)$$

$$= \sum_{\gamma \subset \Gamma} \sum_{\varphi \subset \gamma} (-1)^{e(\varphi)} (-1)^{e(\Gamma-\gamma)}$$

$$= (-1)^{e(\Gamma)} \sum_{\gamma \subset \Gamma}{}' \sum_{\varphi \subset \Gamma} (-1)^{e(\varphi)} (-1)^{e(\gamma)}$$

$$= (-1)^{e(\Gamma)} \sum_{\varphi \subset \Gamma} \sum_{\gamma \supset \varphi} (-1)^{e(\varphi)} (-1)^{e(\gamma)}$$

$$= \pm \sum_{\varphi \subset \Gamma} e^{(\varphi)} \sum_{\gamma \supset \varphi} (-1)^{e(\gamma)}$$

The set of subgraphs containing φ is just the set of subsets of $E(\Gamma) - E(\varphi)$, so $\sum_{\gamma \supset \varphi} (-1)^{e(\gamma)} = 0$ as in the proof of 1

So $$= \pm \sum_{\varphi \subset \Gamma} (-1)^{e(\varphi)} \cdot 0 = 0$$

Corollary In $\mathbb{Q}[[z]]$

$$1 = \sum_{\Gamma \subset \mathcal{G}} \sum_{\theta \subset_c \Gamma} \frac{\tau(\theta)\sigma(\Gamma/\theta)}{|\mathrm{Aut}\,\Gamma|} z^{|\Gamma|}$$

core

$$\left[\begin{array}{l} = 0 \text{ if } \Gamma \neq \emptyset \\ \text{and } \tau(\emptyset) = 1 \end{array} \right]$$

Now instead of summing over all pairs (Γ, Θ) with $\Theta \subset \Gamma$ core, we want to sum over all (core) $\Delta = \Gamma/\Theta$ and core subgraphs <u>with legs</u> that you could insert into the vertices of Δ

The claim is that the equation in the corollary above becomes

$$1 = \sum_{\Delta \in \mathcal{G}} \left(\prod_{v \in V(\Delta)} T_{|v|}(z) \right) \frac{\sigma(\Delta)}{|\mathrm{Aut}\,\Delta|} z^{e(\Delta)}$$

where $T_k(z) = \sum_{n=1}^{\infty} c_{n,k} z^n$

$$c_{n,k} = \chi(A_{n+1,k})$$

$A_{n,k} = \pi_0 HE(R_{n,k}, \partial)$ is the group we've been studying

In particular, from the short exact sequence $1 \longrightarrow F_{n+1}^{k} \longrightarrow A_{n+1,k} \longrightarrow A_{n+1,0} \longrightarrow 1$

we get

$$\chi(A_{n+1,k}) = n^{k}\chi(CaF_{n+1})$$
$$= n^{k} c_n$$

Now we can get a different formula for the coefficients of $\exp(T_k(z))$ by counting graphs. This involves factorials ($= \Gamma$-functions). By manipulating these and a variation on Sterling's asymptotic series for Γ-functions we obtain an asymptotic series for $\sqrt{2\pi}\left(\frac{n}{e}\right)^{n}$.

(Sorry for all the missing details!)