## Lecture 2 — Graph homology.

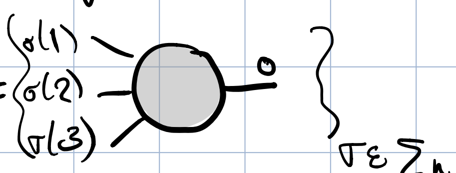
**Last time:** Defined 2 flavors of graph complex. Both generated by finite admissible graphs

odd : orient by ordering vertices, orienting edges

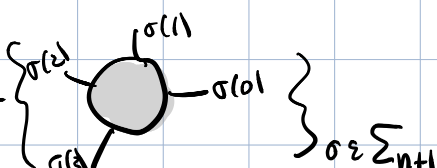
even : orient by ordering edges

Differential given by summing over all ways to collapsing a (non-loop) edge.

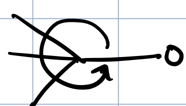
Then began to describe Kontsevich's Lie algebras, constructed using a symplectic vector space $V$ and a cyclic operad $\mathcal{O}$ (esp Comm, Ass, Lie)

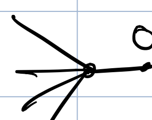
$P(n) =$ 


$P(i) \circ_i P(j) \to P(i+j-1)$

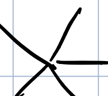
$\mathcal{O}(n) =$ 


Any slot can serve as the output slot.

**Ass** : $\mathrm{Ass}(n)/\Sigma_n =$    Cyclic version 

**Comm** : $\mathrm{Comm}(n)/\Sigma_n =$    $\Sigma_n$-action is trivial

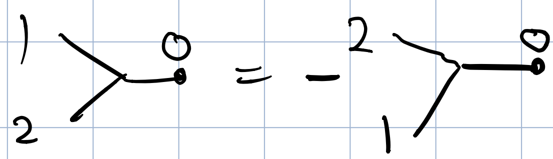
= ass plus comm

cyclic version: 

**Lie:** not commutative or associative

**Anti-symmetry:** $[a,b] = -[b,a]$

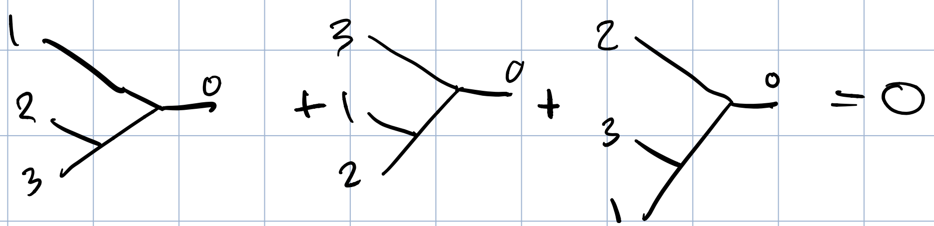
in pictures :



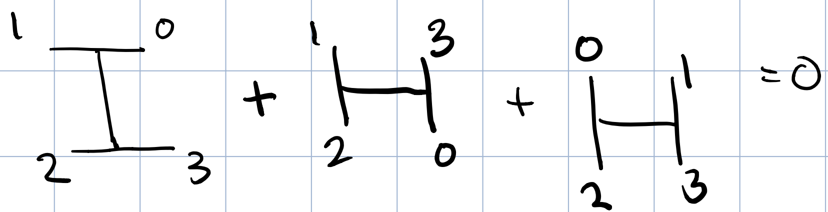
$[a,[b,c]] + [c,[a,b]] + [b,[c,a]] = 0$

in pictures:

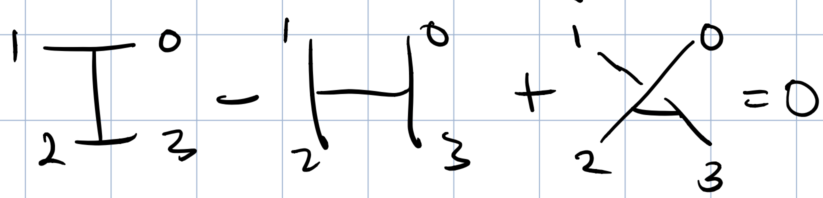
**Jacobi**



$\parallel$



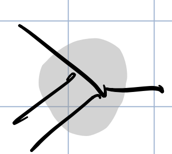
**IHX-rel'n**



so Lie(n) is generated by rooted planar trivalent graphs with n labeled leaves modulo IHX and AS

The action of $\Sigma_n$ extends to an action of $\Sigma_{n+1}$

$\mathrm{Lie}(3)/\Sigma_4 \ni$



mod AS, IHX

(not quite such a neat way of picturing it)

For each of these - (and for any cyclic operad $\mathcal{O}$ such that $\mathcal{P}[1]$ contains only the operad unit)

And for a symplectic vector space $V_k$ w/ symplectic basis $\mathcal{B}_k = \{p_1, \dots, p_k, q_1, \dots, q_k\}$

Want to define a Lie algebra $h_k$

Reference: On a theorem of Kontsevich, by J. Conant and KV (2003)

For $\mathcal{O} = \mathrm{Comm}$, this will be the algebra of polynomial functions on $V_k$ that have no constant or linear terms, with Poisson bracket.

This can also be described as the "Derivations of free polynomial algebra that preserve $\sum dp_i \wedge dq_i$ and the ideal $(p_1, \dots, p_k, q_1, \dots, q_k)$"

Still for $\mathcal{O} = \mathrm{Comm}$, generators of the free Lie algebra (ie monomials) can be pictured as rooted trees labeled by elements of $\mathcal{B}$

Poisson bracket can be described in terms of these pictures.

For general $\mathcal{O}$, can imitate this pictorial description to construct "noncommutative" analogs of Poisson bracket.

The Natural inclusions $V_k \longrightarrow V_{k+1}$ will induce
$h_k \hookrightarrow h_{k+1}$

Then $h_\infty$ = direct limit $\varinjlim h_k$

$h_\infty^+$ : only allow spiders w/ $\geq 3$ legs.

# What does this have to do with graph homology?

(1) There is a (co)homology theory for Lie algebras defined by Chevalley-Eilenberg

(If $G$ = compact ss Lie group w/ Lie algebra $\mathfrak{g}$, then $H_*(G;\mathbb{R}) = H_*(\mathfrak{g})$

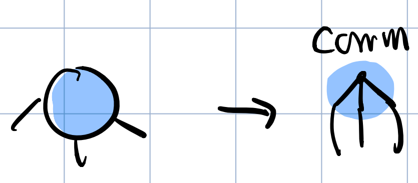
But makes sense for any Lie algebra.)

(2) If you have a cyclic operad $\mathcal{O}$, you can decorate the vertices $v$ of an admissible graph with generators of $\mathcal{O}(|v|-1)/\Sigma_{|v|}$ to get an $\underline{\mathcal{O}\text{-graph}}$:

($|v|$ = valence of $v$)

For Comm, the decoration is trivial. so this is just a graph.

It is non-trivial for Ass and Lie




"Comm graphs"
= graphs

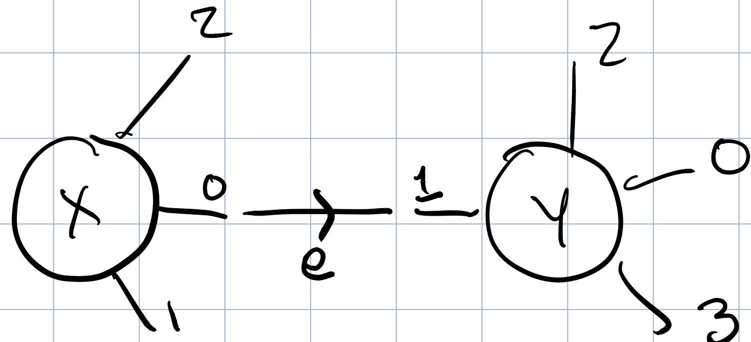

"Ass graphs"
= ribbon graphs


"Lie graphs"
more complicated.

The "$\Theta$-graph" complex $CG_*$ is generated by $\Theta$-graphs modulo $(G, or) = -(G, -or)$

The differential is given by edge-collapse:

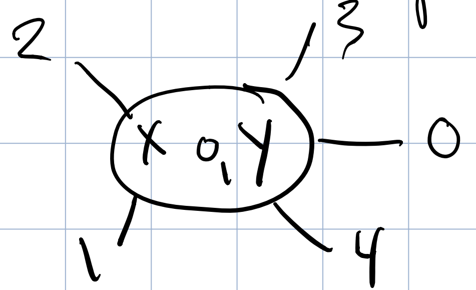
When you collapse an edge, you apply the operad composition (in the direction of the arrow) to merge the vertices:




choose a labeling s.t. $i(e) =$ output slot

$c(e) =$ an input slot (say 1)

compose in the direction of the arrow




(then forget the labels)

**Thm** (Kontsevich) Computing the homology of the odd $\mathcal{O}$-graph complex $\mathcal{CG}_*$ is equivalent to computing the Chevalley-Eilenberg homology of $\mathfrak{h}_\infty$

(more precisely: the "primitive part $PH_*^{CE}(\mathfrak{h}_\infty)$ is $\cong$ to $H_*^{CE}(\mathfrak{sp}_\infty) \oplus H_*(\mathcal{CG}_*)$ )

Furthermore:

For $\mathcal{O} = \mathrm{Ass}$, $H_k(\mathcal{CG}_*) = \bigoplus_{g,s \geq 1} H^{d-k}(\mathrm{Mod}(S_{g,s}); \mathbb{R})$

For $\mathcal{O} = \mathrm{Lie}$, $H_k(\mathcal{CG}_*) = \bigoplus_{n \geq 2} H^{2n-3-k}(\mathrm{Out}(F_n); \mathbb{R})$

For $\mathcal{O} = \mathrm{Comm}$, $H_k(\mathcal{CG}_*)$ contains invariants of odd-dimensional homology spheres

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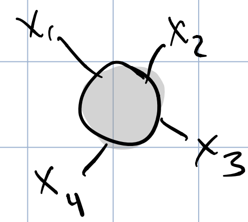
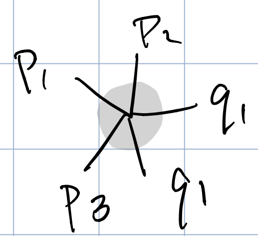
So, it's time to define the Lie algebra $\mathfrak{h}_k$ based on

- the cyclic operad $\mathcal{O}$ and
- a symplectic vector space $V_k$, with symplectic basis $\mathcal{B} = \{p_1 .. p_k\, q_1 \dots q_k\}$

A generator of $h_k$ is a symplectic $\mathcal{O}$-spider 

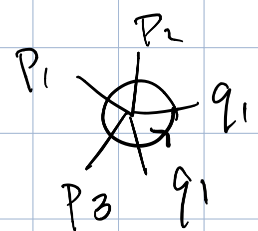

= element of $\mathcal{O}[n] / \Sigma_{n+1}$ with legs decorated by elts of $B$

eg $\mathcal{O} = \mathrm{Comm}$. A symplectic commutative spider

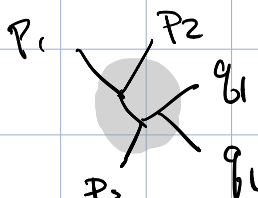
is 

[$\Leftrightarrow$ the monomial $p_1 p_2 p_3 q_1^2$, ie a generator of the free polynomial algebra on $B$]

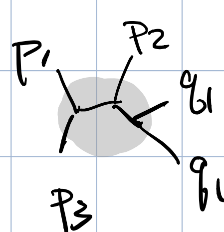
eg $\mathcal{O} = \mathrm{Ass}$

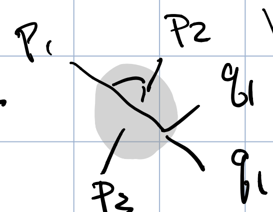


is an associative spider

$\mathcal{O} = \mathrm{Lie}$ 

is a Lie spider

( $\sim$ to 

$-$ 

)

Next, we need to define a

Bracket $[S, T]$ of two symplectic $\mathcal{O}$-spiders.

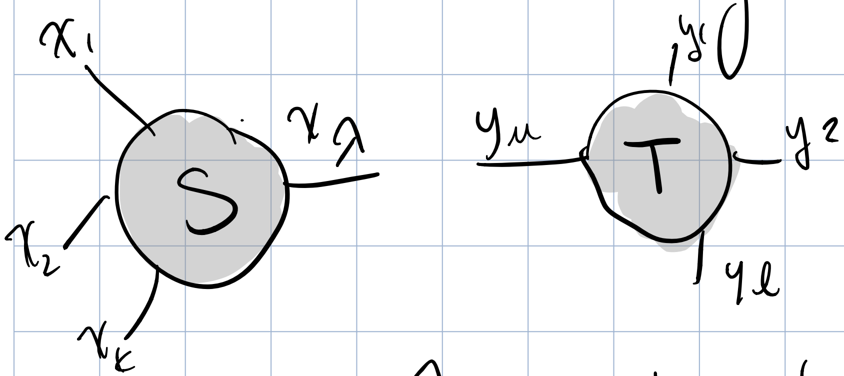
to make this a Lie algebra:

Given $\lambda$ = leg of $S$, labeled by $x_\lambda \in \mathbb{B}$

$\mu$ = leg of $T$, labeled by $y_\mu \in \mathbb{B}$

can <u>mate</u>

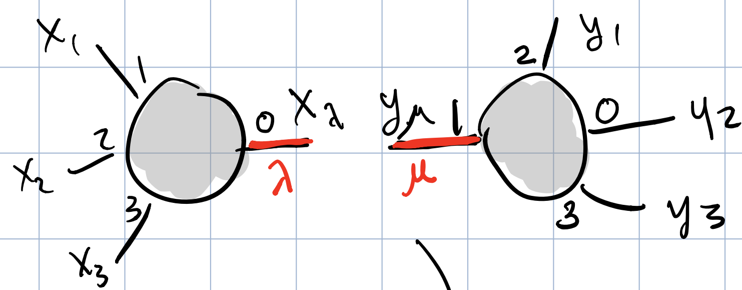
$S$ and $T$ using $\lambda$ and $\mu$:



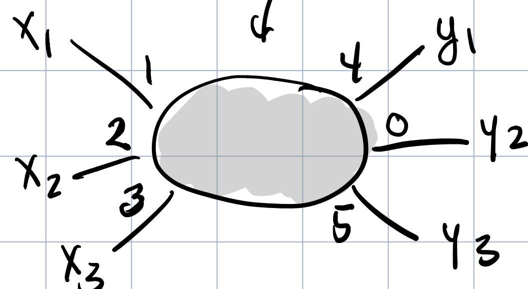

$x_i, y_i \in \mathbb{B}$

use $\lambda$ as the output slot

$\mu$ as an input slot

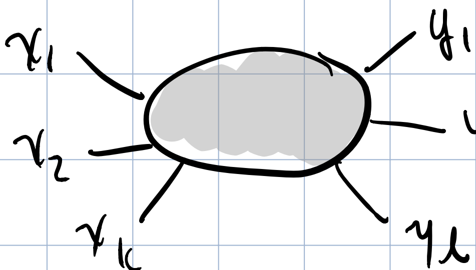



Perform the operad composition




Now lose the slot numbers, remember the $\mathbb{B}$-labels, multiply by $\langle x_\lambda, y_\mu \rangle$:

Define $(ST)_{\lambda\mu} := \langle x_\lambda, y_\mu \rangle$



Now define $[S,T] = \sum_{\substack{\lambda \in S \\ \mu \in T}} (ST)_{\lambda\mu}$

**Exercise** Anti-symmetric, satisfies Jacobi identity

**Exercise**: For Comm, show $[S,T] = \{S,T\}$ (Poisson bracket)

**Claim**: I can think of an $\Theta$-spider as a derivation of the free $\Theta$-algebra.

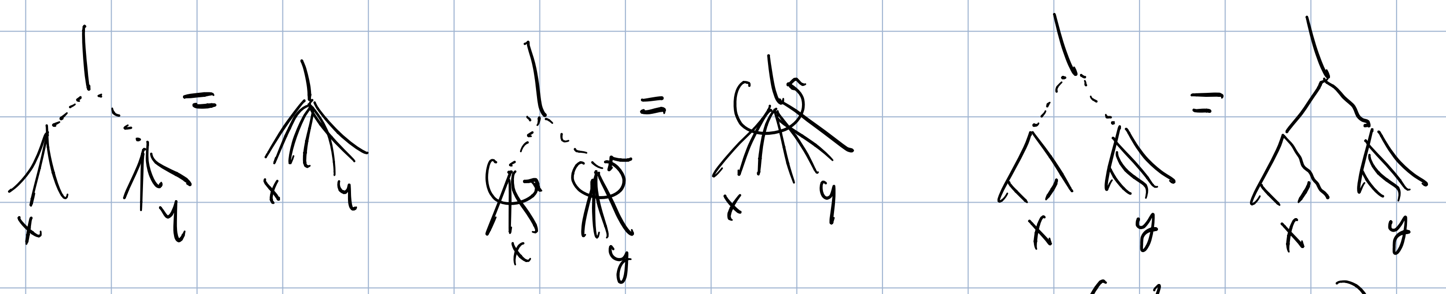
**Free $\Theta$-alg**: generated by monomials in $\mathbb{B}$

eg

| Comm | Ass | Lie |
|---|---|---|
| $p_1^2 q_1 q_2$ | $p_1 q_1 p_1 q_2$ | $[[p_1,q_1],[p_1,q_2]]$ |

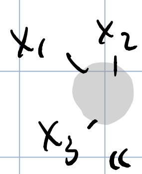

Product in the free $\Theta$-algebra is performed by combining roots:

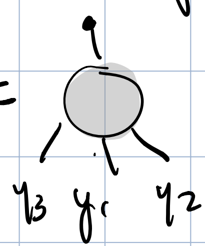
Comm Ass Lie




( /AS, IHX )

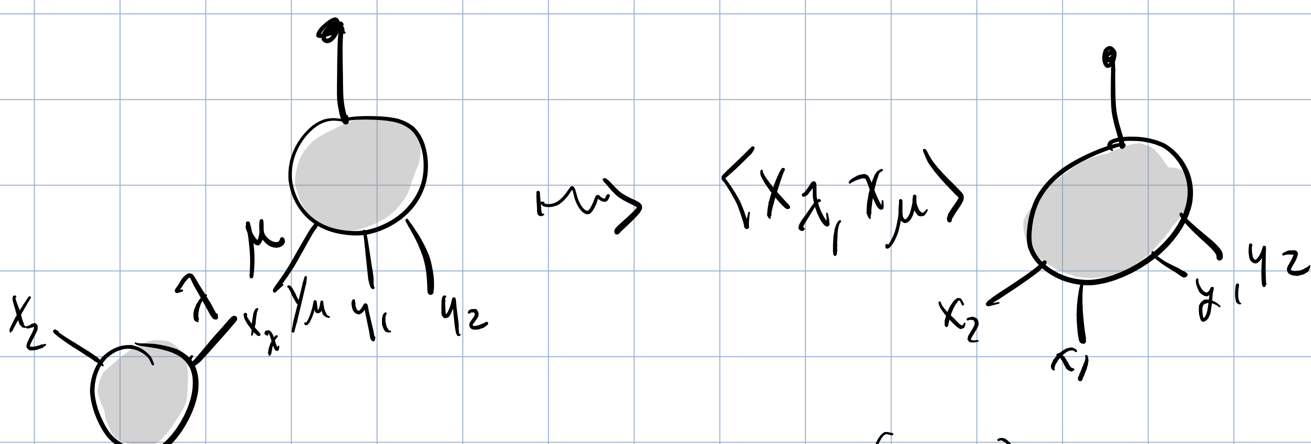
How does a spider D act on an algebra generator A?

$D =$ 

, $A =$ 


$D \cdot A = ?$

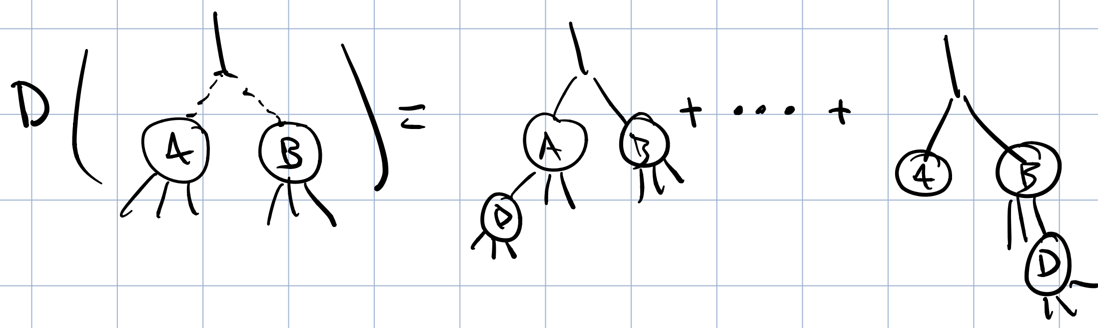
can "mate" a leg of D with a leg of A in the same way we mated spiderlegs:




$:= (DA)_{\lambda\mu}$

Then $D \cdot A = \sum_{\lambda,\mu} (DA)_{\lambda\mu}$

It is clear this is a derivation
(ie $D(AB) = DA \cdot B + A \cdot DB$):



**Exercise** In general,

Derivations form a Lie algebra:

$$[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$$

Identify this bracket w/ the above bracket.

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So, now we have a Lie algebra $h_k$.
Since $h_{k+1}$ is defined by simply allowing more labels on spider legs, we have $h_k \hookrightarrow h_{k+1}$
Define $h_\infty = \varinjlim h_k$

To prove Kontsevich's theorem, need to define Lie algebra homology:

How do you compute homology of a Lie algebra

(and why is it defined this way?)

Answer: Lie algebra = tang. space to id in a Lie group

= linear approximation to the Lie group.

If the Lie group is compact & simply-connected, it is determined by its Lie algebra, so you should be able to compute its cohomology from the Lie algebra, too

Lie algebra cohomology was defined to do this

$H^*$ Lie group = de Rham cohomology

ie: Chains are differential forms $f\, dx_1 \wedge \dots \wedge dx_k$

etc.

This motivates defin of Lie algebra (co)homology

$\mathfrak{h}$ = Lie algebra $H_*^{CE}(\mathfrak{h})$ = homology of $C_k(\mathfrak{h}) = \Lambda^k \mathfrak{h}$

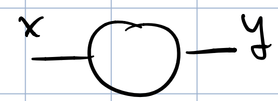
with boundary $\partial : C_k \longrightarrow C_{k-1}$

$$x_1 \wedge \dots \wedge x_k \longmapsto \sum_{i<j} [x_i, x_j] \wedge x_1 \wedge \dots \wedge \hat{x}_i \wedge \dots \wedge \hat{x}_j \wedge \dots \wedge x_k.$$

In order to prove Kontsevich's theorem, we need to enlarge our set of "admissible" graphs

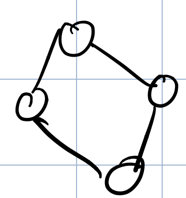
- Allow bivalent vertices
- Allow disconnected graphs

Get "full" graph complex $fCO_*$

$h_\infty$ contains two-legged spiders 

$x, y \in B$.

and we now have $O$-graphs of rank 1:

 = polygons.

If you make a two-legged spider with a $k$-legged spider you get a sum of $k$-legged spiders.

<u>In particular</u>: The 2-legged spiders form a sub-Lie algebra. $h^{(2)}_k$

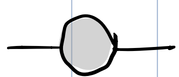
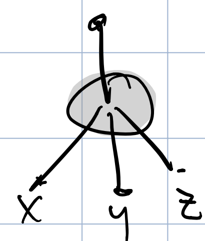
**Claim** $h^{(2)}_k \cong \mathfrak{sp}_k$ = Lie algebra of $Sp_k$.

**Recall** $Sp_k = 2k \times 2k$ matrices $A$
st $A^t J A = J$, $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$

so $\mathfrak{sp}_k = 2k \times 2k$ matrices $A$
st. $A^t J + J A = 0$

with Lie bracket $[A, B] = AB - BA$

In $h^{(2)}$:

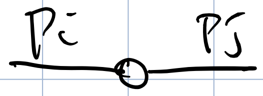
$p_i$  $q_j$ acts on monomials 



by changing $p_j \to -p_i$
$q_i \mapsto q_j$

ie corresponds to matrix
$$\begin{array}{c@{}c} & \begin{array}{cccc} i & j & i & j \end{array} \\ \begin{array}{c} i \\ j \\ i \\ j \end{array} & \left[\begin{array}{cc|cc} & -1 & & \\ & & & 0 \\ \hline & 0 & & \\ & & 1 & \end{array}\right] \end{array}$$

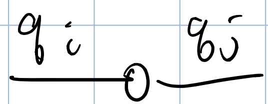
$$= \begin{bmatrix} -E_{ij} & 0 \\ 0 & E_{ji} \end{bmatrix} \in \mathfrak{sp}_k$$




changes $q_i \to p_j$

$q_j \to p_i$

$$= \begin{bmatrix} 0 & E_{ji}+E_{ij} \\ 0 & 0 \end{bmatrix}$$




$p_i \to -q_j$

$p_j \to -q_i$

$$= \begin{bmatrix} 0 & 0 \\ -E_{ji}-E_{ji} & 0 \end{bmatrix}$$

These matrices generate $sp_{2k}$, bracket is given by $[A,B] = AB-BA$.

$h_k^{(2)}$ acts on $h_k$, so $h_k$ is an $sp_{2k}$ module, splits as

$$h_\infty = h_\infty^{(2)} \oplus h_\infty^{+} \cong sp_\infty \oplus h_\infty^{+}$$

$sp_\infty$ acts on $h_\infty^{+}$.