Lecture 5 Graph homology

Last time,

defined $\mathcal{MG}_n$ = moduli space of graphs G

w/ G admissible := connected, $|v| \geq 3$, $\chi = 1-n$

$= CV_n / \mathrm{Out}\, F_n$

also CV_n^*, $\mathcal{MG}_n^* = CV_n^* / \mathrm{Out}\, F_n$

$\partial CV_n^* = CV_n^* \setminus CV_n$

$\partial \mathcal{MG}_n^* = \mathcal{MG}_n^* \setminus \mathcal{MG}_n$

We observed

$$\mathcal{CG}_* = \bigoplus_{n \geq 2} \mathcal{CG}_*^{(n)} \qquad (\chi = 1-n)$$

and identified

$$C_*(\mathcal{MG}_n^*, \partial \mathcal{MG}_n^*) \leftrightarrow \mathcal{CG}_*^{(n)} \qquad (\Theta = \mathrm{Comm}, \text{ even or})$$

$\uparrow$ grading here is by #edges -1 $\qquad$ $\nwarrow$ grading here is by #vertices

If G has $k+1$ edges and $\chi = 1-n$, then

G has $2+k-n$ vertices

so:

$$C_k(\mathcal{MG}_n^*, \partial \mathcal{MG}_n^*) \leftrightarrow \mathcal{CG}_{2+k-n}^{(n)}$$

on both sides, a generator G is zero if it has an odd automorphism.

Since every graph in CV_n has $\geq n$ edges, the entire $(n-2)$-skeleton of CV_n^* is contained in ∂CV_n^*.

So $C_k(MG_n^*, \partial MG_n^*) = 0$ if $k \leq n-2$

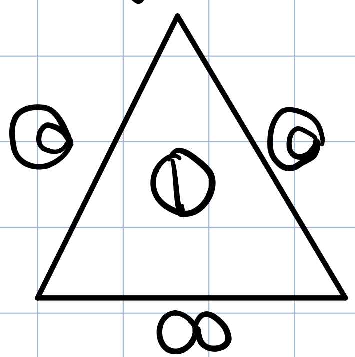
$$\begin{array}{c} \| \\ CG^{(n)}_{2+k-n} \end{array} = 0 \quad \text{if } k \leq n-2$$

ie $CG_0 = CG_{-1} = CG_{-2} = \cdots = 0$.

The entire chain cplex $C_* G$ looks like

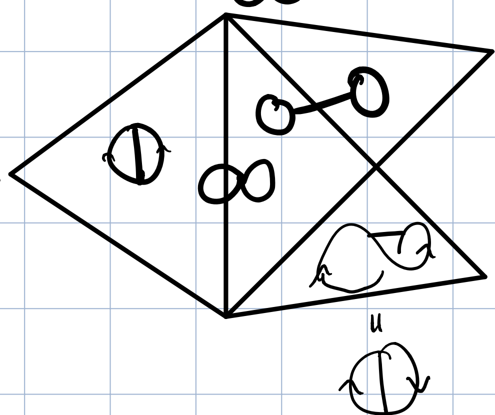
$$\begin{array}{ccccccccccc}
C_6 g & & C_5 g & & C_4 g & & C_3 g & & C_2 g & & C_1 g \\
 & & & & & & & & C_2 g^{(2)} & \xrightarrow{\partial} & C_1 g^{(2)} \\
 & & & & & & & & \oplus & & \oplus \\
 & & & & C_4 g^{(3)} & \xrightarrow{\partial} & C_3 g^{(3)} & \xrightarrow{\partial} & C_2 g^{(3)} & \xrightarrow{\partial} & C_1 g^{(3)} \\
 & & & & \oplus & & \oplus & & \oplus & & \oplus \\
C_6 g^{(4)} & \xrightarrow{\partial} & C_5 g^{(4)} & \xrightarrow{\partial} & C_4 g^{(4)} & \xrightarrow{\partial} & C_3 g^{(4)} & \xrightarrow{\partial} & C_2 g^{(4)} & \xrightarrow{\partial} & C_1 g^{(4)} \\
\oplus & & \oplus & & \oplus & & \oplus & & \oplus & & \oplus \\
C_6 g^{(5)} & \xrightarrow{\partial} & C_5 g^{(5)} & \xrightarrow{\partial} & C_4 g^{(5)} & \xrightarrow{\partial} & C_3 g^{(5)} & \xrightarrow{\partial} & C_2 g^{(5)} & \xrightarrow{\partial} & C_1 g^{(5)} \\
\oplus & & \oplus & & \oplus & & \oplus & & \oplus & & \oplus \\
C_6 g^{(6)} & \xrightarrow{\partial} & C_5 g^{(6)} & \xrightarrow{\partial} & C_4 g^{(6)} & \xrightarrow{\partial} & C_3 g^{(6)} & \xrightarrow{\partial} & C_2 g^{(6)} & \xrightarrow{\partial} & C_1 g^{(6)}
\end{array}$$

∂ operator collapses edges:



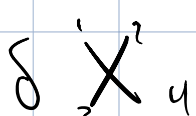
2 vertices become one vertex, lose an edge

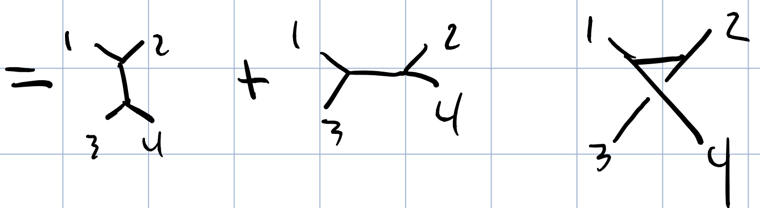
δ operator splits vertices:



one vertex splits into 2 vertices, add an edge

eg δ





both ∂ and δ preserve $\chi = 1-n$

If you grade Cg_* so that $\deg G = v-(n+1)$ instead of v you get

$$C^{-1} \qquad C^{-2}$$

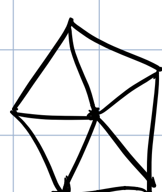
$$\begin{array}{ccccccccccc}
 & & & & & & C_2 g^{(2)} & \xleftarrow{\delta} & C_1 g^{(2)} \\
 & & C_4 g^{(3)} & \xleftarrow{\delta} & C_3 g^{(3)} & \xleftarrow{\delta} & C_2 g^{(3)} & & C_1 g^{(3)} \\
C_6 g^{(4)} & \leftarrow & C_5 g^{(4)} & \leftarrow & C_4 g^{(4)} & \leftarrow & C_3 g^{(4)} & \leftarrow & C_2 g^{(4)} & \leftarrow & C_1 g^{(4)} \\
C_6 g^{(5)} & \leftarrow & C_5 g^{(5)} & \leftarrow & C_4 g^{(5)} & \leftarrow & C_3 g^{(5)} & \leftarrow & C_2 g^{(5)} & \leftarrow & C_1 g^{(5)} \\
C_6 g^{(6)} & \leftarrow & C_5 g^{(6)} & \leftarrow & C_4 g^{(6)} & \leftarrow & C_3 g^{(6)} & \leftarrow & C_2 g^{(6)} & \leftarrow & C_1 g^{(6)}
\end{array}$$

$$C^0 \qquad C^1 \qquad C^{-3} \qquad C^{-4} \qquad C^{-5} \qquad C^{-6}$$

You get Willwacher's graded (co) chain complex

$$H^0 = C_{n+1} g^{(n)} \Big/ \operatorname{im}(C_n g^{(n)})$$

$$C_0 = \ker\left(C^{(n)}_{n+1} \xrightarrow{\partial} C^{(n)}_n\right)$$

contains $W_n =$  .

n even $\Rightarrow (W_n, or) = 0$

n odd $\Rightarrow (W_n, or) \neq 0$

Willwacher found cocycles $\sigma_n \in C^0$, n odd

Che identified H^0 with $\mathfrak{grt}_1$

= "Grothendieck-Teichmüller Lie algebra (unipotent version)"

F. Brown proved $\mathfrak{grt}_1$ contains a free Lie algebra generated by odd classes $\sigma_3, \sigma_5, \dots$

Willwacher showed $\sigma_n(\omega_n) \neq 0$ (n odd)

If G has k vertices and rank n, it has $k+n-1$ edges so $\dim \sigma(G, g) = k+n-2$

Willwacher's result has $k = n+1$, so translates to

$$H^{2n-1}(\mathcal{M}g_n^*, \partial \mathcal{M}g_n^*) \neq 0 \quad (n \geq 3)$$

Willwacher also showed $H^k(\mathcal{CG}_*) = 0$ for $k < 0$

This means $H^i(\mathcal{M}g_n^*, \partial \mathcal{M}g_n^*) = 0$ for $i < 2n-1$

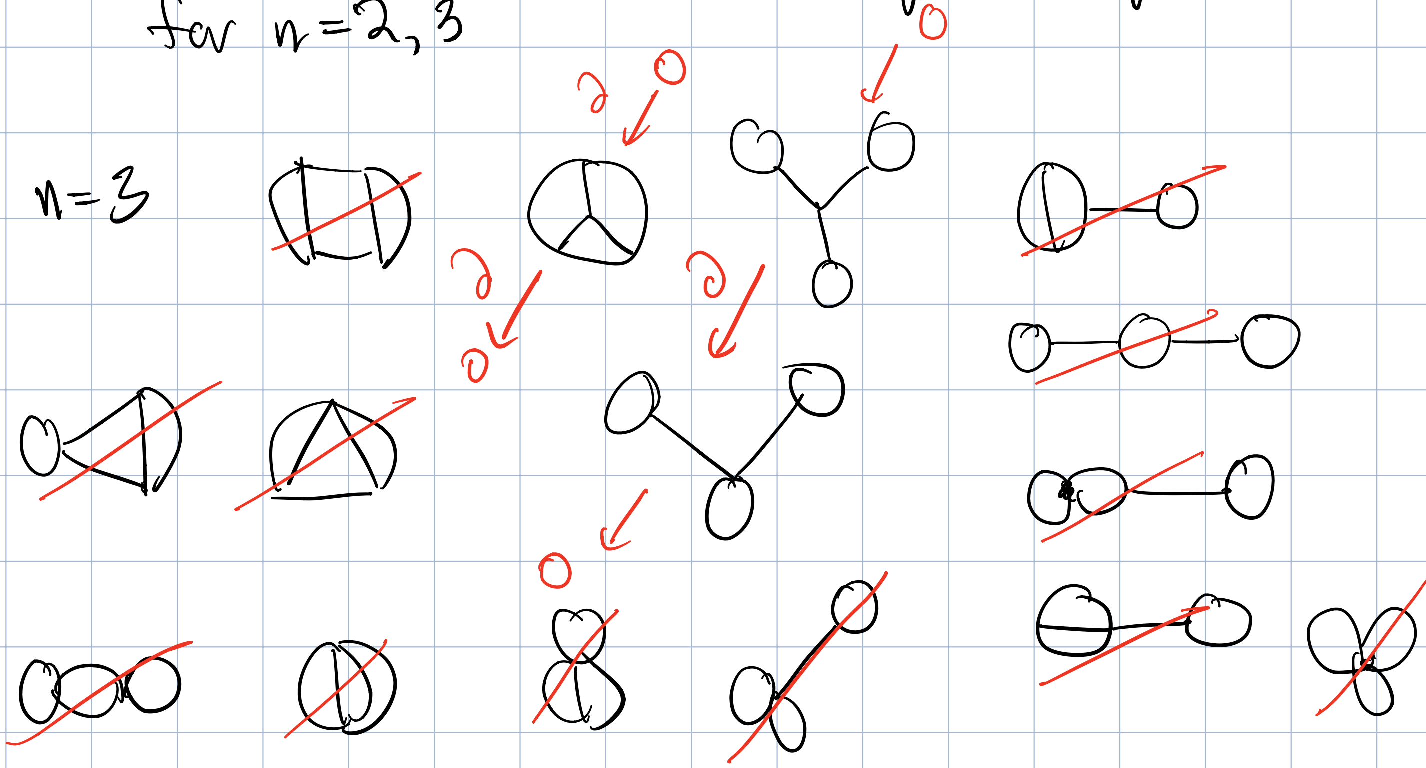
eg for $n=3$, $\dim \mathcal{M}g_n^* = 5$, $H^i = 0$ for $i < 5$

so only cohomology is in dim 5.

All non-zero homology lies in H_k, $2n-1 \le k \le 3n-4$

<u>Exercise</u>: Compute $H_*(\mathcal{MG}^*, \partial \mathcal{MG}^*)$ for $n=2,3$

$n=3$



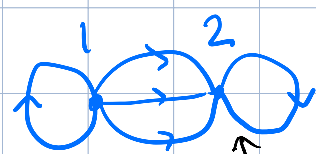
Back to Kontsevich's theorem:

$$\mathcal{O} = \mathrm{Lie} \Rightarrow H_d(\mathcal{CG}_*^{(n)}) \cong H^{2n-2-d}(\mathrm{Out}\, F_n)$$

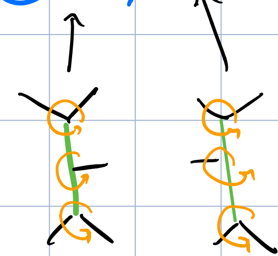
where the orientation on $\mathcal{O}$-graphs is the odd orientation

A generator of $\mathcal{CG}_*$ is a complicated object:

An odd-oriented graph $X =$



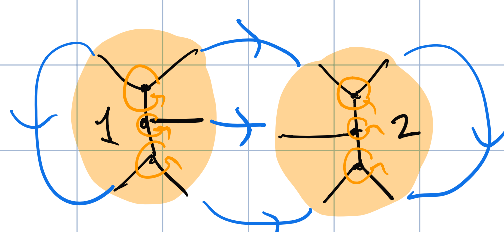
, with vertices decorated with Lie trees



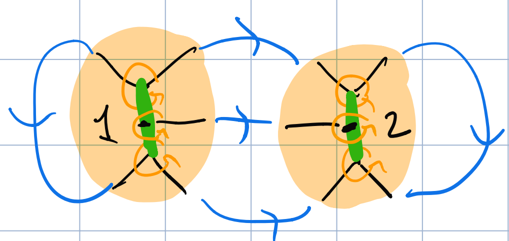
T_1 "tree at 1" T_2 "tree at 2"

These are planar binary trees modulo AS, IHX on internal edges

$(X, \{T_v\}, or) =$

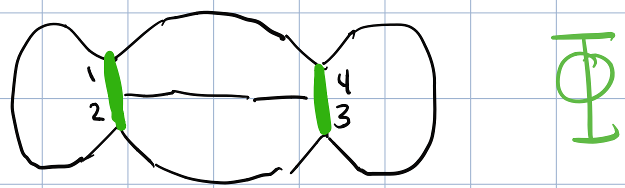


Inside each generator is a natural forest Φ (= union of trees) containing all vertices.



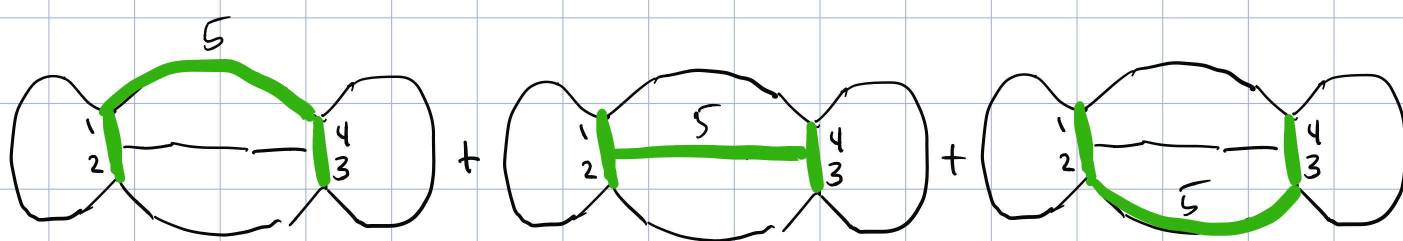
Φ = union of internal edges of the trees T_v

Theorem (Conant-V): All of this orientation data is equivalent to ordering the edges of the forest

$(G, \Phi, or) =$ 

Furthermore, the ∂ operator sums over adding an edge to the forest (labeled by next number)

$\partial(G, \Phi, or) =$



Exercise $\partial^2 = 0$

Proof of Theorem Is linear algebra

(I don't know a direct way of seeing this)

Recall an <u>orientation</u> determined by an ordering of $S = \{x_1, \dots, x_n\}$ can be described as a choice of unit vector in $\Lambda^n \mathbb{R}S$, where $\mathbb{R}S$ is the v. space w/ basis S.

(ordering the x_i gives $x_1 \wedge \dots \wedge x_n \in \Lambda^n \mathbb{R}S \cong \mathbb{R}$
$:= \det \mathbb{R}S$)

so an orientation on an Θ-graph is a unit vector in

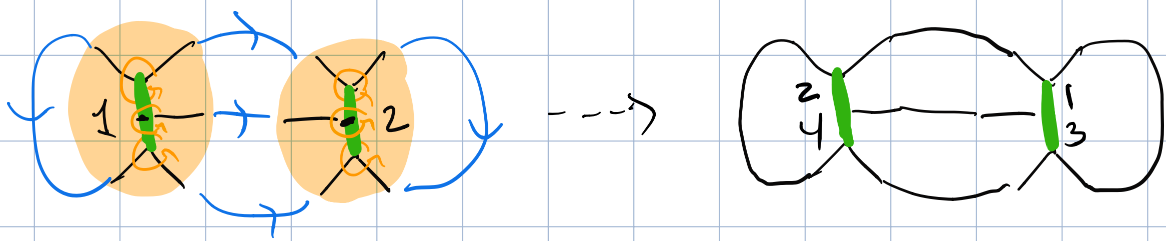
$$\det(\mathbb{R}V(X)) \otimes \bigotimes_{e \in E(X)} \det \mathbb{R}H(e) \otimes \bigotimes_{v \in V(X)} (\otimes_{u \in V(T_v)} \det \mathbb{R}H(u))$$

order $V(X)$; order each pair $H(e)$ of half-edges for $e \in X$; order the half-edges at each vertex of each tree T_v

We are claiming there is a canonical isomorphism of the above expression with

$\det \mathbb{R}E(\Phi)$ $\quad$ ($E(\Phi)$ = edges of Φ)

(ie picking a unit vector in either side determines a unit vector in the other side)



I don't know a direct way to see this

Instead we use two lemmas

Lemma 1: $S = \coprod_{i=1}^{k} S_i$ a finite set

Then $\exists$ a canonical isomorphism

$$\bigotimes_i \det S_i \otimes \det\left(\bigoplus_{|S_i| \text{ odd}} \mathbb{R}\right) \xrightarrow{\cong} \det \mathbb{R}S$$

order elts of each S_i — order the set of S_i with $|S_i|$ odd. $\to$ total ordering of S

Pf switching $x, y \in S_i$ or $S_i \leftrightarrow S_j$ with $|S_i|, |S_j|$ odd

changes the sign of the total ordering of S

$S_i \leftrightarrow S_j$ doesn't change the sign if $|S_i|$ or $|S_j|$

is even.

(Map the other way "groups" the elements of S)

Lemma 2 $0 \to U \xrightarrow{f} V \xrightarrow{p} W \to 1$

a short exact sequence of finite-dim'l vector spaces. Then $\det V$ is canonically isomorphic to $\det U \otimes \det W$.

Pf

Choose a splitting $V \underset{p}{\overset{s}{\rightleftarrows}} W$

The isomorphism is given by

$$\det U \otimes \det W \longrightarrow \det V$$
$$u \otimes w \longmapsto f(u) \wedge s(w)$$

This is independent of s, since $ps = id$

Exercise $0 \to U \to V \to W \to Z \to 0$

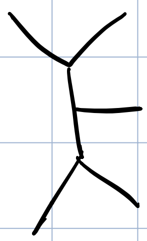
a short exact seq of finite-dim'l v. spaces

$\Rightarrow \exists$ canonical isomorphism

$$\det U \otimes \det W \cong \det V \otimes \det Z$$

Hint: split the sequence into two short exact sequences)

How to use these lemmas:

eg 

T a binary tree

claim: cyclic orderings around each vertex $\iff$ ordering all edges

(The "obvious" correspondence doesn't work)

Use the augmented chain complex of T:

$$0 \longrightarrow C_1 \longrightarrow C_0 \longrightarrow \mathbb{R} \longrightarrow 0$$

This is exact $(\tilde{H}_* T = 0)$

$$C_0 = \mathbb{R}V \qquad \det C_0 = \det \mathbb{R}V$$

To give a chain in C_1, you need to prescribe orientations on the edges, $(\partial e = t(e) - i(e))$

$$\det C_1 = \det \mathbb{R}E \otimes \bigotimes_e \det H(e)$$

Now use Lemma 1:

only one ordering of a one-element set!

$$\det \mathbb{R}V \cong \det \mathbb{R}E \otimes \bigotimes_e \det H(e) \otimes \cancel{\det \mathbb{R}}$$

and Lemma 2 $\cong \det \mathbb{R}E \otimes \det \mathbb{R}H$

Tensor both sides w/ $\det \mathbb{R}V$, $\det \mathbb{R}E$

$$\det \mathbb{R}E \cong \det \mathbb{R}V \otimes \det \mathbb{R}H$$

Now use Lemma 2 again

$$\det \mathbb{R}E \cong \det \mathbb{R}V \otimes \Big(\bigotimes_v \det H(v) \otimes \det \big(\bigoplus_{|v| \text{ odd}} \mathbb{R} \big) \Big)$$

all vertices are odd!

$$\cong \det \mathbb{R}V \otimes \Big(\bigotimes_v \det H(v) \Big) \otimes \det \mathbb{R}V$$

$$\cong \bigotimes_v \det H(v) \quad \checkmark$$

The rest of the proof that

$$(X, \{T_v\}, or_X) \sim (G, \Phi, or_\Phi)$$

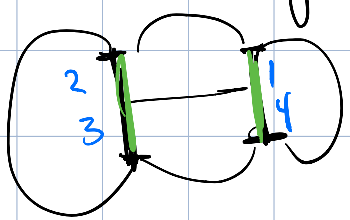
is similar.

(Note that the half-edges of X are exactly the leaves of the T_v ...)

Also have to check:

$$\partial(X, \{T_v\}, or_X) = \sum_{\substack{\Phi \cup e \\ \text{a forest}}} (G, \Phi \cup e, or_{\Phi \cup e})$$

A generator of CG_* is now

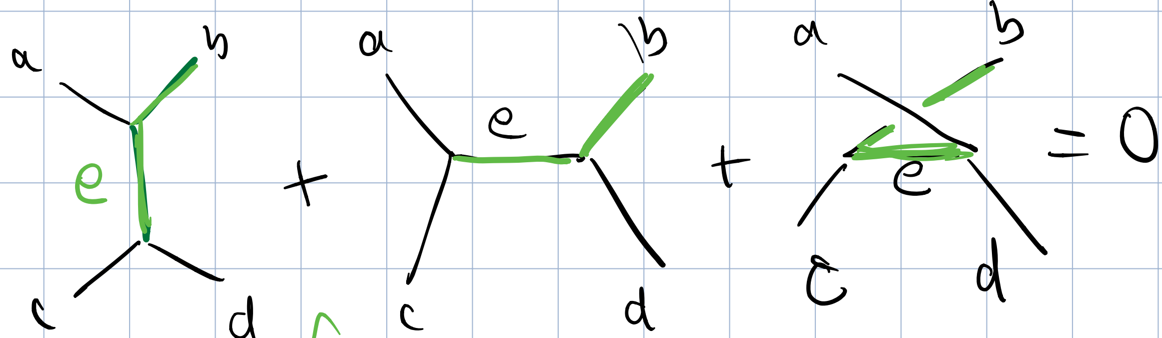


(G, Φ, or_Φ)

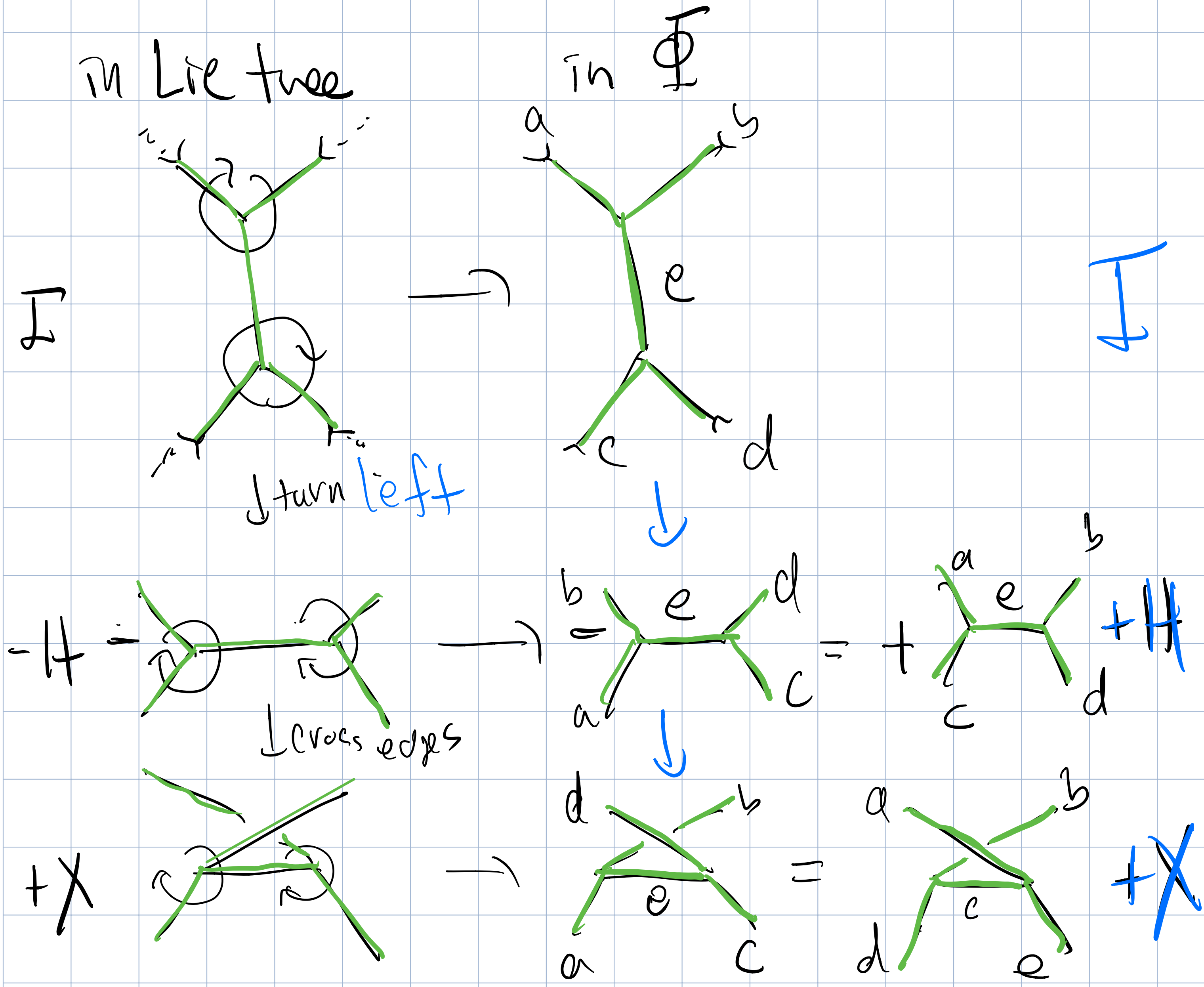
Φ = forest containing all vertices of G

$or \in \det \mathbb{R}E(\Phi)$

IHX translates into



note + sign.


in Lie tree
in Φ
I
a
b
e
c
d
I
turn left
-H
b
e
d
a
c
=
+
a
e
b
c
d
+H
cross edges
+X
d
b
e
a
c
=
a
b
d
c
e
+X

Back to Outer space CV_n.

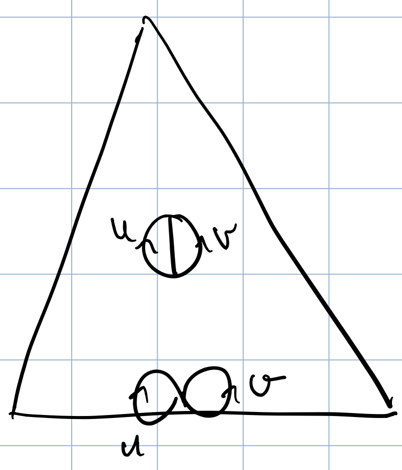
To get the isomorphism

$$H_d(\mathcal{CG}_*^{(n)}) \underset{K}{\cong} H^{2n-2-d}(\mathrm{Out}F_n) \cong H^{2n-2-d}(\mathcal{M}g_n)$$
$$= H^{2n-2-d}(CV_n/\mathrm{Out}F_n)$$

It's easiest to look at CV_n first:

CV_n = disjoint union of open simplices $\sigma(G,g)$, G = admissible, rank n

(connected, $|v| \geq 3$ $\forall v$)

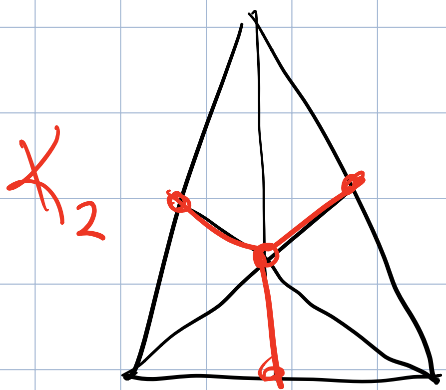


some faces are missing.

$CV_n \subset CV_n^*$ = simplicial completion

$\{u, v\}$ = basis for F_n

Def $K_n \subset (CV_n^*)'$ (barycentric subdivision). Vertex = simplex of CV_n^*.



= span of vertices in CV_n
($\leftrightarrow$ open simplices in CV_n)

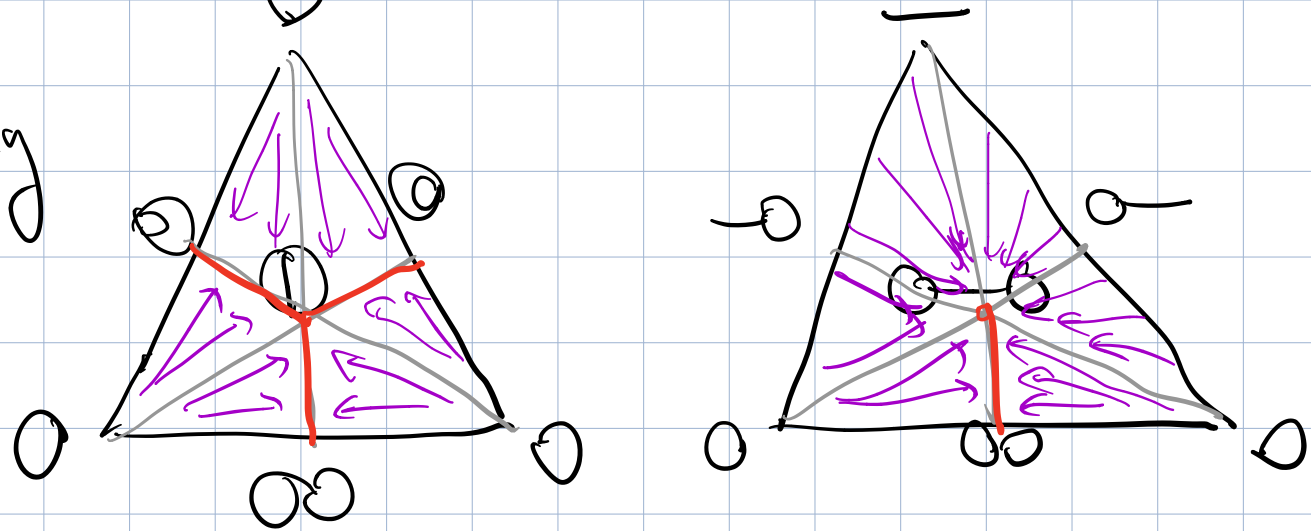
= geometric realization of poset of $\sigma(G,g)$

All maximal simplices of $(CV_n^*)'$ have a missing face (at least a vertex) and a face in K_n (at least a trivalent graph)

$CV_n \searrow K_n$ (deformation retract)

by linearly retracting each maximal simplex of $(CV_n^*)'$ to its face in K_n

eg



The action of $\mathrm{Out}(F_n)$ permutes the $\sigma(G,g)$, so K_n is invariant under the action

K_n is a def. retract of CV_n, so is contractible

$$\Rightarrow H^*(\mathrm{Out}\,F_n) = H^*(K_n/\mathrm{Out}\,F_n)$$

K_n is a simplicial complex

maximal simplex = chain of edge-collapses.

($n=2$ picture is not enlightening here! Try an $n=3$ picture)