We've shown the homology of
Kontsevich's Lie graph complex (odd version) $\mathcal{CG}_*^{(n)}$
is equal to the cohomology of $K_n/\mathrm{Out}\,F_n$

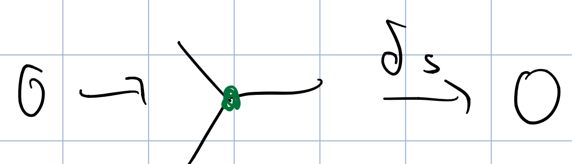
(though the chain complexes are not the same,
and in fact look very different at first)

We left out one step.. showing the vertical columns
in the double complex have no homology except in
the top dimension.

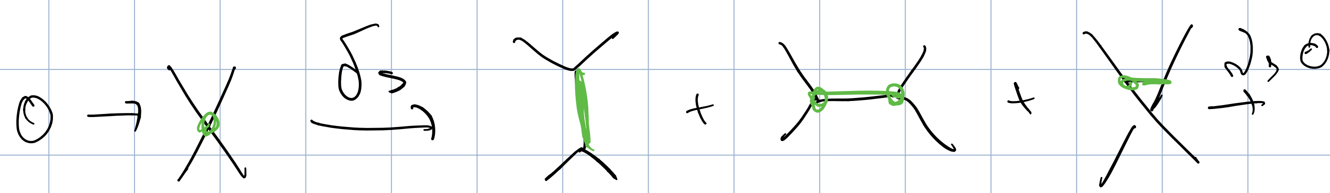
The vertical maps δ_s split one vertex-into two, then add.
What happens at one vertex is independent of what
happens at another.

So let's just look what happens at one vertex v in one
graph G.

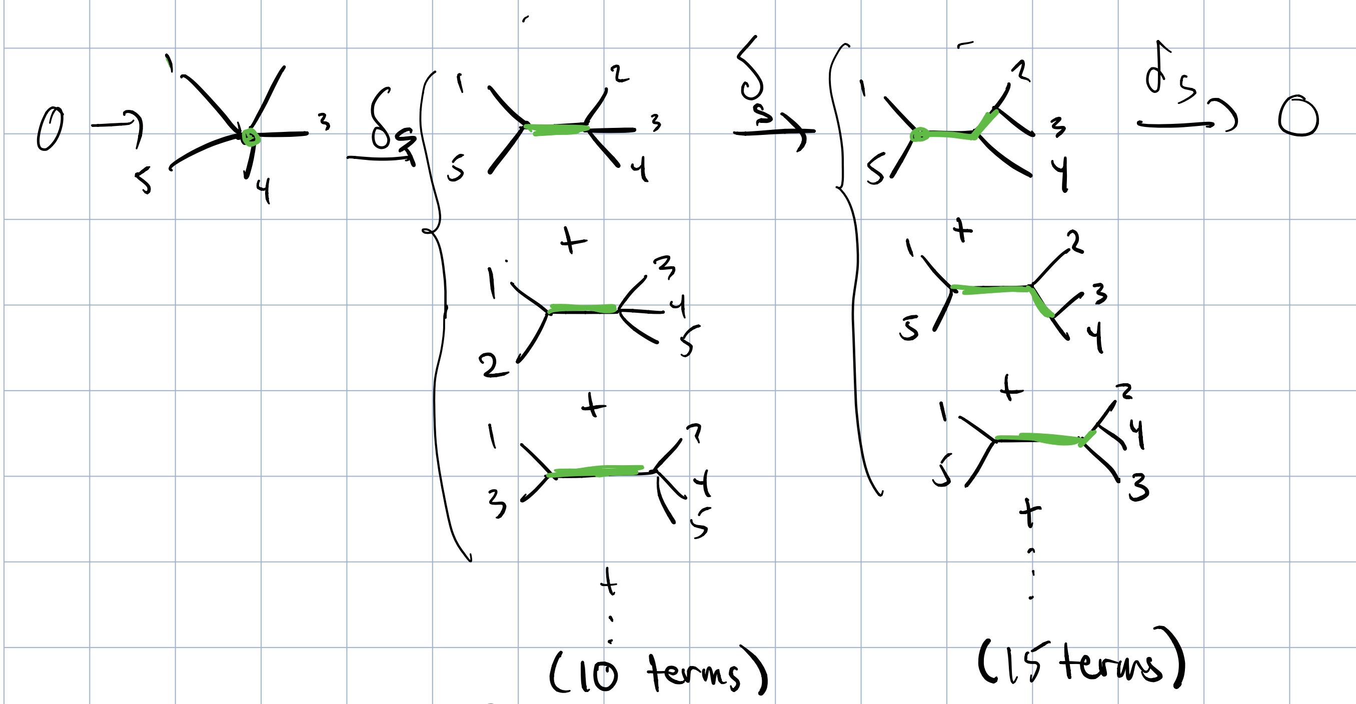
If $|v| = 3$, δ_s doesn't affect it.



If $|v| = 4$



If $|v| = 5$, you can split twice:



$$(*)\quad 0 \to \mathbb{R} \xrightarrow{\delta_s} \mathbb{R}^{10} \xrightarrow{\delta_s} \mathbb{R}^{15} \to 0$$

$(*)$ is the augmented cochain complex of a simplicial complex T_n with

- a 0-cell for each 1-edge tree with n labeled leaves
- a 1-cell for each 2-edge tree with n labeled leaves

⋮

- an $(n-4)$ cell for each $(n-3)$-edge tree with n labeled leaves.

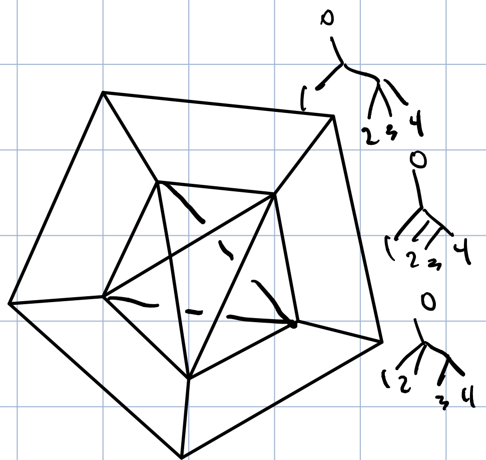
eg

with $n=4$, this s.c has 3 0-cells

$$\simeq S^0 \vee S^0$$

with $n=5$ we have 10 0-cells

15 1-cells,



looks like

(Petersen graph)

Prop $T_n \simeq \bigvee S^{n-4}$ for any n.

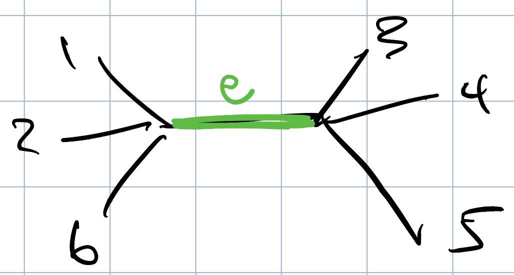
(ie has $\tilde{H}_k = 0$ unless $k = n-4$)

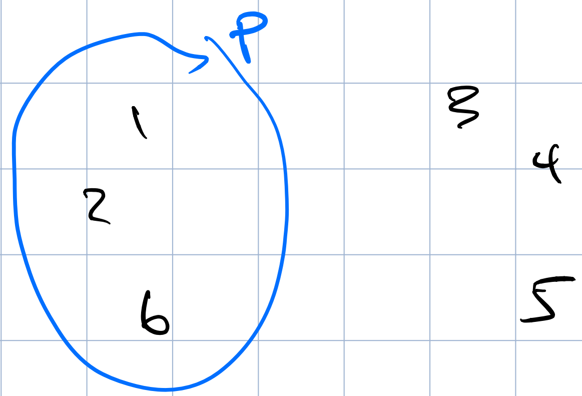
Pf

Another way to describe a tree with n labled leaves and 1 edge is as a thick partition of the set $\{1, \dots, n\}$:

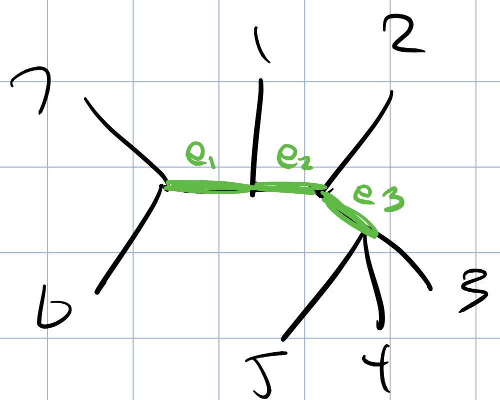
(thick: each side has ≥ 2 elements)

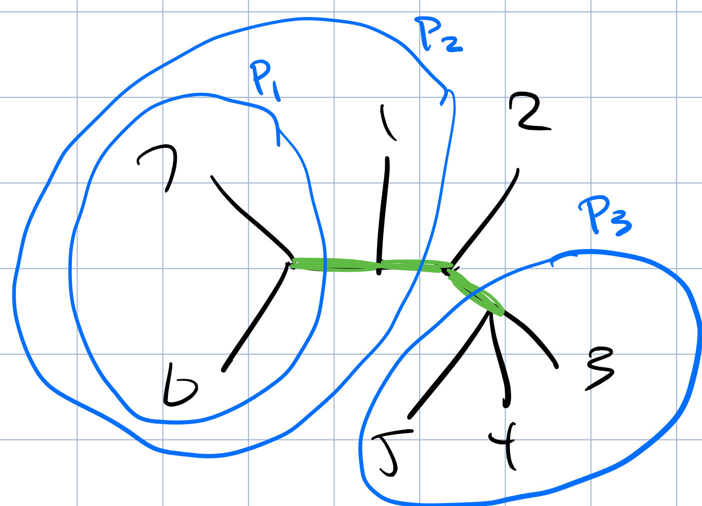
eg





with k edges, you get k pairwise compatible partitions:





(P, Q compatible
$\iff$ same side of P is disjoint from some side of Q)

$\iff$ partitions can be drawn in the plane so that the circles don't intersect.

Now let's reconsider the simplicial complex T_n:

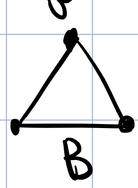
- vertex for each partition (= 1-edge tree)
- edge for each pair (P,Q) of compatible partitions

T_n is flag: $\{P_0, \dots, P_k\}$ are pairwise compatible if and only if they span a k-simplex

Exercise Let $B \subset X$ be a full subcomplex of a flag complex.
If $v \in X \setminus B$ is a vertex with $lk\, v \cap B \neq \emptyset$, and $J \subset B$ is the subcomplex spanned by $lk\, v \cap B$ then the subcomplex $\langle B, v \rangle$ spanned by B and v is equal to $B \cup_J c(J)$.

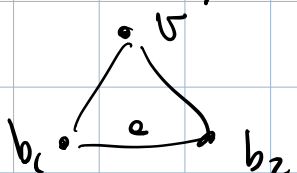
(non-examples: $X = \partial$ of a triangle, (not flag)

B = edge.



Then $J = B$
but $X \neq B \cup_B c(B)$)

X = triangle, B = 2 vertices (not full)



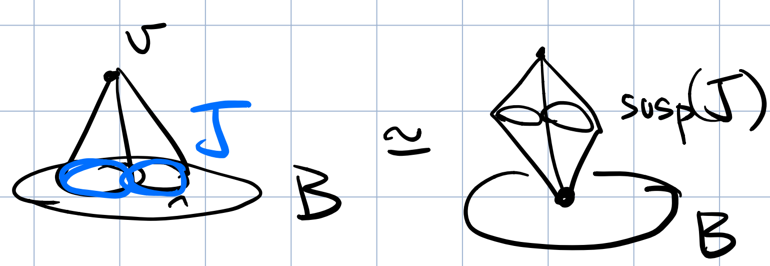
$J = \{b_1, b_2\}$ $cJ =$ 

$B \cup_J c(J) = \partial X$

Corollary B, X, v, as above. If B is contractible then $\langle v, B\rangle \simeq \operatorname{susp} J$.
If $B \simeq \bigvee S^k$ and $J \simeq \bigvee S^{k-1}$ then
$$\langle v, B\rangle \simeq \bigvee S^k.$$

pf Mayer-Vietoris plus van Kampen

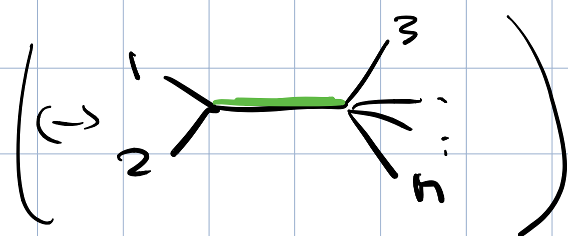
picture:



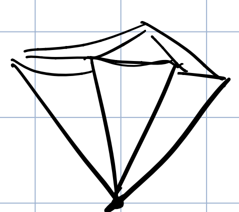
Proof of theorem $(T_n \simeq \bigvee S^{n-4})$

Induction on n (true for $n=4,5$)

Let $P_0 = \begin{matrix} 1 \\ 2 \end{matrix}\ 3 \cdots n$



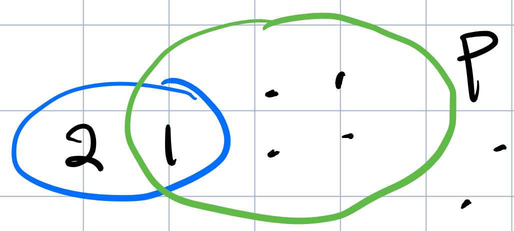
T_n^0 = subcomplex spanned by partitions compatible with P_0



= cone on P_0, so $\simeq$ pt.

Note lk $P_0 \simeq T_{n-1}$. (shrink the leaves 1 & 2 to a pt)

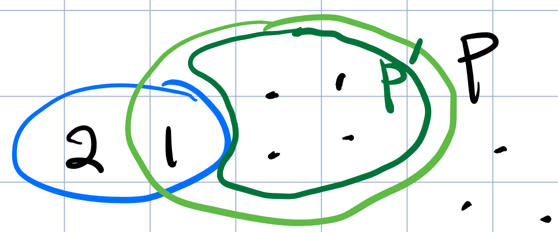
<u>Not in T_n^0</u>: partitions that <u>cross</u> P_0



size 5

Define the <u>size</u> of P to be the number of elements in the side containing 1.

If P crosses P_0, let P′ be the partition obtained from P by putting 1 on the other side.



P′ is compatible with both P <u>and</u> P_0

We will add all P of size > 2 to $c(P_0)$ in order of decreasing size, using the corollary to keep control of the homotopy type.

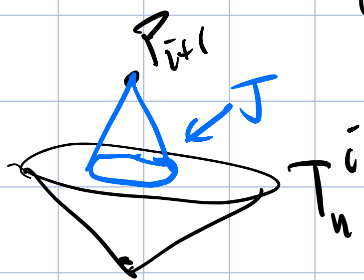
So.. Order the $P \in T_n \setminus c(P_0)$ so that

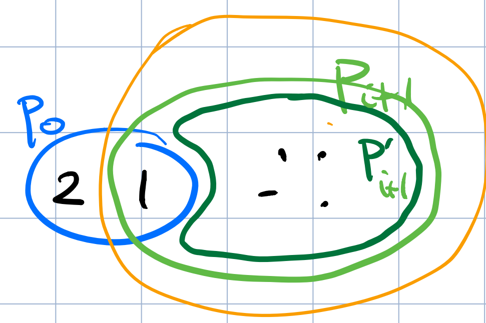
$\text{size}(P_1) \geq \text{size}(P_2) \geq \cdots$ —

Let T_n^i = span of T_n^0 and all P_j with $j \leq i$

Claim: $J = \langle \mathrm{lk}\, P_{i+1} \cap T_n^i \rangle$ is a cone on P'_{i+1} so is contractible.

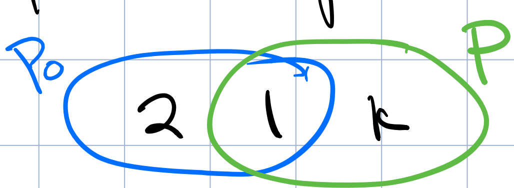
Pf: $Q \in J$ if Q is compatible with P_{i+1} and either has size $\geq$ size P_{i+1} (which $\Rightarrow$ size $>$ size P_{i+1}) or is compatible with P_0

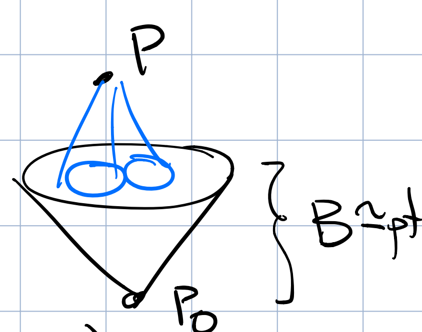




In either case, Q is compatible with P'_{i+1}
so $J = c(P'_{i+1})$

The only vertices we haven't yet included are partitions of size 2 that cross P_0!





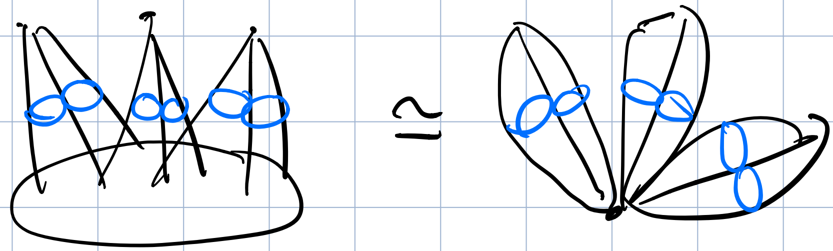
But $\mathrm{lk}\, P \cap B = \mathrm{lk}\, P$

$\simeq T_{n-1}$ (as we saw)

$\simeq \vee S^{n-5}$ by induction

so adding P gives $\mathrm{susp}(\vee S^{n-5}) \simeq \vee S^{n-4}$.

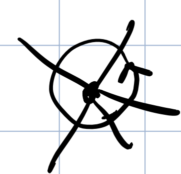
Adding all other $P = 1k$ still gives $\vee S^{n-4}$:



Next: We've related commutative and Lie graph complexes to the topology of moduli spaces of graphs.

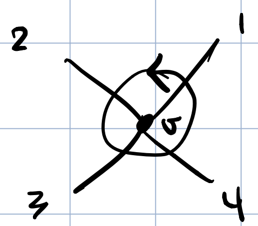
What about associative graphs?

Structure at a vertex = cyclic ordering of adjacent edges



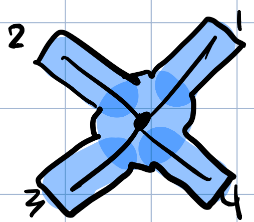
A graph with a cyclic ordering of the edges at each vertex can be "fattened" into a unique oriented compact connected surface

First put a neighborhood of each vertex into the plane, oriented counterclockwise



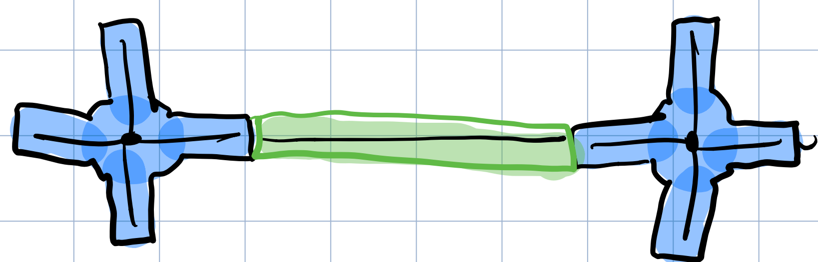
$\subset \mathbb{R}^2$

Then fatten this into a disk with tabs

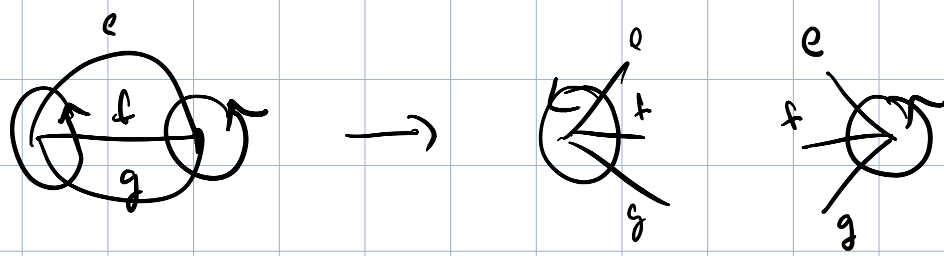


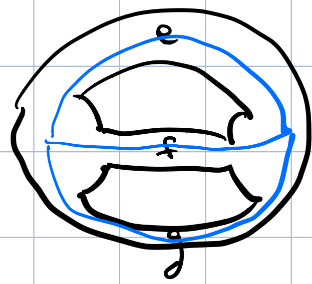
The edges of the graph give a pairing of the tabs

Connect each pair with an (oriented) rectangle

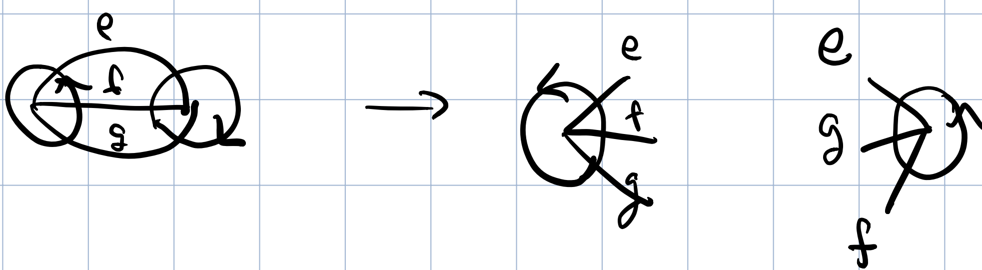


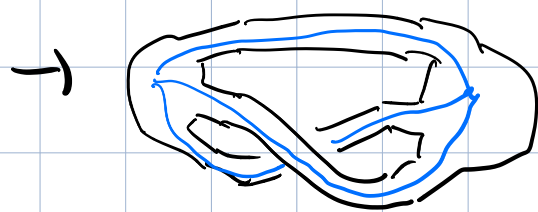
eg





$= S_{0,3}$





$= S_{1,1}$

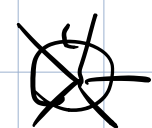
Collapsing an edge doesn't change the homeomorphism type of the surface, so the Associative graph complex breaks up into a direct sum

$$CG_* = \bigoplus_{S = \text{surface}} CG_*^S$$

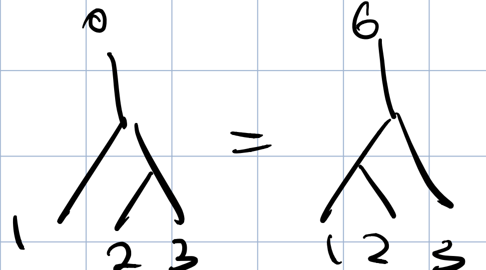
(compact, oriented, connected, with ∂)

We've already done almost all the work needed to relate $\mathcal{CG}_*$ to a moduli space of graphs, when we studied Lie graphs

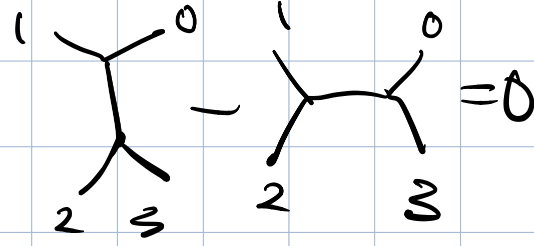
We described generators of the Lie operad as planar trivalent trees modulo AS and IHX relations

We described generators of the Associative operad as planar "stars" 

But back up... this is equivalent to planar trivalent trees modulo the associative relation



, which we read as

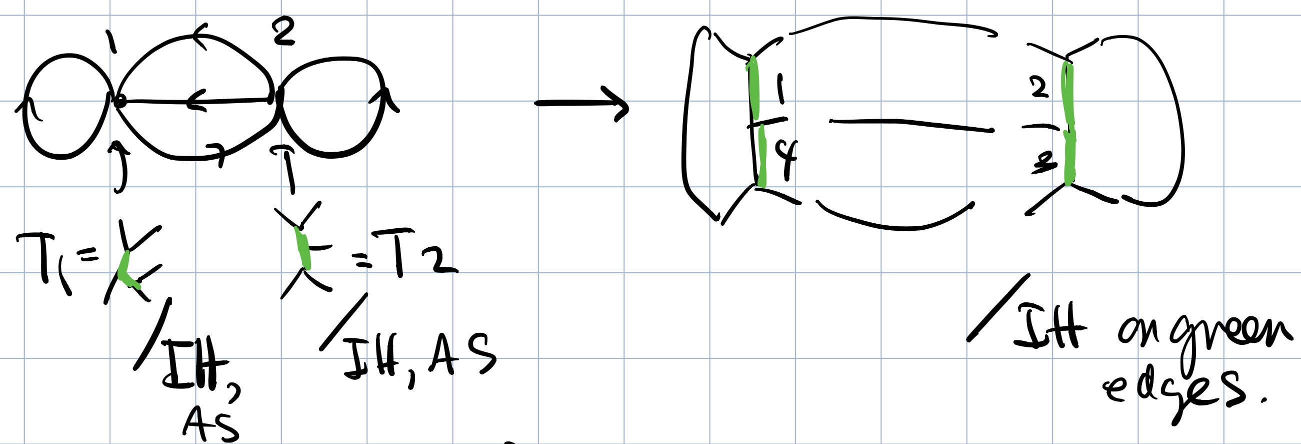


"IH"-relation"

A cyclic ordering of the edges in a star is clearly $\sim$ to a cyclic ordering of the leaves of this trivalent tree

We saw this is ~ ordering the edges of the tree

An associative structure on an odd-oriented graph is



$(X, or, \{T_i\})$ $\qquad$ (G, Φ, or_Φ)

Associative graph $\longrightarrow$ fg_* forested graph

We need to check the orientation lemma

<u>Lemma</u>: The orientation data on $(X, or, \{T_i\})$ graph is equivalent to an ordering of the edges of Φ in (G, Φ, or), and the ∂ operator is the same

(We didn't completely prove this in the Lie case — I just gave you the tools. Ditto here.)

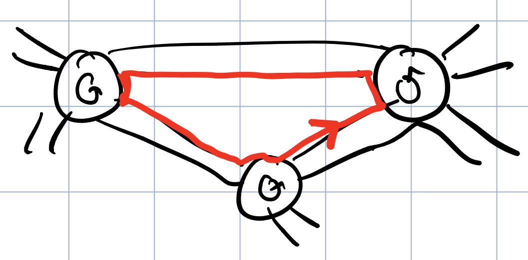
This shows Cg_*^S is isomorphic to fg_*^S, so they have the same homology

We now claim fg_*^S has the same homology as the cochain complex $C^*(K^S/\Gamma(S))$, where K^S = graphs that fatten to S. We have

$K^S \hookrightarrow K_n$ ($\subseteq CV_n$, $\pi_1 S \cong F_n$) by forgetting the cyclic order at each vertex of a ribbon graph.

and

$\Gamma(S) \hookrightarrow Out(F_n)$ is the subgroup that "preserves the cyclic order at each vertex", ie $\Gamma(S)$ preserves the "boundary words", the cyclic words in F_n corresponding to the boundary curves in S:



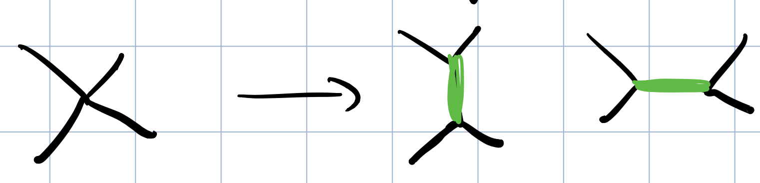
To prove the claim, we decompose the δ operator in $C^*(K^S)$ as before,

$\delta = \delta_a + \delta_s$ $\qquad$ δ_a adds an edge to $\emptyset$
δ_s splits a vertex of G

and, as before, we show the vertical subcomplexes (using δ_s) have homology only in the top dimension

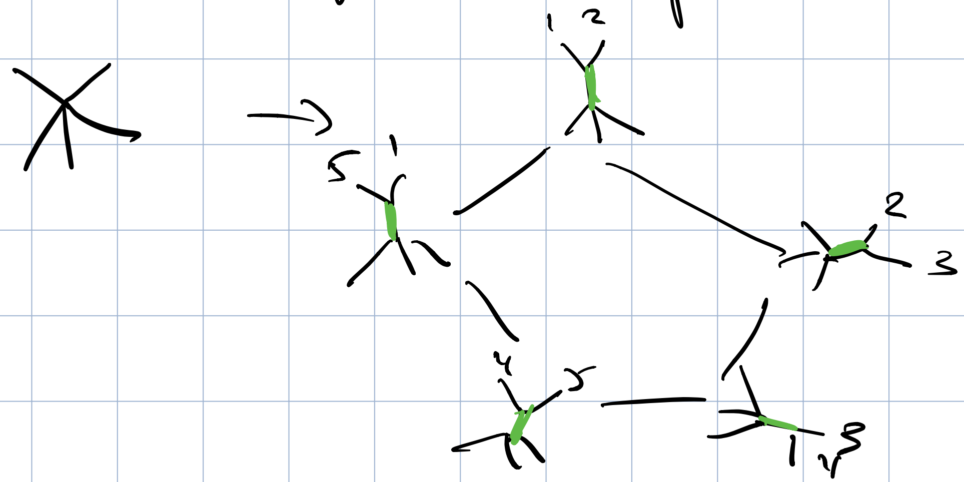
<u>Exercise</u> Identify the vertical (co)chain complex as the cochain complex of a sphere:
look at the trees you can get by IH splitting, starting at a single vertex v of valence $|v|$.

If $|v| = 4$, there are now only two possible splittings without leaving S:



so cplex is $0 \to \mathbb{R} \to \mathbb{R}^2 \to 0$
$=$ augmented cochain cplex of S^0

If $|v|=5$, get the ∂ of a pentagon



<u>**Now**</u>

(1) Zieschang proved $\Gamma(S) \cong \mathrm{Mod}(S)$

(2) The proof that K_n is contractible restricts without change to K_S

$\therefore$ $K_S / \Gamma(S)$ is a (rational)

$K(\mathrm{Mod}(S), 1)$,

i.e. $H_d(\mathcal{C}\mathcal{G}^S_r) = H_d(f\mathcal{G}^S_*) = H^{2r-2-d}(\mathrm{Mod}(S))$