Compactifications of moduli spaces of tori and graphs

If you plan to attend, please register at:
graduate.studies@maths.ox.ac.uk

Moduli spaces are paramater spaces for geometric or algebraic structures

Prototypical example

Let $\mathbb{H}$ = upper half-plane

$$= \{x+iy \mid y>0\} = \{z \mid \operatorname{Im}(z) > 0\}$$

$SL_2(\mathbb{Z})$ acts on $\mathbb{H}$

usually written $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$

Exercise: this preserves $\mathbb{H}$

note this is a <u>left</u> action:

<u>ie</u> To apply AB, you first apply B, then A:

$$AB \cdot z = A(Bz)$$

We will also be interested in <u>right</u> actions

<u>ie</u> TO apply AB, you first apply A, then B

$$z \cdot AB = (z \cdot A)B$$

To get a <u>right</u> action of $SL_2\mathbb{Z}$ on $\mathbb{H}$,

$$z \cdot A = \frac{az+c}{bz+d}$$

(same as acting by A^t in the old formula)

<u>Exercise</u>: This is a right action

Using either action, the quotient M is a moduli space for many different structures:

Things paramatrized by M:

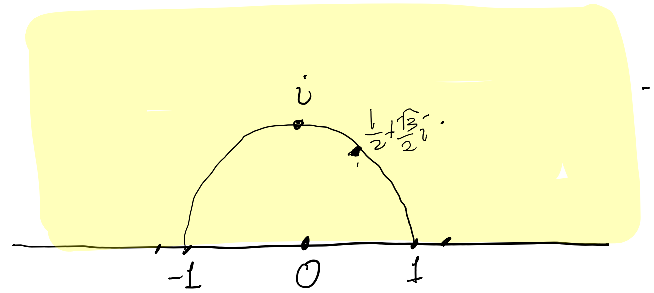
* lattices in $\mathbb{R}^2$ mod rotation, reflection and homothety
* flat tori of area 1
* hyperbolic structures on a once-punctured torus.
* positive definite quadratic forms of det 1 in 2 variables
* elliptic curves
* metric graphs with no separating edges, volume 1 and $\pi_1 \cong F_2$
* systems of weighted non-separating 2-spheres in a doubled handlebody D of genus 2, up to homeomorphism

I'll explain these different interpretations of M in a little while; for now, let's examine the (right) action and the quotient space

Exercise. $\pm I = \pm\begin{pmatrix}1&0\\0&1\end{pmatrix}$ acts trivially

$$\mathrm{stab}(i) = \left\{\pm\begin{pmatrix}1&0\\0&1\end{pmatrix}, \pm\begin{pmatrix}0&1\\-1&0\end{pmatrix}\right\} \cong \mathbb{Z}/4\mathbb{Z}$$

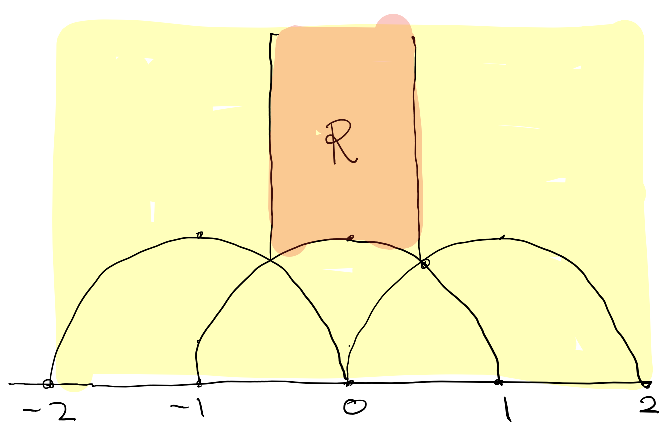
$\begin{pmatrix}0&1\\-1&0\end{pmatrix}: z \longmapsto \dfrac{-1}{z}$ preserves $|z|=1$



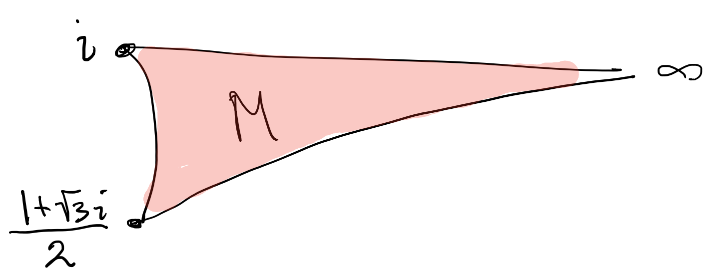
$$\mathrm{stab}\left(\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i\right) = \left\{\pm\begin{pmatrix}1&0\\0&1\end{pmatrix}, \pm\begin{pmatrix}0&-1\\1&1\end{pmatrix}, \pm\begin{pmatrix}1&1\\-1&0\end{pmatrix}\right\} \cong \mathbb{Z}/6\mathbb{Z}$$

A (non-strict) fundamental domain for the action is

$$R = \left\{ z \in \mathbb{H} \;\middle|\; |z| \geq 1,\ |\mathrm{Re}(z)| \leq \tfrac{1}{2} \right\}$$



The quotient M looks like:



Note M is not compact (it has a cusp) –

- in particular it is not a projective variety (this makes algebraic geometers unhappy)

- The action of $SL_2\mathbb{Z}$ on $\mathbb{H}$ is proper (stabilizers are finite) but not cocompact, so $\mathbb{H}$ is not quasi-isometric to $SL_2\mathbb{Z}$ – (this makes geometric group theorists unhappy)

- $\mathbb{H}$ is contractible and $SL_2\mathbb{Z}$ acts properly, so $H^*(SL_2\mathbb{Z}) = H^*(\mathbb{H}/SL_2\mathbb{Z})$ but non-compactness $\Rightarrow$ can't use Poincaré duality to establish a connection between H^* and H_* (this makes algebraic topologists (mildly) unhappy)

Ways to compactify M

First idea: Add the missing cusp point

M is defined as the quotient of $\mathbb{H}$ by the $SL(2,\mathbb{Z})$-action.

The $SL_2\mathbb{Z}$-action extends to the real line by the same formula $x\cdot\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \frac{ax+c}{bx+d}$

if we allow ∞ as a value

eg $0\cdot\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} = \frac{1}{0} = \infty$

and define $\infty\cdot\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \frac{a}{b}$

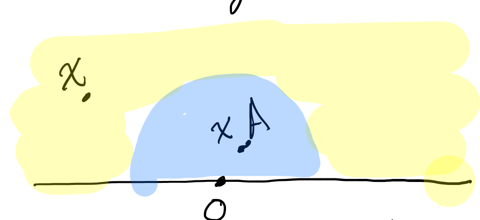
$\mathbb{H}\cup\mathbb{R}\cup\infty$ is compact, w/ topology as subspace of $\mathbb{R}^2\cup\infty$ (nbhd basis for ∞ = complements of balls)

but we've added too many points ($\mathbb{R}/\mathbb{Q}$ of them) and the quotient is not even Hausdorff

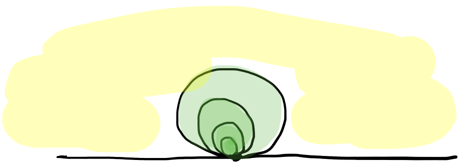
(orbit of 0 is dense in $\mathbb{R}\cup\infty$, you can't separate orbits in $\mathbb{R}\cup\infty$)

Even adding just the orbit of 0, quotient is still not Hausdorff:

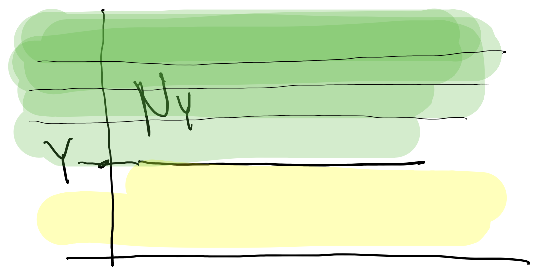
If we try to use translates of (balls in $\mathbb{R}^2$) $\cap\,\mathbb{H}$ as a basis for the topology, the quotient will not be Hausdorff: You can't separate 0 from any interior point



Instead, use horoballs as a nbd basis for points on ∂:



or, at ∞ =



Exercise

using horoballs, can separate the orbit of any interior point from the orbit of ∞.

This is the Satake topology, makes $\mathbb{H}/SL_2\mathbb{Z}$ into a compact Hausdorff space $\overline{M}^S$

(In fact, $\overline{M}^S$ can be given the structure of a projective variety)

Upstairs, in $\mathbb{H}$, have added the orbit of ∞ $= \mathbb{Q} \cup \infty$.

$$\operatorname{stab}(\infty) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : b = 0 \right\} = \left\{ \pm \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} \middle| n \in \mathbb{Z} \right\}$$

is no longer finite, so $\mathbb{H} \cup \mathbb{Q} \cup \infty$ is not quasi-isometric to $SL_2\mathbb{Z}$, also, the quotient M does not compute $H^*(SL_2\mathbb{Z})$

Next idea: (Borel-Serre, but ideas go back to Siegel at least)

preserve the topology of $\mathbb{H}/SL_2\mathbb{Z}$

by adding a <u>circle</u> instead of a point

Lesson from Satake: have to be careful about the topology

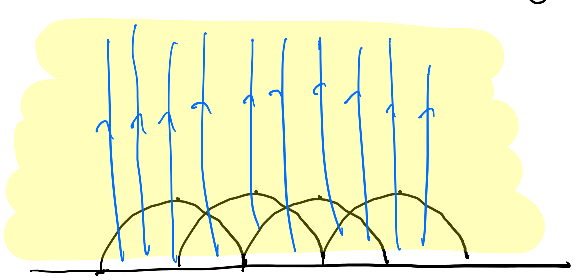
Consider a flow on $\mathbb{H}^2$ defined as follows:

$$g_t = \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}, \quad t > 0 \qquad \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}\begin{pmatrix} x & 1 \\ y & 0 \end{pmatrix} = \begin{pmatrix} x/t & 1/t \\ ty & 0 \end{pmatrix}$$

$$x+iy \mapsto \begin{pmatrix} x & 1 \\ y & 0 \end{pmatrix} \qquad \sim \begin{pmatrix} x & 1 \\ t^2 y & 0 \end{pmatrix}$$

At time t, $x+iy \rightsquigarrow x+it^2y$

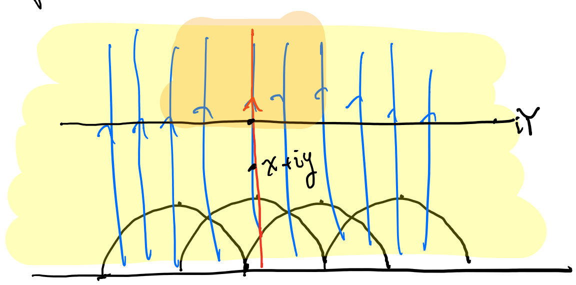
(if you want it to go at unit speed in the hyperbolic metric, write $g_t = \begin{pmatrix} e^{-t/2} & 0 \\ 0 & e^{t/2} \end{pmatrix}$)



Let e = orbit space of this flow

Add a point to $\mathbb{H}$ for each orbit, i.e. add a copy of e

ρ = geodesic ray thru $x+iy$. $N(\rho,\varepsilon,Y) =$



(rays thru $x'+iy$ | $|x-x'|<\varepsilon$) $\cap$ (horoball at height Y)

$N(\rho,\varepsilon,Y)$

we've added a "line at ∞"

Now add a line for every $SL_2\mathbb{Z}$-translate of this picture, ie a line at every rational point

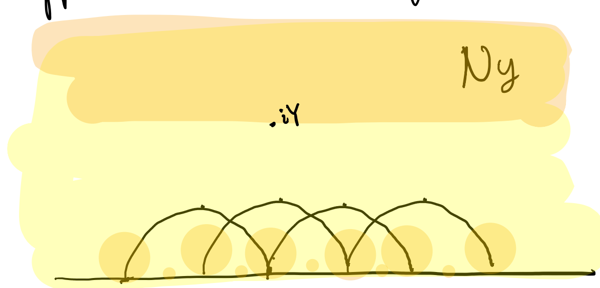
The $N(\rho,\varepsilon,Y)\cdot g$ and balls in $\mathbb{H}$ form a basis for the topology of $\overline{\mathbb{H}}^{BS}$ = Borel-Serre bordification of $\mathbb{H}$.

Thm The quotient by the $SL_2\mathbb{Z}$-action is compact.

Third idea: Find a cocompact equivariant subspace of $\mathbb{H}^2$

Take 1 Instead of adding lines at ∞, simply cut off horoballs equivariantly

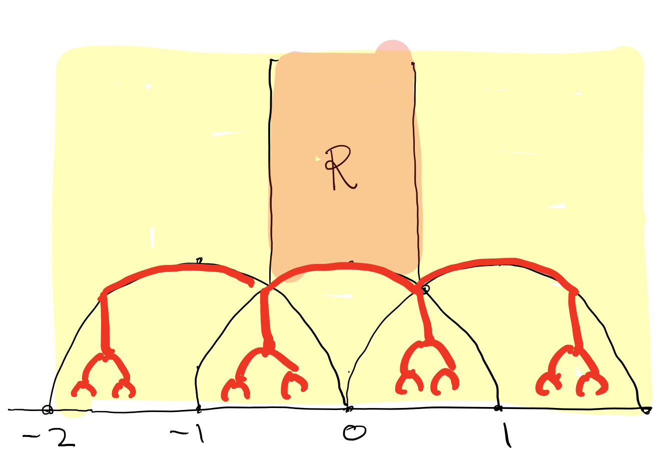
Remove N_Y for $Y>1$ and all its translates



This is a manifold with boundary, and compact quotient

Take 2

Take lower ∂ of the fundamental domain and all of its translates



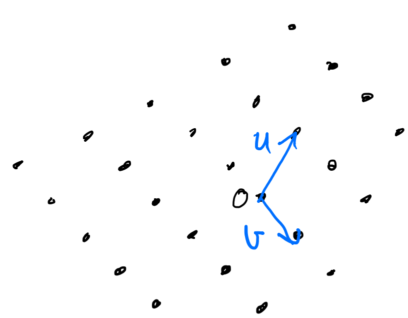
This is cocompact, but not a manifold (no Poincaré duality available here)

I claimed that $\mathbb{H}/SL_2\mathbb{Z}$ is a moduli space for various things. Now want to justify that

What is the

Moduli space of lattices in $\mathbb{R}^2$?

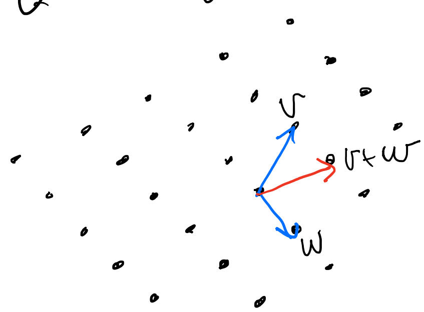
Lattice = subgroup $\mathbb{Z}v \oplus \mathbb{Z}w$, v and w independent vectors.



write v, w as column vectors in a matrix

$$L = \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix} \in GL(2,\mathbb{R})$$

Changing basis gives you the same subgroup



- this is doing column operations on L
  ie multiplying L on the right by an elt of $GL_2\mathbb{Z}$

so we may take $GL_2\mathbb{R} / GL_2(\mathbb{Z})$ as

a moduli space for lattices

If we want to consider lattices only up to reflection, rotation and scaling, we can multiply L on the left by

a reflection matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

a rotation matrix $r_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

or a scale matrix

$$h = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \quad \lambda > 0$$

Then get a moduli space

$$O(2)\cdot H \backslash GL_2\mathbb{R} / GL_2\mathbb{Z}$$

If we want to keep the information about the generators of the lattice, get

$$X = O(2)\cdot H \backslash GL_2\mathbb{R}$$

= moduli space of marked lattices modulo isometry and homothety