Lecture 3

Doing **Borel-Serre for GL_n**

Notation

$G = GL_n\mathbb{R}$ $\quad K = O(n)$ $\quad H = \left\{ \begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{pmatrix} \middle| \lambda > 0 \right\}$

$P < G$ a parabolic subgp, $K_P = K \cap P$

$X = KH \backslash G$ $\qquad \Gamma = GL_n\mathbb{Z}$

We want to add a euclidean space $e(P)$ to X for each rational parabolic supgroup P

$\left(P = \text{stab } V_1 \subset \cdots \subset V_k, \quad V_i \subset \mathbb{R}^n \text{ rational} \right)$

changing basis conjugates P

Can choose a basis w/ $v_1, \ldots v_{i_1} \in V_1$

$v_{i_1+1} \ldots v_{i_2} \in V_2$

etc

Change basis so $v_i \longmapsto e_i$

Then P is conjugate to $\mathrm{stab}(E_1 \subset \cdots \subset E_k)$

which is block lower triangular.

i^{th} diagonal block of size $\dim(E_i) - \dim(E_{i-1})$

$$\left(\begin{array}{c|c|c|c} * & & & \\ \hline * & * & & \\ \hline * & * & * & \\ \hline * & * & * & * \end{array}\right)$$

The block lower triangular matrices are the standard parabolics

For $n=3$ they are

$$P_1 = \left(\begin{array}{c|cc} * & & \\ \hline * & * & * \\ * & * & * \end{array}\right) = \mathrm{stab}\langle e_1 \rangle$$

$$P_2 = \left(\begin{array}{cc|c} * & * & \\ * & * & \\ \hline * & * & * \end{array}\right) = \mathrm{stab}\langle e_1, e_2 \rangle$$

$$B = \left(\begin{array}{c|c|c} * & & \\ \hline * & * & \\ \hline * & * & * \end{array}\right) = P_1 \cap P_2 = \mathrm{stab}\, \langle e_1 \rangle \subset \langle e_1, e_2 \rangle$$

Prop: $P \hookrightarrow G$ induces a homeomorphism $HK_P\backslash P \longrightarrow HK\backslash G$

Pf We may assume P is standard. The right action of G on $X = HK\backslash G$ restricts to a right action of P.

By Gram-Schmidt (also known as the "QR" algorithm) every $g \in GL(n,\mathbb{R})$ can be written $g = qr$, with $q \in O(n)$ and r lower triangular.

So each coset HK is equal to $HKqr = HKr$, $r \in \underline{\underline{P}}$!

ie P acts transitively on X.

The stabilizer of HK is

$$HK \cap P = H \cdot K_P$$

so $\quad HK \backslash G = X = HK_P \backslash P$

Fundamental fact about group actions:
If G acts on X then the orbit of $x \in X$ can be identified with $G/\mathrm{stab}\, x$

Note $\quad g \in HK_P = HK \cap P$

$\Rightarrow$ rows of g are orthogonal (in std form)

eg
$$\left(\begin{array}{c|cc} * & 0 & 0 \\ \hline x & * & * \\ y & * & * \end{array}\right) \in HK_{P_1} \Rightarrow x = y = 0$$

in general g is <u>block diagonal</u>.

$$HK_p \subset \begin{pmatrix} * & & & \\ & * & & \\ & & * & \\ & & & * \end{pmatrix} =: L_p$$

"Levi component of P"

The center of L_p is diagonal matrices which are constant in each block.

<u>Def</u> $$A_p = \left\{ \begin{pmatrix} \lambda_1 I & & & \\ & \lambda_2 I & & \\ & & \ddots & \\ & & & \lambda_{k+1} I \end{pmatrix} \text{ w/ } \lambda_i > 0 \right\}$$

so $\subseteq$ center of L_p (so in particular commutes with K_p.)

Now can see that there is a left action of A_p on X!

Left Action of A_p:

write $x \in X$ as $HK_p p$, $p \in P$

If $a \in A_p$, $a \cdot HK_p p = HK_p a p$

$(\neq HK_p p a!)$

This is <u>not</u> the same as the right action

We think we understand $n=2$, so:

How does this play out for $n=2$?

$$P = \left(\begin{array}{c|c} * & 0 \\ \hline * & * \end{array}\right) = \operatorname{stab}\langle e_1 \rangle$$

$$K_p = P \cap O(2) = \begin{pmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{pmatrix} \quad \varepsilon_i = \pm 1$$

$$\left[\text{pf: } \underset{\substack{\Uparrow \\ P \cap O(2)}}{\begin{pmatrix} x & 0 \\ y & z \end{pmatrix}} \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} = \begin{pmatrix} x^2 & xy \\ xy & y^2+z^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]$$

$g \in GL_2\mathbb{R}$ $\qquad HKg \in X$

we represented this by $HK\begin{pmatrix} x & 1 \\ y & 0 \end{pmatrix}$ with $y>0$

$\Leftrightarrow x+iy \in \mathbb{H}$

We want to write it as $HK_p\, p \in HK_p \backslash P$

But $\begin{pmatrix} x & 1 \\ y & 0 \end{pmatrix}$ is not in P (it's not in <u>any</u> parabolic!)

So choose a different coset representative which <u>is</u> in P (we know it exists)

$$HKg = HK\begin{pmatrix} x & 1 \\ y & 0 \end{pmatrix} = HK\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x & 1 \\ y & 0 \end{pmatrix}$$

$$= HK\begin{pmatrix} -y & 0 \\ x & 1 \end{pmatrix}$$

$$\leftrightarrow HK_p\begin{pmatrix} -y & 0 \\ x & 1 \end{pmatrix} = \left\{\pm\begin{pmatrix} -y & 0 \\ x & 1 \end{pmatrix}, \pm\begin{pmatrix} y & 0 \\ x & 1 \end{pmatrix}\right\} \in HK_p \backslash P$$

$a \in A_p = \begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix} \quad s, t > 0$

$$\begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix} H K_p \begin{pmatrix} -y & 0 \\ x & 1 \end{pmatrix} = H K_p \begin{pmatrix} -sy & 0 \\ tx & t \end{pmatrix}$$

$$= H K_p \begin{pmatrix} -s/t\, y & 0 \\ x & 1 \end{pmatrix}$$

$$\longleftrightarrow \quad x + i \tfrac{s}{t} y \quad \left(\tfrac{s}{t} > 0\right)$$

ie the orbits of A_p are the vertical lines in $\mathbb{H}$

The stabilizer in A_p of HK_p is H:

(pf: $a \cdot HK_p = HK_p \Rightarrow a \in HK_p$

$\Rightarrow a \in HK_p \cap A_p$.

Since $K_p \cap A_p = \langle I \rangle$, $a = \lambda I \in H$.)

So each orbit is $\Leftrightarrow H \backslash A_p$

Note $A_P \approx (\mathbb{R}_{>0})^{k+1}$

so $\quad {}_H\backslash A_P \approx (\mathbb{R}_{>0})^{k}$

The action of A_P on X is called the geodesic action associated to P

The orbits of A_P foliate $X \approx \mathbb{R}^d$ $\left(d = \frac{n(n+1)}{2} - 1\right)$, each orbit is $\approx \mathbb{R}^k$

Claim the orbit space $\quad {}_{A_P}\backslash X := e(P)$

is homeomorphic to $\mathbb{R}^{d-k}$.

To see this note $\quad A_P \backslash X = A_P H K_P \backslash P = A_P K_P \backslash P$

$(H < A_P)$

eg $P = \mathrm{stab}\ e_1 \subset e_1 e_2 \subset \cdots \subset e_1 \ldots e_{n-1}$

$$P = \begin{pmatrix} * & & & & 0 \\ & * & & & \\ * & & * & & \\ & & & * & \\ & & & & * \end{pmatrix} \qquad K_P = \begin{pmatrix} \varepsilon_1 & & \\ & \ddots & \\ & & \varepsilon_n \end{pmatrix}, \ \varepsilon_i = \pm 1$$

$$A_P = \begin{pmatrix} s_1 & & \\ & \ddots & \\ & & s_n \end{pmatrix} \ s_i > 0$$

$$K_P A_P \backslash P = \begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ * & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \approx \mathbb{R}^{\frac{n(n-1)}{2}}$$

$$\left[\begin{aligned} d - k &= \left(\frac{n(n+1)}{2} - 1\right) - (n-1) \\ &= \frac{n^2 + n - 2 - 2n + 2}{2} \\ &= \frac{n^2 - n}{2} \quad \checkmark \end{aligned}\right]$$

eg $P = \mathrm{stab}\ \langle e_1 \ldots e_k \rangle$

$$p = \left(\begin{array}{c|c} * & \\ \hline * & * \end{array}\right) \in P \qquad K_P = \left(\begin{array}{c|c} O(k) & 0 \\ \hline 0 & O(n-k) \end{array}\right)$$

$K_P\, p$ has a ! coset representative $\left(\begin{array}{c|c} \begin{matrix} a_1 & 0 \\ * & a_k \end{matrix} & 0 \\ \hline * & \begin{matrix} b_1 & 0 \\ * & b_k \end{matrix} \end{array}\right)$

$A_p K_p p$ has ! coset rep $\left(\begin{array}{c|c} \begin{matrix} 1 & & 0 \\ & * & \\ k & * & x \end{matrix} & 0 \\ \hline * & \begin{matrix} 1 & & 0 \\ * & * & \\ x & & x \end{matrix} \end{array}\right)$

$$\approx \mathbb{R}^{\frac{n(n-1)}{2}} \times \mathbb{R}_{>0}^{k-1} \times \mathbb{R}_{>0}^{n-k-1} \approx \mathbb{R}^{\frac{n(n-1)}{2}+n-2}$$

codimension 1? $\left(\frac{n(n+1)}{2} - 1\right) - \left(\frac{n(n-1)}{2} + n - 2\right)$

$$= \frac{\cancel{n^2} + \cancel{n} - 2 - \cancel{n^2} + \cancel{n} - \cancel{2n} + 4}{2}$$

$$= \frac{2}{2} = 1 \checkmark$$

<u>Exercise</u>: General case.

note: for $n > 2$
The subgroups intersect, so the orbits will too.

P acts on $e(P) = A_p K_p \backslash P$ (on the right):

$$(K_p A_p p) \cdot q = K_p A_p pq$$

Specialize to $n=3$ so we can draw pictures. The standard parabolics are:

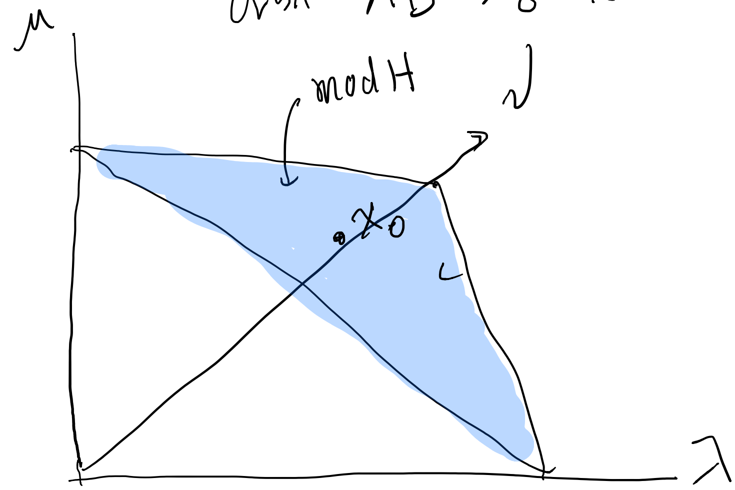
$$P_1 = \left(\begin{array}{c|cc} * & 0 & 0 \\ \hline * & * & * \\ * & * & * \end{array}\right) \qquad P_2 = \left(\begin{array}{cc|c} * & * & 0 \\ * & * & 0 \\ \hline * & * & * \end{array}\right) \qquad B = \left(\begin{array}{c|c|c} * & 0 & 0 \\ \hline * & * & 0 \\ \hline * & * & * \end{array}\right)$$

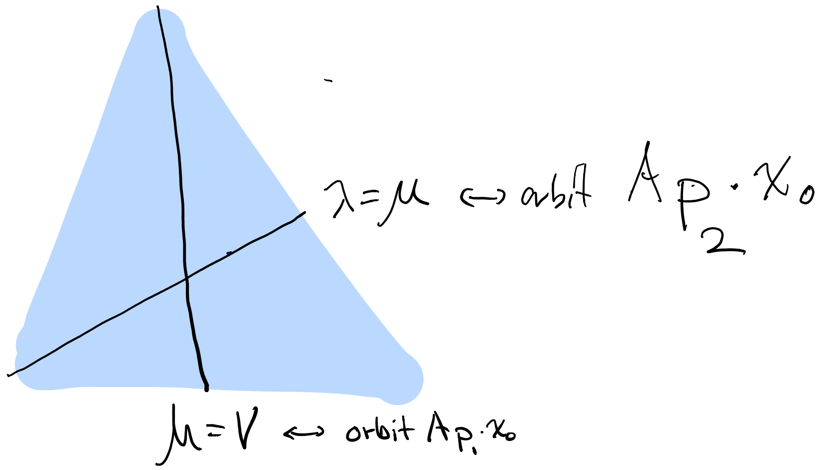
$(B = P_1 \cap P_2)$

$$A_{P_1} = \begin{pmatrix} \lambda & & \\ & \mu & \\ & & \mu \end{pmatrix} \qquad A_{P_2} = \begin{pmatrix} \lambda & & \\ & \lambda & \\ & & \mu \end{pmatrix} \qquad A_B = \begin{pmatrix} \lambda & & \\ & \mu & \\ & & \nu \end{pmatrix}$$

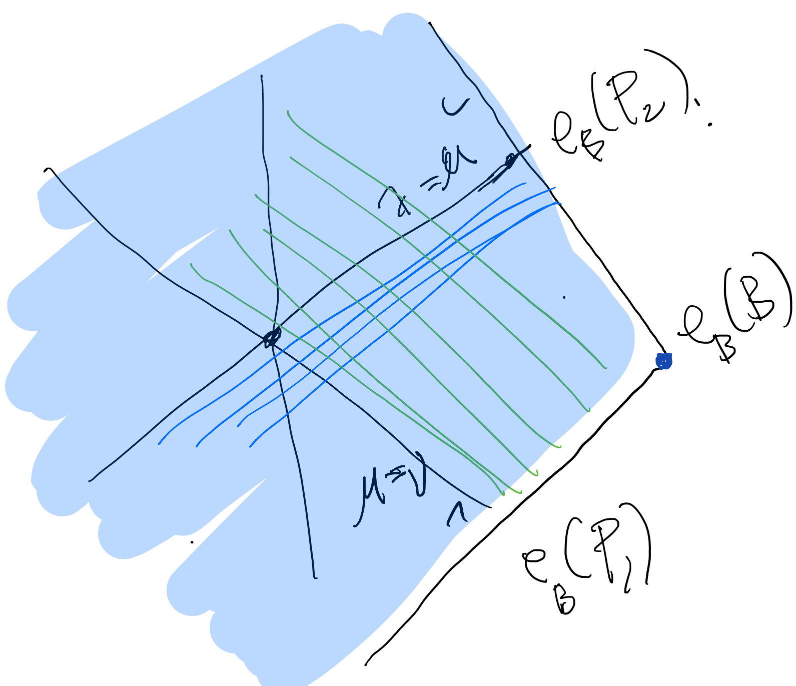
orbit of $x_0 = KH_e$ under A_B

orbit $A_B \cdot x_0 \approx A_B/H$





Better picture



blue = orbit $A_B x_0$

green lines = orbits $A_{P_1} x$ for $x \in A_B x_0$

blue lines = orbits $A_{P_2} x$ for $x \in A_B x_0$

$e_B(B)$ is one point, for the orbit $A_B x_0$

$e(B)$ has one point for each A_B orbit in X, $\approx \mathbb{R}^3$

$e_B(P_i)$ has one point for each orbit in this picture so is $\approx \mathbb{R}$

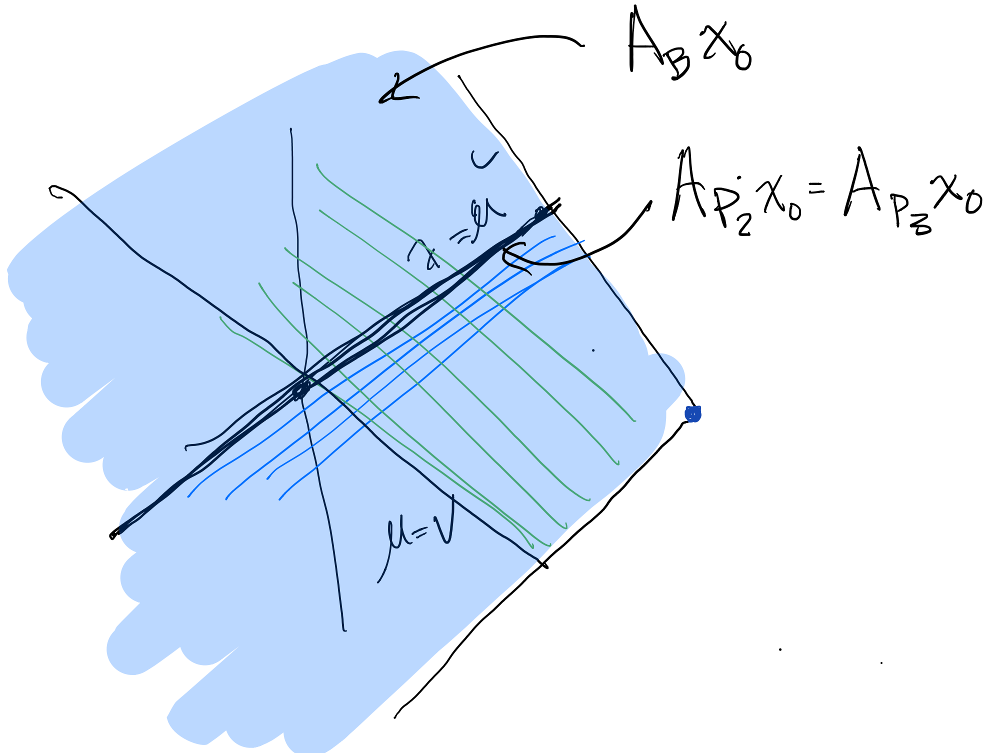
$e(P_i) = e_B(P_i) \times \mathbb{R}^3 \approx \mathbb{R}^4$.

We want to add an $e(P)$ for <u>every</u> parabolic

<u>eg</u> $P_3 = \operatorname{stab}\langle e_3 \rangle = \left(\begin{array}{cc|c} * & * & * \\ * & * & * \\ \hline 0 & 0 & * \end{array}\right)$

But note: $A_{P_3} = \begin{pmatrix} \lambda & & \\ & \lambda & \\ & & \mu \end{pmatrix} = A_{P_2}$!

so $A_{P_3} \cdot KH = A_{P_2} \cdot KH$ = same line in $A_B x_0$.



but action of A_{P_3} is in opposite direction

$$\begin{pmatrix} \lambda & & \\ & \lambda & \\ & & \mu \end{pmatrix} \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{A}} & 0 \\ & & 0 \\ \hline x & y & b \end{array}\right) = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{λA}} & 0 \\ & & 0 \\ \hline \mu x & \mu y & \mu b \end{array}\right)$$

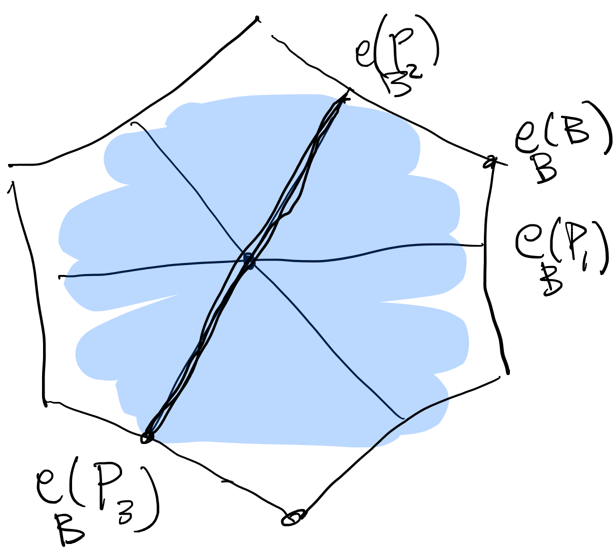
$$\begin{pmatrix} \lambda & & \\ & \lambda & \\ & & \mu \end{pmatrix} \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{A}} & x \\ & & y \\ \hline 0 & 0 & b \end{array}\right) = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{λA}} & \lambda x \\ & & \lambda y \\ \hline 0 & 0 & \mu b \end{array}\right)$$

(compare rank 2 :)

$$\begin{pmatrix} \lambda & \\ & \mu \end{pmatrix}\begin{pmatrix} -y & 0 \\ x & 1 \end{pmatrix} = \begin{pmatrix} -\lambda y & 0 \\ \mu x & \mu \end{pmatrix} \sim \begin{pmatrix} -\frac{\lambda}{\mu} y & 0 \\ x & 1 \end{pmatrix}$$

$$\begin{pmatrix} \lambda & \\ & \mu \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & -y \end{pmatrix} = \begin{pmatrix} \lambda & \lambda x \\ 0 & -\mu y \end{pmatrix} \sim \begin{pmatrix} 1 & x \\ 0 & -\frac{\mu}{\lambda} y \end{pmatrix}$$

$$\frac{\lambda}{\mu} \to \infty \iff \frac{\mu}{\lambda} \to 0$$



So want to glue a <u>hexagon</u> to $A_B x_0$

recall X is foliated by the A_B-orbits, so $e_B(B)$ is a point but $e(B) = pt \times \mathbb{R}^3$

$e_B(P_i)$ is a line but

$e(P_i) = \text{line} \times \mathbb{R}^3 \approx \mathbb{R}^4$

<u>note</u>: $B = \text{stab } e_1 \subset e_1e_2 = \text{stab } e_1 \cap \text{stab } e_1e_2$

P_1 and P_2 are the <u>only</u> parabolics containing B

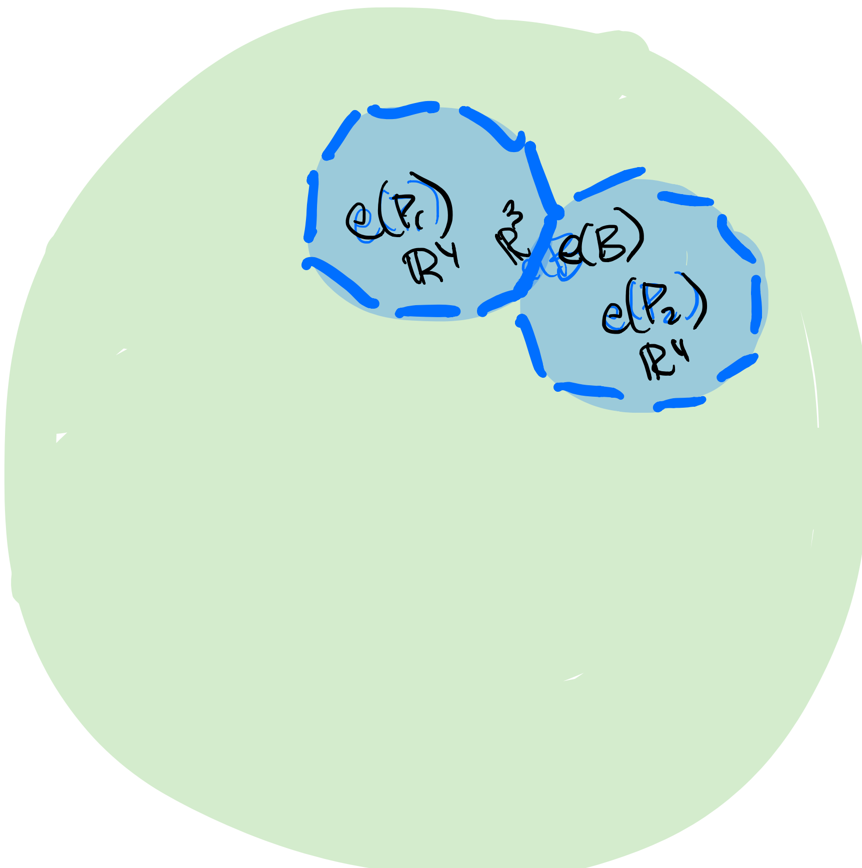
so $e(P_1)$ and $e(P_2)$ are the only $e(P)$'s containing $e(B)$.

on the other hand, P_2 contains <u>lots</u> of conjugates of B:

Take any line $\ell \in \langle e_1 e_2 \rangle$

$B_\ell = \text{stab}(\ell \subset \langle e_1 e_2 \rangle) \subseteq P_2$

so $e(P)$ contains lots of $e(B)$'s



$X \approx \mathbb{R}^5$

Define

$$\bar{X} = X \cup \bigcup_P e(P)$$

To Do.

* Describe the topology on $\overline{X}$ (what is a neighborhood of a point?)

Then show

* the enlarged space $\overline{X}$ is still contractible
* the action extends
* stabilizers are finite and
* the quotient is compact

That's the tricky part