Compactifications of moduli spaces

Lecture 4

<u>Where we are:</u>

We're studying the Borel-Serre bordification for GL_n.

$G = GL_n\mathbb{R}$, $K = O(n)$, $H = \{\lambda I_n \mid \lambda > 0\} \subseteq Z(G)$

$X = HK\backslash G = HK_P\backslash P$ for any parabolic P ($K_P = K \cap P$)

$A_P \cong \mathbb{R}_{>0}^{k+1}$ acts on X on the <u>left</u>

$P = \operatorname{stab} E_1 \subset \dots \subset E_k$ a <u>standard</u> parabolic ($E_i = \langle e_1, \dots e_{j_i} \rangle$) (block lower triangular)

$x_0 = HK \in HK\backslash G$

<u>orbit</u> $A_P x_0 \cong A_P/H \cong \mathbb{R}_{>0}^{k}$

<u>orbit space</u> $e(P) = A_P\backslash X \cong \mathbb{R}^{d-k}$.

$\left(d = \dim X = \frac{n(n+1)}{2} - 1\right)$

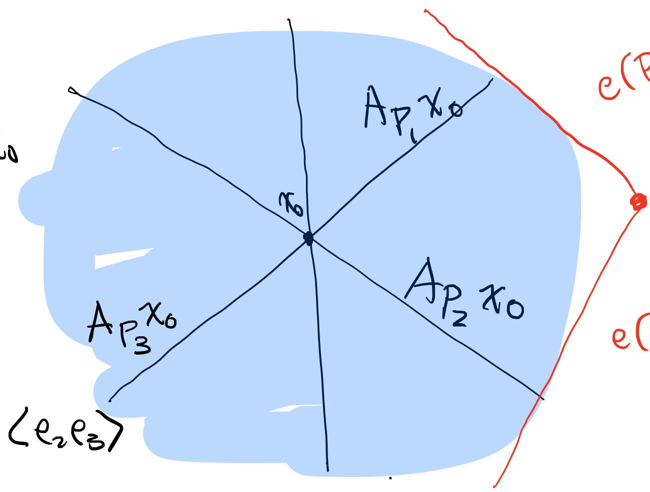
We've specialized to $n=3$ so we can draw pictures

$P_1 = \text{stab}\langle e_1\rangle$ $\qquad$ $A_{P_1} = \left\{ \begin{pmatrix} \lambda & & \\ & \mu & \\ & & \mu \end{pmatrix} \middle| \lambda,\mu>0 \right\}$

$P_2 = \text{stab}\langle e_1 e_2\rangle$ $\qquad$ $A_{P_2} = \left\{ \begin{pmatrix} \lambda & & \\ & \lambda & \\ & & \mu \end{pmatrix} \middle| \lambda,\mu>0 \right\}$

$B = \text{stab}\langle e_1\rangle \subset \langle e_1 e_2\rangle$ $\qquad$ $A_B = \left\{ \begin{pmatrix} \lambda & & \\ & \nu & \\ & & \mu \end{pmatrix} \middle| \lambda,\mu,\nu>0 \right\}$

$A_B \cdot x_0$



$e(P_1) = \mathbb{R}^{3\times} \approx \mathbb{R}^4$

$\dim = 2$

$e(B) = \mathbb{R}^{3\times} \cdot$

$e(P_2) = \mathbb{R}^{3\times} \approx \mathbb{R}^4$

$P_3 = \text{stab}\langle e_2 e_3\rangle$

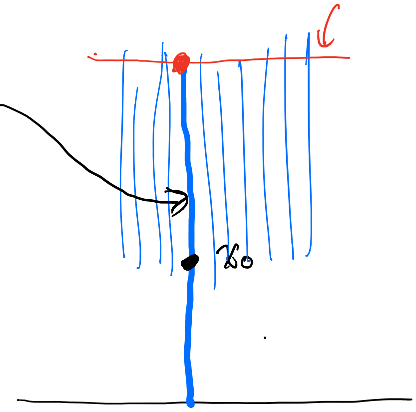
$A_{P_3} = A_{P_1}$

$e(B) = \mathbb{R}^{\times} \cdot$

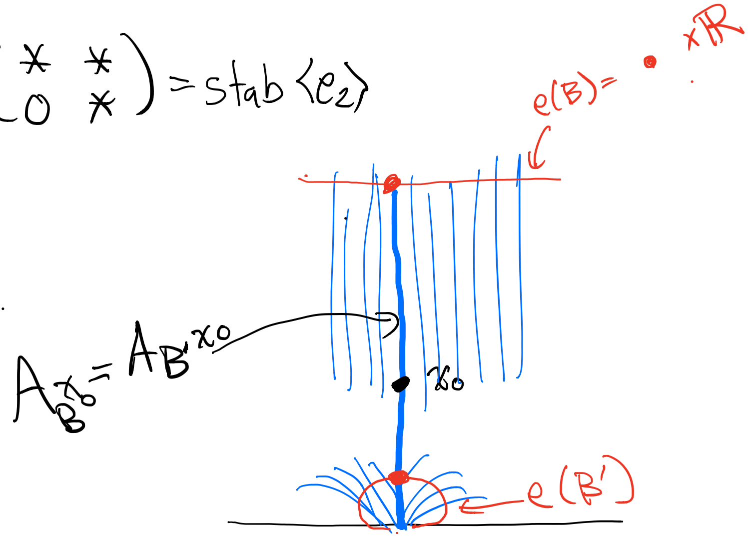
in $n=2$ picture $A_B x_0$

$B = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$

$= \text{stab}\langle e_1\rangle$



If $B' = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} = \mathrm{stab}\langle e_2 \rangle$
then
$A_B = A_{B'}$



We want to define $\overline{X}$ by adding $e(P)$ for every rational parabolic P.

$$\overline{X} = X \cup \bigcup_P e(P).$$ This is a disjoint union.

need a topology on $\overline{X}$, ie need to know how to glue the $e(P)$'s to X and to each other

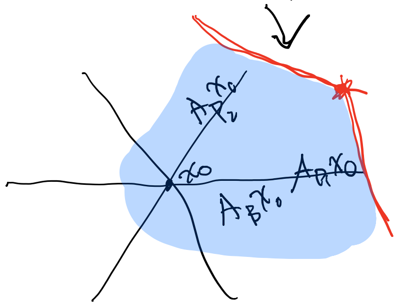
We know the picture for the orbit of x_0 under the standard parabolics:

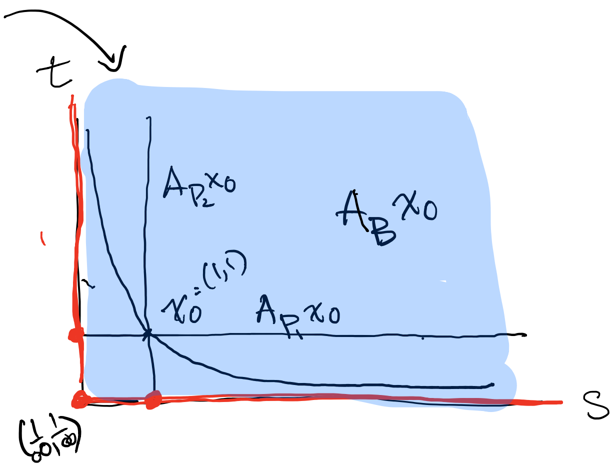
We've identified $A_P x_0$ with $\mathbb{R}^k_{>0}$ $(k = 1 \text{ or } 2)$
we think of gluing $e(P)$ "at ∞", but could also invert the picture
$(s_1, \dots, s_k) \in \mathbb{R}^k_{>0} \mapsto (\frac{1}{s_1}, \dots, \frac{1}{s_k})$ to see it better:

$\underline{n=3}$ $\quad A_B x_0 = \left\{ \begin{pmatrix} s & & \\ & 1 & \\ & & t \end{pmatrix} \right\}$ $\quad A_{P_1} x_0 = \left\{ \begin{pmatrix} s & & \\ & 1 & \\ & & 1 \end{pmatrix} \right\}$ $\quad A_{P_2} x_0 = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & t \end{pmatrix} \right\}$

$s, t > 0$

Identify $A_P x_0$ with $\mathbb{R}^k_{>0}$





Define $\hat{A}_P x_0 := \mathbb{R}^k_{\geq 0}$

Action of A_P on $A_P x_0$ extends to $\hat{A}_P x_0$

eg $\quad P = P_1$

$$A_P x_0 = A_P/H = \mathbb{R}_{>0}$$

$$\begin{pmatrix} \lambda & & \\ & \mu & \\ & & \mu \end{pmatrix} \cdot \begin{pmatrix} s & & \\ & 1 & \\ & & 1 \end{pmatrix} = \begin{pmatrix} \lambda s & & \\ & \mu & \\ & & \mu \end{pmatrix}$$

$$\sim \begin{pmatrix} \frac{\lambda}{\mu} & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$\hat{A}_P x_0 \approx \mathbb{R}_{\geq 0}$

$$= \begin{pmatrix} \lambda/\mu\, t & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

works for $t = 0$ too.
(fixes point if $t=0$)

eg $P = B$

$$A_B \qquad\qquad A_B x_0 \approx \mathbb{R}^2_{>0}$$
$$\ni \qquad\qquad\qquad \ni$$
$$a \qquad\qquad\qquad\qquad \tau$$

$$\begin{pmatrix} \lambda & & \\ & \nu & \\ & & \mu \end{pmatrix} \begin{pmatrix} s & & \\ & 1 & \\ & & t \end{pmatrix} = \begin{pmatrix} \frac{\lambda}{\nu} s & & \\ & 1 & \\ & & \frac{\mu}{\nu} t \end{pmatrix}$$

works if $s=0$ or $t=0$ (or both) too

Now take a copy of $\hat{A}_P x_0$ for each $x \in X$, identify them if they're in the same A_P-orbit:

In $X \times \hat{A}_P x_0$, say $(x,\tau) \underset{P}{\sim} (y,\sigma)$
if $(y,\sigma) = (ax, a\tau)$
for some $a \in A_P$

Define $\boxed{X(P) = X \times \hat{A}_P x_0 \big/ \sim_P}$

$[x,\tau]_P =$ equiv. class of (x,τ)

$(y,\tau) \sim_p (\tau^{-1}y, 1)_p$ if $\tau \in A_p x_0 = A_p/H$

(so τ is invertible)

so get one point of X for each $\sim$ class $[y,\tau]_p$ with τ invertible,

$$[y,\tau]_p = [\tau^{-1}y, 1]_p \rightsquigarrow \tau^{-1}y \in X$$

ie

$x \longmapsto [x,1]_p := x_p$ embeds a copy of $X \hookrightarrow X(P)$

also $(x,0) \sim (ax,0)$ if $a \in A_p$,

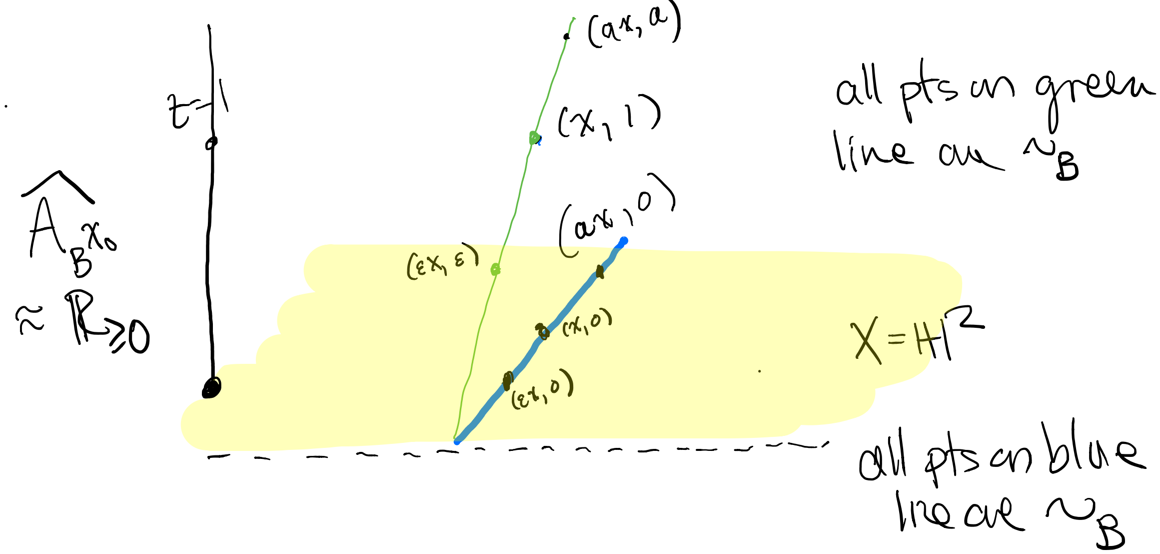
ie get a point $[x,0]_p$ in $X(P)$ for each A_p-orbit

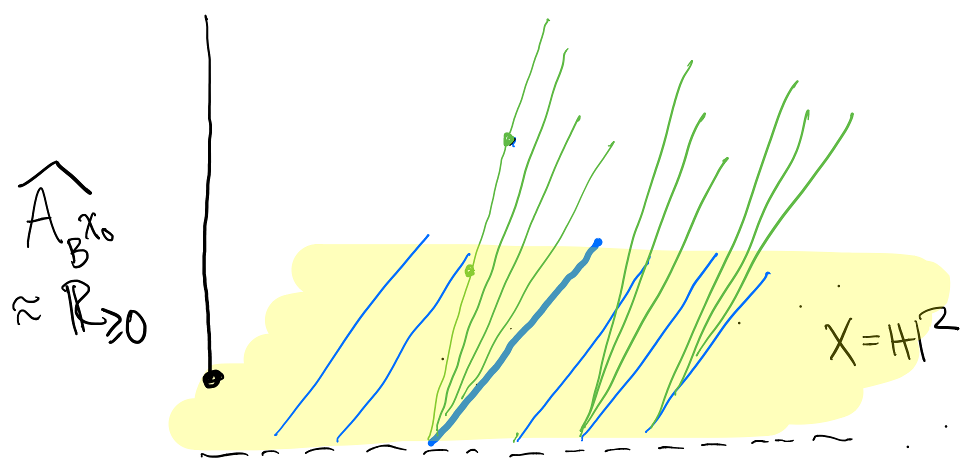
so $X(P)$ contains both X and $e(P) = A_p \backslash X$

we've glued $e(P)$ to X.

Picture for $n=2$

$B = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} = \mathrm{stab}\langle e_1 \rangle$, $\hat{A}_B = \mathbb{R}_{\geq 0}$

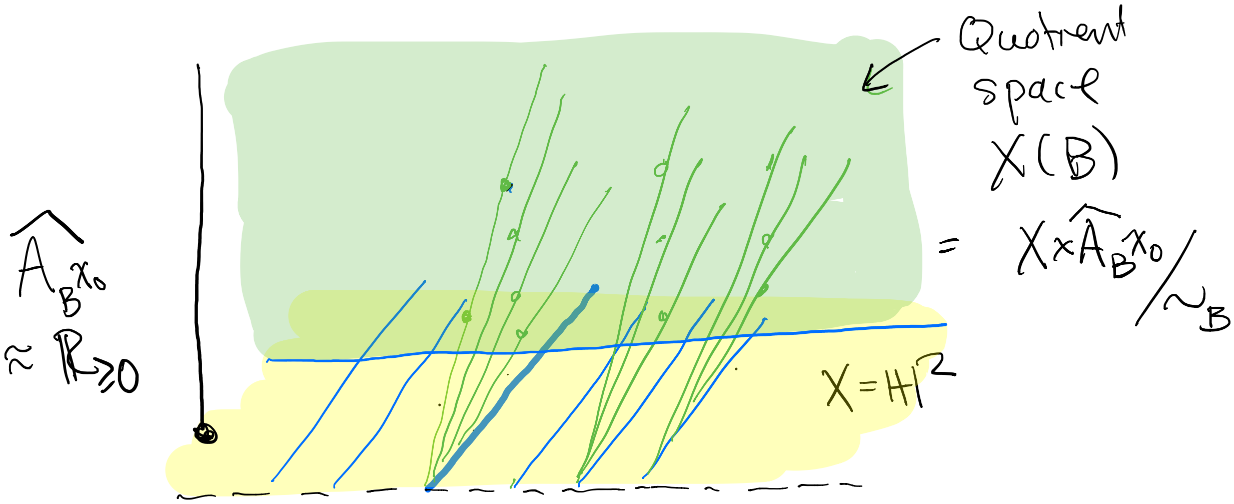




green & blue lines = orbits of A_B

= equiv classes $(x, t) \sim_B (ax, at)$

for $a \in A_B$



one point in green half-plane for each orbit of A_P.

$$X(P) = X \cup e(P)$$

$$\underbrace{\{[x,1]_P\}}_{= X} \cup \underbrace{\{[x,0]_P\}}_{= A_P \backslash X}$$

For $n > 2$ the picture is more complicated because parabolics form chains:

If $P \supset Q$, then $A_P < A_Q$

so $A_P x_0 \subseteq A_Q x_0$, $\widehat{A_P x_0} \subseteq \widehat{A_Q x_0}$

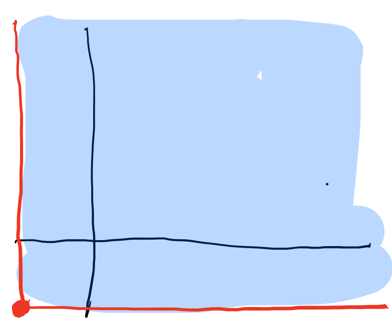
$$X \times \widehat{A}_P x_0 \subset X \times \widehat{A}_Q x_0$$

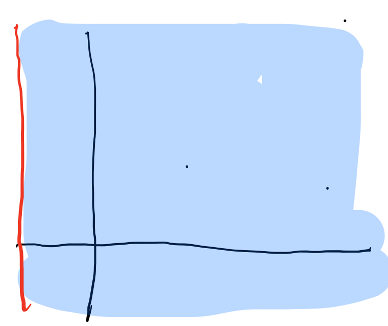
and $(x,\tau) \sim_P (x',\tau') \Rightarrow (x,\tau) \sim_Q (x',\tau')$

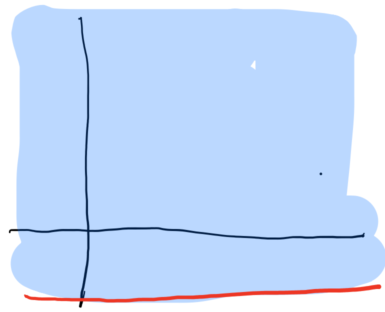
so $X(P) \hookrightarrow X(Q)$

In fact $X(Q) = X \cup \underset{A_Q \backslash X}{\underset{\parallel}{e(Q)}} \cup \bigcup_{P \supset Q} \underset{A_P \backslash X}{\underset{\parallel}{e(P)}}$

In our running $n=3$ example

$X(B) =$  $\times \mathbb{R}^3$

$X(P_1) =$  $\times \mathbb{R}^3$

$X(P_2) =$  $\times \mathbb{R}^3$

$e(P_1) =$  $\times \mathbb{R}^3$

$e(P_2) =$  $\times \mathbb{R}^3$

$e(B) =$  $\times \mathbb{R}^3$

So $X(B) = X \cup e(B) \cup e(P_1) \cup e(P_2)$

<u>Note</u>: $e(B) = \overline{e(P_1)} \cap \overline{e(P_2)}$

To get $\overline{X}$, glue the $X(P)$ along their copies of X:

$$\overline{X} = X \cup \bigcup_{P \text{ rational}} X(P) \Big/ x \sim x_P \ \forall P$$

$$\overline{X} = X \cup \bigcup_{P \text{ rational}} e(P)$$

$\overline{X}$ is a manifold with $\partial = \bigcup_P e(P)$

$\partial \overline{X}$ has a collar, so $\overline{X}$ is homotopy equivalent to its interior X

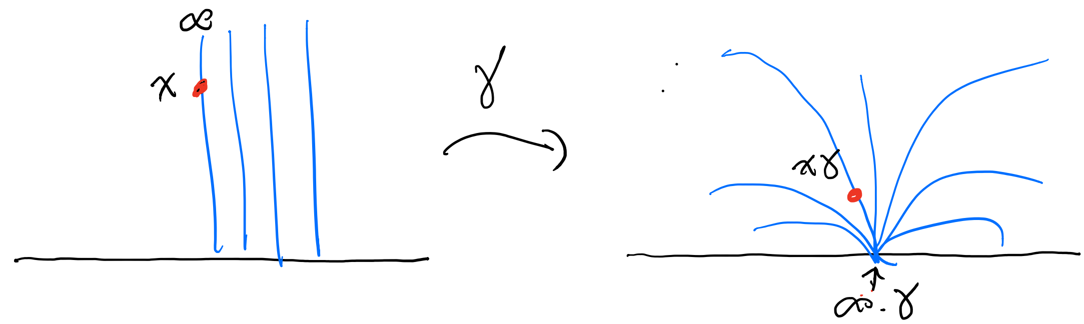
ie $\overline{X}$ is contractible

Action of $\gamma \in GL_n\mathbb{Z}$ on $\overline{X}$:

This is a right action; it should extend the usual right action on $X = HK\backslash G$:

$$(HKg)\cdot \gamma = HKg\gamma$$

on $e(P)$: recall the $n=2$ picture



want to send $e(B) = A_B \backslash X$ $\quad B = \mathrm{stab}\, e_1$

to $e(\gamma^{-1} B \gamma)$, $\gamma^{-1} B \gamma = \mathrm{stab}\, e_1 \gamma$

In general, on $e(P) = A_P \backslash X$

define $(A_P x) \cdot \gamma = A_{\gamma^{-1} P \gamma}\, x \gamma$

note $\mathrm{stab}(e(P)) = \{\gamma \mid \gamma^{-1} P \gamma = P\}$
$= P$ (exercise)

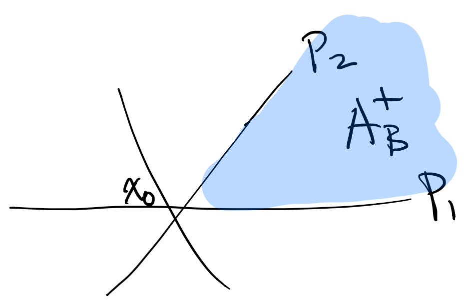
Γ acts transitively on rational parabolics $P = \mathrm{stab}(V_1 \subset \cdots \subset V_k)$ of the same type (determined by the dimensions of $V_1, \ldots, V_k$)

Why is this action cocompact?

$$\bar{X} = X \cup e(B)\cdot\Gamma \cup e(P_1)\cdot\Gamma \cup e(P_2)\cdot\Gamma$$
$$= \left(X \cup e(B) \cup e(P_1) \cup e(P_2)\right)\cdot\Gamma$$

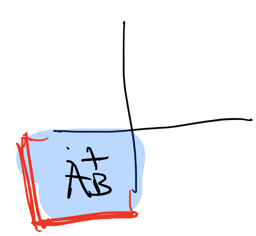
Classical reduction theory finds a fundamental domain for the action of Γ on X

This is contained in $C \times A_B^+$ for $\underline{\text{compact}}$ C. where $A_B^+ \subseteq A_B x_0$ is the "positive cone" bounded by



$A_P x_0$ with $P \supset B$, B the standard minimal parabolic

We've compactified A_B^+ by taking a homeomorphism to $\mathbb{R}^k_{>0}$ and adding a corner: consisting of points in $\underline{x(B)}$



$$\overline{X} = \left(A_B^+ \times C \cup e(B) \cup e(P_1) \cup e(P_2) \right) \cdot \Gamma$$

$$= \left(\underbrace{\overline{A_B^+} \times C}_{\text{compact}} \cup \overline{e(B)} \cup \overline{e(P_1)} \cup \overline{e(P_2)} \right) \Gamma$$

$$= \left(\overline{A_B^+} \times C \cup \overline{e(P_1)} \cup \overline{e(P_2)} \right) \cdot \Gamma$$

since $e(B) \subset \overline{e(P_1)}$.

So to show the quotient is compact, suffices to show $\overline{e(P_i)} / \operatorname{stab} e(P_i)$

is compact.

eg $\overline{e(P_2)}$:

$$\Gamma_p = \operatorname{stab} e(P_2) = P_2 \cap GL_3\mathbb{Z}$$

$$= \left\{ \left(\begin{array}{cc|c} \multicolumn{2}{c|}{A} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline m & n & \pm 1 \end{array} \right) \;\middle|\; \begin{array}{l} A \in GL_2\mathbb{Z} \\ m, n \in \mathbb{Z} \end{array} \right\}$$

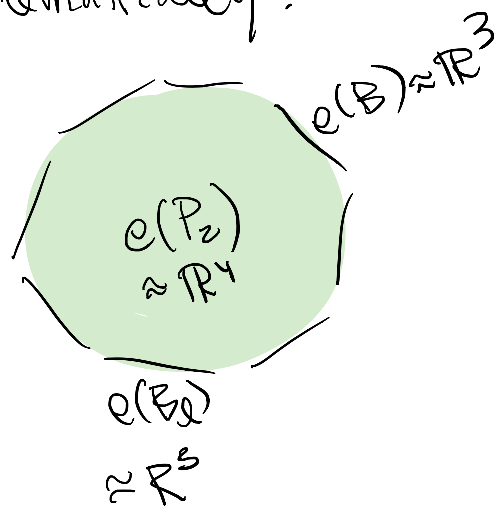
Γ_p acts transitively on rational lines ℓ in $\langle e_1, e_2\rangle$, ie on flags $\ell \subset \langle e_1 e_2\rangle$

So $B_\ell := \mathrm{stab}(\ell \subset \langle e_1 e_2\rangle) \subseteq P_2$ for all $\ell \subset \langle e_1 e_2\rangle$ rational

so $e(B_\ell) \subset \overline{e(P_2)}$ for all ℓ

In fact $\overline{e(P_2)} = e(P_2) \cup \bigcup_{\ell \subset P_2} e(B_\ell)$

Schematically:



looks like our picture of $\mathbb{H}^2$ with lines added at every rational point.

Claim $\overline{e(P_2)}$ is a fibration

$$\begin{pmatrix} I & \begin{matrix}0\\0\end{matrix} \\ \hline * * & \pm 1 \end{pmatrix} = \mathbb{R}^2 \to \overline{e(P_2)}$$
$$\downarrow$$
$$\text{B-S bordification of } \mathbb{H}^2 = \overline{X}_2$$

Γ_{P_2} acts cocompactly on both the base and the fiber

$$\Rightarrow \overline{e(P_2)} / \Gamma_{P_2} \text{ is compact.}$$

To see this, recall (set $P = P_2$)

$$e(P) = A_P \backslash X = A_P K_P \backslash P$$

Here $K_P = P \cap K = \begin{pmatrix} O(2) & \begin{matrix}0\\0\end{matrix} \\ \hline 0 \quad 0 & \pm 1 \end{pmatrix}$

Given a point $A_p K_p \left(\begin{array}{cc|c} * & * & 0 \\ * & * & 0 \\ \hline * & * & * \end{array}\right)$ in $e(P_2)$,

can choose a new coset representative of the form $\left(\begin{array}{cc|c} * & 0 & 0 \\ * & * & 0 \\ \hline * & * & * \end{array}\right)$ using K_p.

Then change it to $\left(\begin{array}{cc|c} -y & 0 & 0 \\ x & 1 & 0 \\ \hline u & v & 1 \end{array}\right)$ using A_p.

Then by K_p to $\left(\begin{array}{cc|c} x & 1 & 0 \\ y & 0 & 0 \\ \hline u & v & 1 \end{array}\right)$: the top

2×2 corner is how we represented points $z = x+iy \in \mathbb{H}^2$.

Now can map $e(P) \longrightarrow \mathbb{H}^2$

by sending $\left(\begin{array}{cc|c} x & 1 & 0 \\ y & 0 & 0 \\ \hline u & v & 1 \end{array}\right) \longmapsto x+iy$

The fiber is $\left(\begin{array}{c|c} I & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline u \ v & 1 \end{array}\right) \cong \mathbb{R}^2$

ie we have a fibration

$$\begin{array}{ccc} \mathbb{R}^2 \approx \left(\begin{array}{c|c} I & \begin{smallmatrix}0\\0\end{smallmatrix} \\ \hline u\ v & 1 \end{array}\right) & \longrightarrow & e(P_2) \\ & & \big\downarrow \\ & & \mathbb{H}^2 \end{array}$$

Γ_P acts by right multiplication.

$$\left(\begin{array}{cc|c} x & 1 & 0 \\ y & 0 & 0 \\ \hline u & v & 1 \end{array}\right) \left(\begin{array}{c|c} A & \begin{smallmatrix}0\\0\end{smallmatrix} \\ \hline m\ n & \pm 1 \end{array}\right) = \left(\begin{array}{c|c} z \cdot A & \begin{smallmatrix}0\\0\end{smallmatrix} \\ \hline (u\ v)A + (m\ n) & \pm 1 \end{array}\right)$$

This preserves the fiber $\left(\begin{array}{c|c} I & \begin{smallmatrix}0\\0\end{smallmatrix} \\ \hline u\ v & 1 \end{array}\right)$

iff $A = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$

in which case the action is $(u,v) \longmapsto (\pm u, \pm v) + (m,n)$

so the quotient is the torus,
in particular it is compact.

The action on the base is the usual action of $GL_2\mathbb{Z}$ on $\mathbb{H}^2$, which is <u>not</u> cocompact.

But in $e(P_2)$ we have added all of the $e(B_e)$'s to $e(P)$

which has the effect of adding lines to the base at every rational point of $\partial\mathbb{H}$,

so the quotient $\overline{e(P_2)}/\Gamma_{P_2}$

is the quotient of the bordified hyperbolic plane, which <u>is</u> compact.