Lecture 6

Correction from last lecture
correspondence between $\partial\overline{X}$ and $T(\mathbb{Q}^{x})$
(Tits building) in Borel-Serre

Exercise $P_V = \text{stab}(V)$
$P_W = \text{stab}(W)$

Then $P_V \cap P_W$ is parabolic if and only if $V \subset W$ or $W \subset V$ in which case it is $\text{stab}(V \subset W)$ (or $W \subset V$)
(Recall parabolic = stabilizer of a flag)

If $V \subset W$, then $\overline{e(P_V)} \cap \overline{e(P_W)} = \overline{e(P_V \cap P_W)}$

In general, $P = \text{stab}\ V_1 \subset \cdots \subset V_k$
$Q = \text{stab}\ W_1 \subset \cdots \subset W_k$
then $P \cap Q$ is parabolic if and only if the V_i and W_i form a single flag

Cover $\overline{\partial X}$ by the $\overline{e(P)}$ with P maximal,

ie $P = \mathrm{stab}\, V$, V a proper rat'l subspace

The nerve of this cover has one vertex for each proper subspace of $\mathbb{Q}^n$

It has an edge whenever $\overline{e(P_V)} \cap \overline{e(P_W)} \neq \emptyset$, which happens iff $V \subset W$ or $W \subset V$

It has a k-simplex for each chain $V_0 \subset \cdots \subset V_k$ of k-inclusions

ie the nerve is equal to $T(\mathbb{Q}^n)$.

If $V \subset W$, then $\overline{e(P_V)} \cap \overline{e(P_W)} = \overline{e(P_V \cap P_W)}$, which is contractible.

In general $\overline{e(P_{V_1})} \cap \cdots \cap \overline{e(P_{V_k})} =$

$\overline{e(P_{V_1} \cap \cdots \cap P_{V_k})}$ whenever $V_1 \subset \cdots \subset V_k$ forms a flag. Since these are all contractible, the nerve of the cover is homotopy equivalent to $\overline{\partial X}$.

Now back to Outer space

We'll actually use <u>reduced</u> outer space $\mathcal{O}_n$

= marked metric graphs (g, G) with no separating edges (or bivalent vertices)

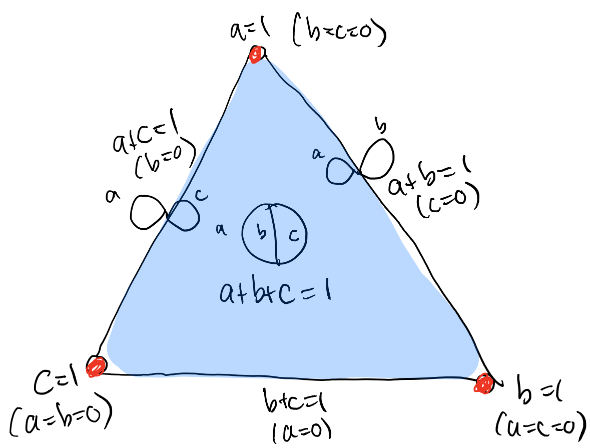
Recall the combinatorial structure:

$\mathcal{O}_n$ decomposes as $\coprod \sigma(g,G)$

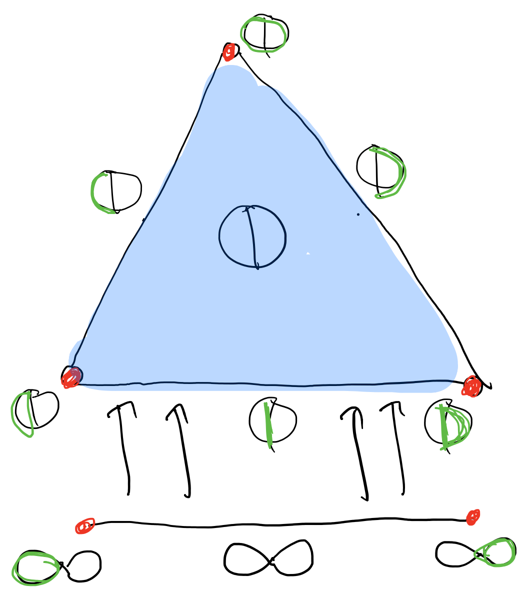
a disjoint union of open simplices

Faces of $\sigma(g,G) \longleftrightarrow$ length 0 on edges of some subgraph G'

A face is <u>at ∞</u> if G' contains a loop.



red faces are at ∞



In Borel-Serre we added a space for every flag of subspaces of $\mathbb{Q}^n$ or, equivalently, for every rational subtorus of $T^n = \mathbb{R}^n/\mathbb{Z}^n$.

eg $\underline{n=2}$: rational lines give simple closed curves on T^2.

In $\mathcal{O}_n$, we will add cells one $\sigma(g,G)$ at a time. The construction will not depend on g, so we will (temporarily) drop g from the notation, just write σ_G

In $\mathcal{O}_n$, if you are near ∞ in some σ_G then some essential subgraph G_1 is very small, where "essential" means non-trivial H_1.

Renormalize so G_1 has volume 1.
This gives you a point in a simplex associated to G_1
If you are near ∞ in σ_{G_1}, there is a subgraph $G_2 \subset G_1$ which is small
Renormalize so G_2 has volume 1
etc. This has to stop since rank H_1 decreases at each step.

ie at a point near ∞, we have a "flag" of subgraphs

$$G = G_0 \supset G_1 \supset \cdots \supset G_k .$$

with (renormalized) metrics of volume 1

Each gives a point in a simplex σ_{G_i} of dimension $e(G_i) - 1$

Let $r_i : \sigma_G \longrightarrow \sigma_{G_i}$ be the renormalization map and set

$$r \doteq \prod r_i : \sigma_G \longrightarrow \prod \sigma_H \subset \prod \overline{\sigma}_H$$

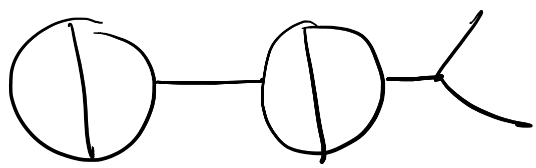
<u>Want</u> to define Σ_G = closure of image of σ_G

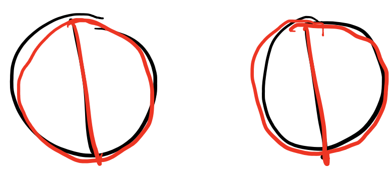
It will be useful to ignore subgraphs with separating edges or isolated vertices
(but allow disconnected subgraphs and bivalent vertices.)

Def A **core graph** is a finite graph with no separating edges or isolated vertices.

Observe

1. Every graph has a maximal core, obtained by removing separating edges and isolated vertices:

$G =$ 

$c(G) =$ 

2. every core subgraph of G is contained in $c(G)$.

3. $\text{Rank } H_1(G) = \text{rank } H_1(c(G))$
$= \#\text{edges} - \#\text{vertices} + \#\text{components}$

4. Removing an edge e from a core graph reduces the rank of $H_1(G)$. (because e doesn't separate G)

Def of Σ_G:

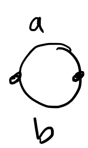
Let $r : \sigma_G \longrightarrow \prod_{\substack{C \subseteq G \\ \text{core subgraph}}} \overline{\sigma}_C$

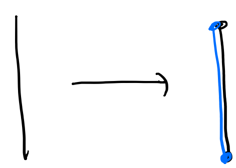
(Here G itself is counted as a core subgraph)

Def Σ_G = closure of $r(\sigma_G)$ in $\prod_{\substack{C \text{ core} \\ \subseteq G}} \overline{\sigma}_C$

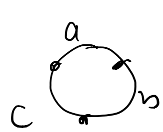
Examples

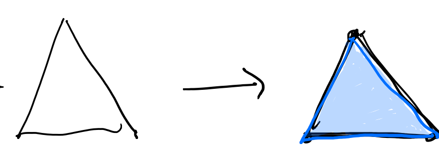
$G =$  $\sigma_G = \bullet = \overline{\sigma}_G = \Sigma_G$

$G =$ 

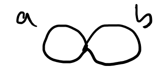
$\sigma_G =$  $= \Sigma_G$

$a+b=1$
$a,b>0$

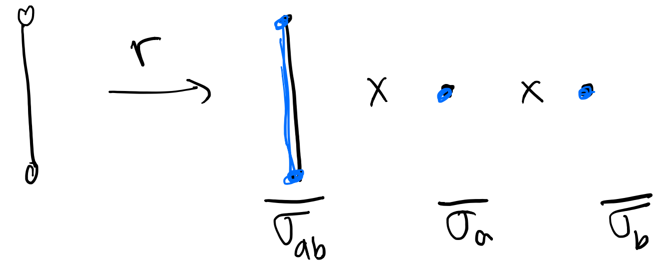
$G =$ 

$\sigma_G =$  $= \Sigma_G$

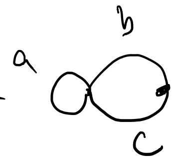
$a+b+c=1$

$G =$ 

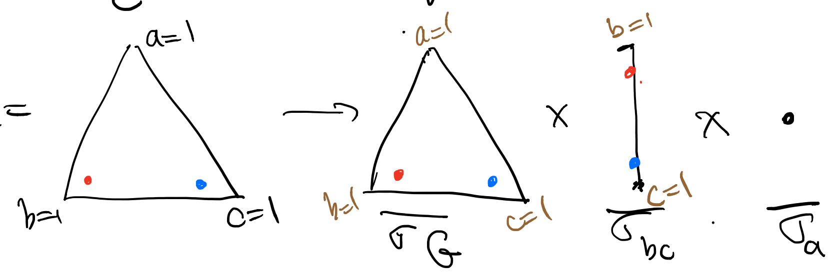
core subgraphs G, a, b

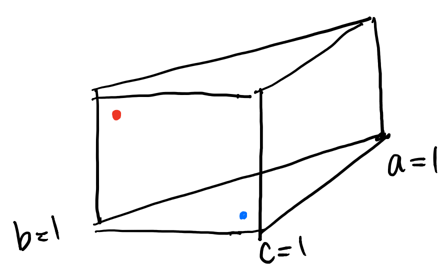
$\sigma_G =$ 

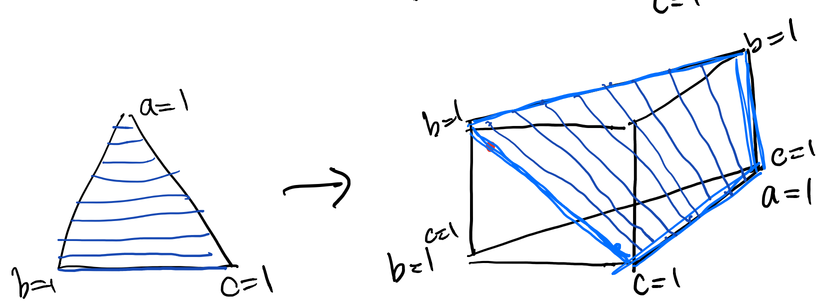
$\Sigma_G \approx$ 

$G =$ 

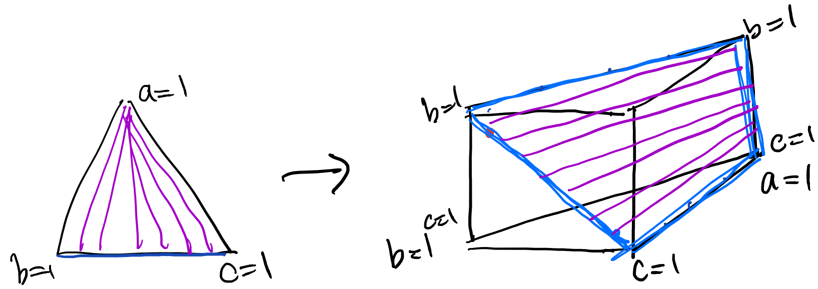
Core subgraphs G, a, bc

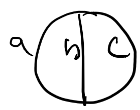
$\sigma_G =$ 



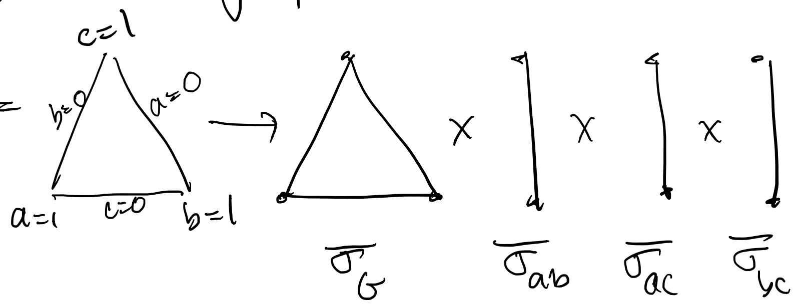


or

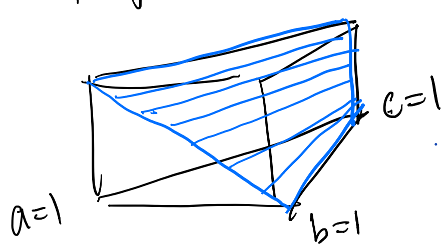


G = 

core graphs are G, ab, ac, bc

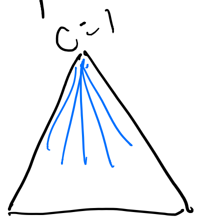
$\sigma_G =$ 

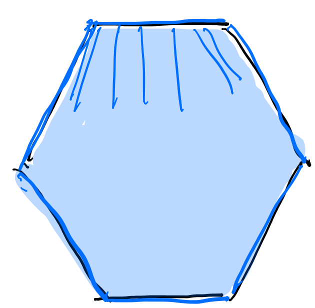
Look at projection onto 1st 2 factors:



Get same picture we had before

ie each corner vertex has been blown up to a line segment

$\sigma_G =$ 

$\Sigma_G =$ 

If the metric on G is non-degenerate (ie there are no edges of length 0) then the metric on every $C \subset G$ is determined by the metric on G.

So it determines a unique point of Σ_G

In fact, if every core graph has a non-zero edge, this determines a unique point of each $\overline{\sigma_C}$, hence a unique point of Σ_G

Let $G_1 \subset G$ be the subgraph spanned by length 0 edges, and $C_1 = \mathrm{core}(G_1)$

To determine a point of $\overline{\sigma_{C_1}}$, need a metric on C_1 with at least one non-zero edge

Every core graph in G either has a non-0 edge or is $\subseteq C_1$, so a non-degenerate metric on C_1, together with the (degenerate) metric on G, determines a point of Σ_G

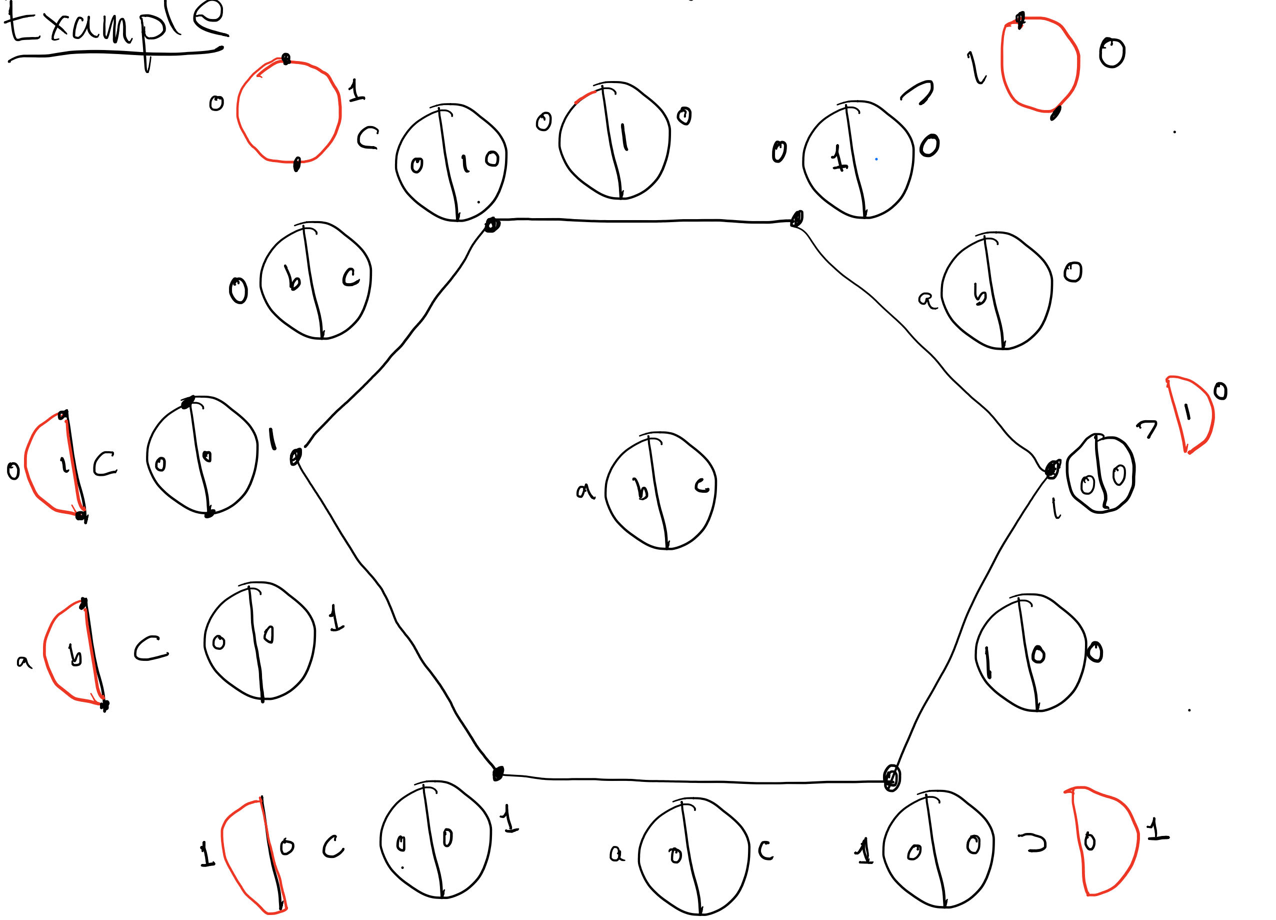
Continuing, we see that a point of Σ_G is determined by a chain of core graphs

$$G = C_0 \supset C_1 \supset \cdots \supset C_k \supset C_{k+1} = \emptyset$$

with metrics of volume 1, such that

$$C_i = \text{core}\,(\text{zero-length subgraph of } C_{i-1})$$

Example

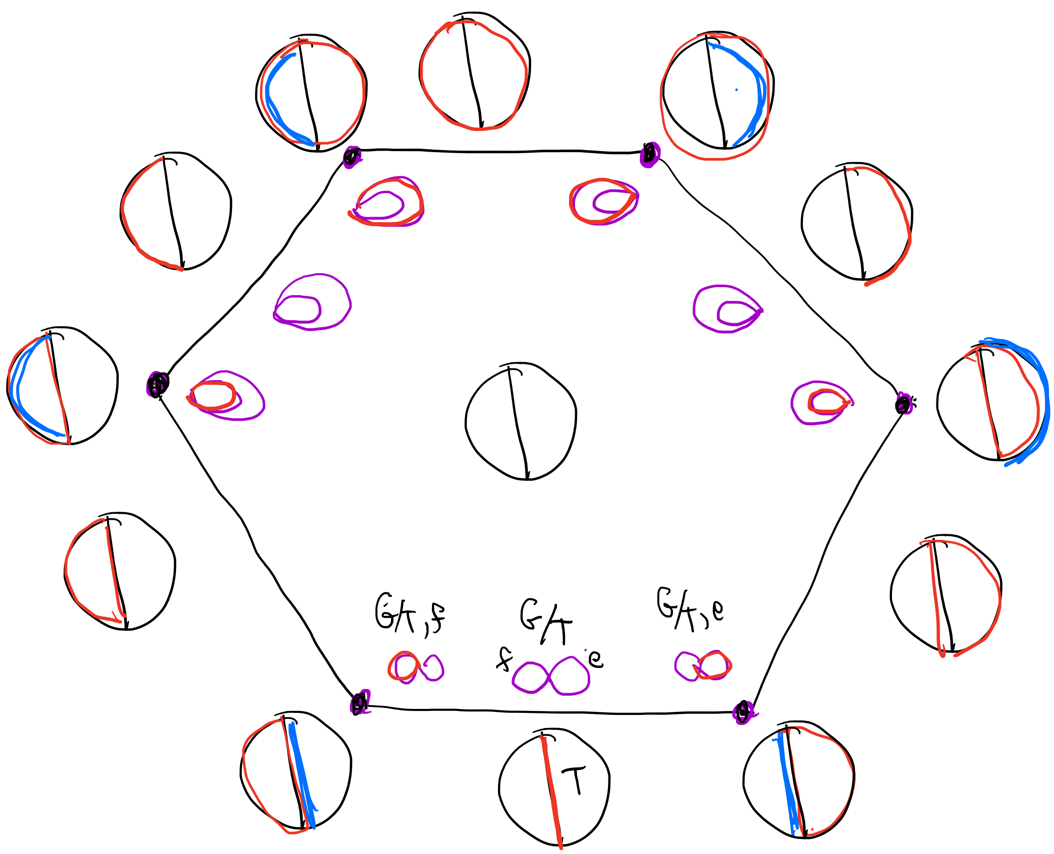


Claim

Vertices of $\Sigma_G \leftrightarrow (G, T; e_1, \dots, e_k)$

T = maximal tree in G

$e_1, \dots, e_k$ = edges of $G \setminus T$, ordered



red edges = length 0 in G

blue edges : length 0 in $C < G$

I will have more to say about the combinatorial description of Σ_G later.

Now put the marking back into the notation

Note If (g',G') is obtained from (g,G) by collapsing a forest Φ: $G \xrightarrow{c_\Phi} G'$
then c_Φ sends core graphs C of G to core graphs C' of G'
and $\sigma_{C'}$ is a face of σ_C

so
$$\underset{\sigma_{C'}}{\underline{\Sigma}}(g',G') \xhookrightarrow{\iota_{G',G}} \underset{\sigma_C}{\underline{\Sigma}}(g,G)$$
$$\sigma_{C'} \hookrightarrow \sigma_C$$

Define $\overline{\Theta}_n = \coprod \Sigma(g,G) \Big/ \text{gluing by } \iota_{G',G}\text{'s.}$

Thm $\overline{\Theta}_n$ is contractible

Pf

The cover by the $\Sigma(g,G)$'s

Has nerve $=$ nerve of cover by $\sigma(g,G$'s$)$

$=$ spine of Outer space

$\simeq pt$

Equivalence of points in $\overline{\Theta_n}$

$(g, G = C_0 \supset C_1 \supset \cdots \supset C_k) \sim (g', G' = C_0' \supset C_1' \supset \cdots \supset C_k')$

if $\exists$ graph automorphism $G \longrightarrow G'$ which is an isometry on each C_i

or if they are identified under face relations

Action of $\varphi \in \mathrm{Aut}(F_n)$ on $\overline{\Theta_n}$:

$$(g, G \supset C_1 \supset \cdots \supset C_k) \circ \varphi$$

$$= (g\varphi, G \supset C_1 \supset \cdots \supset C_k)$$

Prop The action of $\mathrm{Aut}(F_n)$ on $\overline{\Theta_n}$ is proper.

Pf $\mathrm{stab}(g, G = C_0 \supset C_1 \supset \cdots \supset C_k)$

$=$ autos of G which restrict to isometries of C_i for all i

$\subseteq \mathrm{Aut}(G)$ finite.

The action is also <u>cocompact</u>:
it changes the marking, so there is a fundamental domain with one cell for each possible chain of core graphs

$$(G = C_0 \supset C_1 \supset \cdots \supset C_k)$$

There are only finitely many possibilities so the quotient has only finitely many quotients of cells.

So now we have a proper, cocompact action of $Out(F_n)$ on a contractible CW-complex $\overline{\theta}_n$.

In particular we can identify $H^k(\Gamma; \mathbb{Z}\Gamma)$ with $H^k_c(\overline{\theta}_n)$.

Bieri-Eckman tells us that $\Gamma = Out(F_n)$ is a virtual duality group of dim. d if

$$H^k(\Gamma; \mathbb{Z}\Gamma) = \begin{cases} \text{free abelian} & k = d \\ 0 & k \neq d \end{cases}$$

Next task: compute $H^k_c(\overline{\theta}_n)$!