Lectures on Geodesic flows

May 30, 2018

The idea of these lectures is to discuss some classical ideas from ergodic theory and dynamical systems through the lens of a family of classical examples, namely geodesics flows on negatively curved surfaces. These include the notions of:

1. Ergodicity
2. Rigidity
3. Entropy
4. Other connections with fashionable mathematics.

1 Preliminaries

Let us consider some basic notation, definitions and properties.

1.1 Surfaces and their topology

We let V denote a compact connected orientiable surface (i.e., a 2 dimensional manifold). We first need to decide what our surface looks like (at least to help in drawing pictures).

Theorem 1.1 (Poincaré classification theorem). *The surface V is homeomorphic to exactly one of the following:*

1. *the sphere S^2 (genus 0)*
2. *a torus $\mathbb{T}^2$ (genus 1)*
3. *the connected sum of g tori, for $g \geq 2$ (genus g)*

1.2 Surfaces and their Riemannian metrics

Roughly speaking, a Riemannian metric gives a natural distance on the surface based (locally) on patchs of Euclidean space.

A convenient viewpoint is to think of a Riemannian metric $\rho = \{\| \cdot \|_{\rho,x}\}$ $(x \in V)$ as a family of norms on the tangent spaces T_xV (for each $x \in V$) where describe the lengths $\|v\|_\rho$ of tangent vectors $v \in TV$ (which is locally like $V \times \mathbb{R}^2$) has length $\|v\|_{\rho,x}$.

To then define a more familiar notion of distance on V itself, given $x, y \in V$ we can define the (path) metric

$$d(x,y) = \inf_\gamma \left\{ \int_a^b \|\dot\gamma(t)\| dt \right\}$$

where the infimum is over all smooth curves $\gamma : [a,b] \to V$ which start at x and finish at y (i.e., $\gamma(a) = x$ and $\gamma(b) = y$). (The parametrisation is unimportant. and we could take $a = 0$ and $b = 1$,say)

A parameterised curve $\gamma : [0,1] \to V$ is a geodesic if it (locally) minimises the distance (i.e., for sufficiently large N the restriction $\gamma : [i/N, (i+1)/N]$ minimises the distance $\gamma(i/N)$ and $\gamma((i+1)/N))$ in the sense above.

One can conveniently associate notions of area.

2 Geodesic flows

We now introduce some dynamics. The basic dynamical tool is the geodesic flow. This is a flow (hence the name) which takes place not on the two dimensional space V but on the there dimensional space of tangent vectors of length 1 (with respect to the Riemannian metric $\| \cdot \|_\rho$).

2.1 Definition of the geodesic flow

We can now introduce some dynamics. We want to define a flow on the compact three dimensional manifold

$$M = \{v \in TV : \|v\|_\rho = 1\} \qquad \text{(“Sphere bundle”)}$$

To define a geodesic flow $\phi_t : M \to M$ $(t \in \mathbb{R})$ we can take $v \in M$ and choose the unique (unit speed) geodesic $\gamma_v : \mathbb{R} \to V$ such that $\dot\gamma_v(0) = v$. We can then define $\phi_t(v) := \dot\gamma_v(0)$. (This corresponds to "parallel transport" using more elaborate Riemannian metrics).

This is a flow in the usual sense:

1. $\phi_{t=0} = Id$; and

2. $\phi_{t+s} = \phi_t \circ \phi_s$ for $s, t \in \mathbb{R}$.

There is a natural correspondence between(directed) geodesics on V and orbits for the associated geodesic flow. There is a natural correspondence between(directed) closed geodesics on V and closed orbits for the associated geodesic flow.

For convenience and emphasis we record this as follows:

Lemma 2.1. *The (oriented) closed geodesics τ are in one-one correspondence with the closed orbits of the geodesics flow. The lengths and periods agree.*

2.2 Negative curvature

. The properties of the geodesic flow tend to be dynamically more interesting if the curvature is negative. Given a Riemannian metric ρ we can define the curvature function $\kappa : V \to \mathbb{R}$ by

$$\kappa(x) = \frac{12}{\pi}\left(\lim_{r\to 0}\frac{\pi^2 r - \mathrm{Area}(B(x,r))}{r^4}\right)$$

where $B(x,r) = \{y \in V : d(x,y) < r\}$.

We are particularly interested in surfaces for which $\kappa(x) < 0$ for all $x \in V$. In particular, negative curvature corresponds to $\mathrm{Area}(B(x,r))$ being larger than πr^2 for small r.

The most useful result on negative curvature is the foliowing theorem (for triangles).

Theorem 2.2 (Gauss-Bonnet)**.** *If $\Delta \subset V$ is a triangle with geodesic sides and internal angles $0 < \theta_1, \theta_2, \theta_3 < 1$ then*

$$-\int_\Delta \kappa(x) d\mathrm{Area}(x) = \pi - (\theta_1 + \theta_2 + \theta_3)$$

with strict inequality for negatively curved spaces.

Corollary 2.3. *If $\kappa(x) < 0$ then V must be a surface of genus $g \geq 2$.*

This can be seen by writing the sphere and torus as unions of triangles,

Example 2.4 (Non-Euclidean Geometry $\kappa = -1$)**.** *Let $\mathbb{H}^2 = \{x + iy : y > 0\}$. We can then define a metic $ds^s = \frac{dx^2+dy^2}{y^2}$ which means that*

$$d(x_1 + iy_1, x_2 + iy_2) = \cosh^{-1}\left(1 + \frac{|x_1 - x_2|^2 + |y_1 - y_2|^2}{2y_1 y_2}\right)$$

Then

- *this metric has curvature $\kappa = -1$;*
- *the geodesics are semi-circles that meet the real axis perpendicularly or are vertical lines; and*
- *the maps $z \mapsto \frac{az+b}{cz+d}$ $(a, b, c, d \in \mathbb{Z}$ with $ad - bc = 1$ $)$ are orientation preserving isometries (i.e., preserve the distances).*

To obtain a compact surface we can consider a (discrete) subgroup of such isometries Γ for which the quotient $V = \mathbb{H}^2/\Gamma$ is compact. (Then $\Gamma = \pi_1(V)$).

2.3 Spaces of metrics

We can consider different Riemannian metrics on the same (topological) surface V. If we assume that the metrics have $\kappa = -1$ then the space of such metrics is homemorphic to $\mathbb{R}^{6(g-1)}$ (and this is referred to as Teichmüller space).

More generally any metric of variable negative curvature can be reduced to a metric of constant negative curvature $\kappa = -1$:

Lemma 2.5 (Koebe)**.** *If ρ is a metric of negative curvature then it is conformally equivalent to a metric of constant negative curvature ρ_0 with $\kappa = -1$ (i.e., there exists a smooth function $f : V \to \mathbb{R}^+$ such that $\rho = f(x)\rho_0$).*

Since $\mathbb{R}^{6(g-1)}$ and $C^\infty(V, \mathbb{R}^+)$ are both (path) connected we have a simple corollary:

Corollary 2.6. *The space of metrics of variable negative curvature on V is path connected.*

The space $\mathcal{T}$ of metrics of constant negative curvature $\kappa = -1$ is $6(g-1)$-dimensional. In a seemingly exotic development there are a variety of metrics which can be placed on $\mathcal{T}$ (Teichmülcer metric, Weil-Petersson metric and the associated geodesic flow on these spaces can be studied too).

2.4 Properties for negative curvature

We can now emphasise the dynamical importance of the negative curvature. A key tool is the following.

Definition 2.7. *We can define the sets*

$W^s(v) = \{v' \in M\colon d(\phi_t v, \phi_t v') \to 0 \text{ as } t \to +\infty\}$ *(Stable manifold/horocycle)*

$W^u(v) = \{v' \in M\colon d(\phi_{-t} v, \phi_{-t} v') \to 0 \text{ as } t \to +\infty\}$ *(Unstable manifold/horocycle)*

In negative curvature these sets are actually (Immersed) submanifolds.

Lemma 2.8. *The stable and unstable horocycles satisfy the following properties:*

1. *for all $v \in M$ the sets $W^s(v)$ and $W^u(v)$ are one dimensional C^∞ sub manifolds;*
2. *the families $\mathcal{W}^s = \{W^s(v)\}$ and $\mathcal{W}^u = \{W^u(v)\}$ form transverse C^1 foliations; and*
3. *$\mathcal{W}^s$ and $\mathcal{W}^u$ are not uniformly integrable.*

The contraction can also be seen to be exponential in the sense that it we consider the (transverse) tangent spaces $E_v^s = T_v W^s(v)$ and $E_v^u = T_v W^u(v)$ then:

Lemma 2.9 (Anosov Property)**.** *There exists $\lambda > 0$ and $C > 0$ such that $\|D\phi_t | E_v^s\| \le Ce^{-\lambda t}$ and $\|D\phi_{-t} | E_v^u\| \le Ce^{-\lambda t}$, $t > 0$*

Example 2.10 (Donnay-Pugh)**.** *Let $V \subset \mathbb{R}^3$ be a surface constructed by taking two large concentric circles joined by a large number of small tubes with negative curvature.*

2.5 Invariant probability measures

There are many different invariant measures μ for the geodesic flow (i.e., $\mu(M) = 1$ and $\mu(\phi_t B) = \mu(B)$ for any $t \in \mathbb{R}$ and Borel $B \subset M$).

Perhaps the most natural is the Liouville measure ν, which we scale to assume to be normalised (i.e., $\nu(M) = 1$). This corresponds to the Sinai-Ruelle-Bowen measure in the more general context of Anosov flows and more general hyperbolic flows.

Lemma 2.11 (Liouville measure). *There is a unique ϕ-invariant measure on M which is equivalent to the Riemannian volume and called the Liouville measure.*

Remark 2.12. *We can use local coodinates (x_1, x_2) on V and then the natural area on V comes from* $\det(g_{ij})dx_1dx_2$. *The three dimensional manifold M locally looks like $M = V \times \mathbb{T}^1$ and then ν the locally like* $\det(g_{ij})dx_1dx_2d\theta$.

There is a more convenient dynamical way to describe this measure. Any invariant measure can be written locally as $d\mu \times dt$ where

1. dt is the usual lebesgue measure along the orbits of the geodesic flow.and
2. μ is the induced measure on a (local) two dimensional transverse section W. (This is also called a current) by the geometers).

As a special case we can consider a local transverse section by choosing $v_1, v_2, v_3, v_4 \in M$ such that $v_2 \in W^s(v_1)$, $v_3 \in W^u(v_2)$, $v_4 \in W^u(v_3)$ and $\phi_T v_1 = v_4$ (i.e., v_1 and v_4 lie on the same orbit displaced by T). We then let these stable and unstable manifolds as the boundary of the section W.

Lemma 2.13. *We can characterise the Liouville measure of W by setting $\mu(W) := T(W)$ (again suitable normalised).*

The following property is the most fundamental property of the Liouville measure.

Theorem 2.14 (Hopf, 1939). *The geodesic flow is ergodic with respect to the measure ν, i.e., for any $F \in C(M, \mathbb{R})$ we have that for almost every v (with respect to ν):*

$$\lim_{T\to+\infty} \frac{1}{T} \int_0^T F(\phi_t v)dt = \int F d\nu$$

(i.e., temporal average equals the special average).

Proof. The idea of the proof is based on the result of Birkhoff that says: for almost very (ν) $v \in M$ then the future and past averages agree, i.e., for any $F \in C(M)$ we have

$$\lim_{T\to+\infty} \frac{1}{T} \int_0^T F(\phi_t v)dt = \lim_{T\to+\infty} \frac{1}{T} \int_{-T}^0 F(\phi_t v)dt.$$

It remains to see that these limits are independent of v (and thus equal to $\int F d\nu$). This easy enough if points lie on the same horocyle:

- If $v' \in W^s(v)$ then $\lim_{T\to+\infty} \frac{1}{T}\int_0^T F(\phi_t v)dt = \lim_{T\to+\infty} \frac{1}{T}\int_0^T F(\phi_t v')dt$
- If $v' \in W^u(v)$ then $\lim_{T\to+\infty} \frac{1}{T}\int_{-T}^0 F(\phi_t v)dt = \lim_{T\to+\infty} \frac{1}{T}\int_{-T}^0 F(\phi_t v')dt$.

Now we can patch together these results. By the non-uniform integrability property of the foliations we can connect nearby points $v, v' \in M$ by a network of stable and unstable manifolds, i.e., $\exists v_1, v_2 \in M$ such that $v_1 \in W^s(v) \cap W^u(v_2)$ and $v' \in W^s(v_2)$. Thus by the above the limits agree at v and v'. (This implicitly uses that the measure μ is absolutely continuous, i.e., no measure is lost under pushing the measures along the foliations, which here is a consequence of the foliations being C^1). □

The geodesic flow has stronger properties. One of the more famous is that it is exponential mixing, i.e., if $F \in C^\infty(M)$ with $\int F d\nu = 0$ then there exists $\epsilon > 0$ and $C > 0$ such that $\left|\int F \circ \phi_t F d\mu\right| \leq Ce^{-\epsilon t}$ for $t \geq 0$.

3 Classification

It is natural to ask which geodesic flows "look similar" or which surfaces "look identical". In mathematical terms this involved classifying the flows up to flow equivalence, then topological conjugacy and classifying the surfaces up to isometry.

3.1 Flow equivalence

A direct consequence of the Anosov property is the following

Lemma 3.1 (Structural Stability : Flow equivalence)**.** *Let ρ_1 be a metic of negative curvature. Then for a (sufficiently close) metric ρ_2 of negative curvature we have that the geodesic flows ϕ_1 and ϕ_2 are flow equivalent (i.e., there is a a homeomorphism $h : M_{\rho_1} \to M_{\rho_2}$ which takes orbits to orbits and a reparameterization function $r : M \to \mathbb{R}^+$ such that $h(\phi_{1,t}(v)) = \phi_{2,s}(h(v))$ where $s = \int_0^t r(\phi_{1,u}v)du$).*

In particular h carries orbits to orbits, but doesn't necessarily preserve the parameterisation.

One simple way to see how the map $h_0 : M \to M$ might be constructed is as follows. Providing $\epsilon > 0$ is sufficiently and the metrics are sufficiently close then for each $v \in M_1$ we can consider the tassociated geodesic on V_1. There will be a unique orbit or M_2 which stays within distance ϵ.

The usual proofs of structural stability are based on a fixed point theorem for the space of homeomorphisms on M (following work of Mather) or use shadowing properties (Bowen). It comes for free that the flow equivalence h is also Hölder continuous, i.e, there exists $\alpha > 0$ such that $\sup_{x \neq y} \frac{d_{\rho_2}(h(x),h(y))}{d(x,y)^\alpha}$).

Remark 3.2. *The situation is a little simpler in the discrete case of Anosov diffeomorphisms $f : M \to M$. Nearby diffeomorphisms g are topologically conjugate to f (i.e., there exists a homeomorphism $h : M \to M$ such that $f \circ h = h \circ g$).*

We now want to impose some additional conditions which will ensure that the parameterisations of the orbits match up.

3.2 Conjugacy

We begin with the basic definition.

Definition 3.3. *We say that two (geodesic) flows $\phi_{1,t} : M_1 \to M_1$ and $\phi_{2,t} : M_2 \to M_2$ topologically conjugate, i.e., there exists a homeomorphism $h : M_1 \to M_2$ such $h(\phi_{1,t}(v) = \phi_{2,t}h(v)$, for all $t \in \mathbb{R}$ and $v \in M_1$.*

Here we let $M_i = SV_i$ $(i = 1, 2)$ be the unit tangent bundles to the underlying surfaces V_i $(i = 1, 2)$. In particular, we see from topological considerations that V_1 and V_2 must at the very least have the same genus.

To proceed, we want to explore how much information is contained in a knowledge of the lengths of closed orbits. There are infinitely many distinct closed geodesics on V (one in each conjugacy class in $\pi_1(V)$). Furthermore, there are denumerably many (i.e., exactly one in each conjugacy class in $\pi_1(V)$).

Definition 3.4. *We let $\mathcal{C}$ denote the family of all closed geodesics on V (equivalently closed orbits for the associated geodesic flow $\phi_t : M \to M$) which can be thought of conjugacy classes in $\pi_1(V)$, for example. We define the length spectrum $L_\rho : \mathcal{C} \to \mathbb{R}$ to be the map which associates to each closed geodesic (orbit) its length (period) .*

We now want to consider a surface V and two metrics of negative curvature ρ_1 and ρ_2 and their associated geodesic flows $\phi_{\rho_1,t} M_1 \to M_1$ and $\phi_{\rho_1,t} : M_2 \to M_2$.

Theorem 3.5 (Topological Conjugacy)**.** *Assume that $L_{\rho_1} = L_{\rho_2}$ then the geodesic flows are topologically conjugate, i.e., there exists a homeomorphism $h : M_1 \to M_2$ such $h(\phi_t v) = \phi_t h(v)$ for all $v \in M_1$ and $t \in \mathbb{R}$.*

We can use the flow equivalence $h_0 : M_1 \to M_2$ established earlier in Theorem 3.1 (but now denoted h_0 since new now want to reserve h for the conjugacy) and improve it to a conjugacy $h_0 : M_1 \to M_2$. It is the additional assumption $L_{\rho_1} = L_{\rho_2}$ that allows us to go from having flow equivalence of orbits to actually having a conjugacy. The proof is purely dynamical (and a version It remains true in the broader context of Anosov flows).

We can assume without any loss of generality the assumption that f is differentiable in the flow direction (i.e, along the orbits of the flow so that $\frac{dh(\phi_t v)}{dt}$ exists and is continuous). Since we don't need to assume it preserves the parameterzation we can aways smooth it in the flow direction

The key to constructing a conjugacy is to use Livsic's Theorem:

Theorem 3.6 (Livsic)**.** *If $k : M \to \mathbb{R}$ is*

1. *Hölder continuous*
2. *differentiable along the orbits of the flow and*
3. *integrates to zero along closed orbits (i.e., if $\phi_T(v) = v$ with $T > 0$ then $\int_0^T k(\phi_t v) dt = 0$)*

then there exists a continuous function $u : M \to \mathbb{R}$ (differentiable in the flow direction) such that

$$k(\phi_t x) - k(x) = \int_0^t u(\phi_s) ds. \tag{1}$$

(or equivalently $u(v) = \frac{du(\phi_t v)}{dt}|_{t=0}$)

Proof. Using the ergodicity (although it is sufficient to know there is a single dense orbit) we can choose $v \in M$ whose orbit $\mathcal{O} := \cup_{t>0} \phi_{\rho,t}(v)$ is dense in M. We can then define $u(\phi_t v) = \int_0^t k(\phi_s v) ds$ and we see that (1) holds (at least) on $\mathcal{O}$.

Now we need to know that h is uniformly (or even Hölder) continuous in $\mathcal{O}$ and thus extends continuously from the dense set $\mathcal{O}$ to M. In particular, we need to know that if $\phi_{t_1} v$ and $\phi_{t_2} v$ are close then the values $u(\phi_{t_1} v)$ and

$u(\phi_{t_2}v)$ are close. This is where the information about integrating to zero on closed orbits is used, we can write

$$u(\phi_{t_1}v) - u(\phi_{t_2}v) = \int_{t_1}^{t_2} u(\phi_t v)dt \approx 0$$

by approximating last term by the integral around a closed obit (which vanishes by hypothesis) to complete the proof. □

Now that we have Livsic's theorem at our disposal, we can promote the flow equivalence h_0 between $\phi_{1,t} : M_1 \to \mathbb{R}$ and $\phi_{2,t} : M_2 \to \mathbb{R}$ to a topological conjugacy h by a suitable reparamterization.

Proof of topological conjugacy. Let $v' = h_0(v)$ and then we can use the derivative of h in the flow direction to help reparameterize the flow. More precisely, we can write

$$\frac{d}{dt} h_0(\phi_{1,t}v)|_{t=0} = w(v)\dot{\phi}_2(h(v))$$

where $\dot{\phi}_2$ is the vector field for the flow ϕ_2 (tangent to the orbit for $v' = h_0(v)$) giving the "direction" of the derivative of h_0 and $w : M \to \mathbb{R}$ the magnitude.

To apply Livsic's theorem, we can consider the function $k(v) = w(v) - 1$. If $\phi_{1,T}(v) = v$ is a periodic orbit for $\phi_1$of period L then $h_0(v)$ is also a periodic orbit of (by hypothesis) period $\int_0^T w(\phi_s)ds = L$, i.e., $\int_0^T k(\phi_{1,t}(v)dt = 0$. Thus Livsic's theorem applies and we can deduce that there exists a $u : M_1 \to \mathbb{R}$. Setting $h(v) = \phi_{2,u(v)}h_0(v)$ we have the required topological conjugacy. □

In the next subsection we will see (or geodesic flows) how we can deduce more. More precisely, how to show that the underlying surfaces V_1 and V_2 are the same (i.e., isometric).

3.3 Isometries

Given a conjugacy h coming from Theorem 3.5 we would like to know that if $x \in V_1$ then there exists $x \in V_2$ such that $h(S_x V_1) = S_{x'} V_2$, i.e., fibre are mapped to fibres. Of course this will not work for every conjugacy since they are only defined up to a shift under the geodesic flow, i.e., if h is a conjugacy then so is $\phi_{2,t} \circ h$. However, *it suffices to consider just three geodesics passing through the same point and then showing that their images under h again (locally) pass through a single point of V_2.*

Definition 3.7. *Given $v \in V_1$ and $0 < \theta < 2\pi$ we let $R_\theta(v)$ be the vector rotated (in a clockwise direction, say) by an angle θ above the same point $x \in V$. If we consider the images $h(v)$ and $h(R_\theta v)$ then (locally) these too must intersect at an angle which we denote by $0 < \Theta(\theta, v) < 2\pi$.*

if we consider v, $R_{\theta_1} v$ and $R_{\theta_2} v$ (for $\theta_1 \neq \theta_2 > 0$) then these correspond to three geodesics which locally intersect at the same point x, say. But we can assume (for a contradiction) that the images images of these three geodesics may correspond (locally) to geodesics which form a triangle with internal angles

$$\Theta(\theta_1, v), \pi - \Theta(\theta_1 + \theta_2, v) \text{ and } \Theta(\theta_2, R(\theta_1)v).$$

In negative curvature (by Gauss-Bonnet) we know that the sum of the internal angels of any non-degenerate triangle is *strictly less* than π and thus for s this triangle we have

$$\Theta(\theta_1, v) + \Theta(\theta_2, R(\theta_1)(v)) < \Theta(\theta_1 + \theta_2, v)$$

We want to get a contradiction to this strict inequality (thus showing the triangle is degenerate, i.e., triples of intersecting geodesics go to triples of intersecting geodesics as required).

To this end we can average over M_1 with respect to the (normalised) Liouville measure ν to define a function on angles $F : [0, \pi] \to [0, \pi]$ by

$$F(\theta) = \int_{M_1} \Theta(\theta, v) d\nu(v).$$

The following properties can be easily checked:

Lemma 3.8. *The function $F(\theta)$ satisfies*

1. *$F(0) = 0$ and $F(\pi/2) = \pi/2$*
2. *$F(\pi - \theta) = \pi - F(\theta)$,* (symmetry)
3. *for θ_1, θ_2 and $\theta_1 + \theta_2 \in [0, \pi]$ we have*

$$F(\theta_1 + \theta_2) \geq F(\theta_1) + F(\theta_2), \quad \text{(Superadditivity)}$$

To complete the contradiction we need the following (geometry and dynamics free) calculus lemma:

Lemma 3.9 (Calculus Lemma)**.** *Let $F; [0, \pi] \to [0, \pi]$ be any smooth function having the properties in Lemma 3.8 then for any $0 < \theta_0 < \pi$ we have that*

$$\int_0^{\theta_0} \frac{\sin\theta}{\sin F(\theta)} d\theta \leq F(\theta_0).$$

We skip the tedious proof for the moment (= forever).

A bit more calculus gives the following:

Lemma 3.10. *Let $F : [0, \pi] \to [0, \pi]$ be any smooth function having the super additive property in Lemma 3.8. Then either*

1. *There exists a fixed point $0 < \theta_0 = F(\theta_0) \leq \pi$ and $F(x) < x$ for $0 < x < \theta_0$; or*
2. *$F(x) \leq x$ for all $0 \leq x \leq \pi$.*

Lemma 3.9 implies that there is no subinterval interval $(0, \theta_0)$ upon which $F(\theta) < \theta$. Assume for a contradiction this isn't true, and choose $\theta_0 \leq \frac{\pi}{2}$ maximal with this property. Then

$$\int_0^{\theta_0} \frac{\sin\theta}{\sin F(\theta)} d\theta \leq F(\theta_0) = \theta_0.$$

and since on $[0, \theta_0]$ the integrand $\frac{\sin\theta}{\sin F(\theta)} > 1$ we get the required contradiction.

We can now apply Lemma 3.10 to deduce that $F(x) \leq x$ for all $0 \leq x \leq \pi$. By symmetry $F(\pi - \theta) = F(\theta)$ and so $F(\theta) \geq \theta$ for $0 \leq \theta \leq \pi$, i.e., $F(\theta) = \theta$. But then we have equality in the super additivity property, which contradicts the Gauss-Bonnet and negative curvature.

In summary, we have the following (stronger) result because we are dealing with geodesic flows (rather than general Anosov flows):

Theorem 3.11 (Otal's Theorem)**.** *Assume that $L_{\rho_1} = L_{\rho_2}$ then the the underlying surfaces are the same (i.e., isometric): $V_1 = V_2$*

3.4 Sunada's Theorem

We know that the length spectrum $L_\rho : \mathcal{C} \to \mathbb{R}$ determines the surface. However, would it be enough just to know the values it takes? That is, does knowing just the numbers

$$\{L_\rho(c) : c \in V\} \subset \mathbb{R}$$

determine the metric ρ?

In fact is known that it is *not* enough to know these values (i.e., the lengths of closed geodesics without knowing which free homotopy class they came from) to determine ρ.

There is a nice construction due to Sunada of pairs surfaces V_1 nd V_2 which have constant negative curvature $\kappa = -1$ and have the same numerical values for lengths of closed geodesics, but are different surfaces (i.e., not isometric).

A curious feature of their construction is that they are both both quotients of a common third surface $\overline{V}$. Assume that hew have a surface $\overline{V}$ and a finite group of isometries G which acts freely on $\overline{V}$ (i.e., no fixed points). If we have subgroups $G_1, G_2 < G$ then we can consider the quotient surfaces

$$V_1 = \overline{V}/G_1 \text{ and } V_2 = \overline{V}/G_2$$

- The surfaces V_1, V_2 are isometric if and only if the groups are conjugate (i.e., $\exists g \in G$ such that $gG_1g^{-1} = G_2$)

However, we want to consider a weaker condition: We say that G_1, G_2 are *almost conjugate* if for every $g \in G$ we have

$$\mathrm{Card}([g] \cap G_1) = \mathrm{Card}([g] \cap G_2)$$

where $[g] = \{hgh^{-1} : h \in G\}$ is the set of conjugate elements to g.

Theorem 3.12 (Sunada)**.** *If G_1 and G_2 are almost conjugate then the set of values of the length spectra agree*

Proof. This proof is based on an exercise in number theory. The idea is that if we have a closed geodesic γ on V_1 then it projects to a geodesic γ_0 on the common quotient $V_0 = \overline{V}/\Gamma$. The number of lifts to V_i of γ_0 are given by

$$\frac{\mathrm{Card}(g : g^{-1}g_0g \in G_i)}{\mathrm{Card}(G_i)}$$

where g_0 is a representative of γ_0. This is independent of i (by definition) and so the length values match up. □

Thus it suffices to find examples of such groups (and surfaces $\overline{V}$) which are almost conjugate but not isometric.

Example 3.13. *Let $G = SL(3, \mathbb{Z}_2)$ and*

$$G_1 = \begin{pmatrix} 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \text{ and } G_2 = \begin{pmatrix} 1 & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix}$$

4 Entropy

Entropy started off as an invariant (i.e., if two flows are topologically conjugate then they have the same entropy). However, it has many more uses, especially as a quantification of how complicated a dynamical system is.

4.1 Topological entropy

For any flow $\phi_t : M \to M$ we can associate the topological entropy $0 \leq h(\phi)$ (of the time one flow $\phi_{t=1}$).

Definition 4.1. *Given $T > 0$ and $\epsilon > 0$ we let $N(T, \epsilon)$ be the cardinality of the smallest finite set $X = X(\epsilon) \subset M$ such that for any $v \in M$ there exists $v' \in X$ such that $\sup_{0 \leq t \leq T} d(\phi_t v, \phi_t v') < \epsilon$. The topological entropy is then given by*

$$h(\phi) := \lim_{\epsilon \to 0} \limsup_{T \to +\infty} \frac{1}{T} \log N(T, \epsilon).$$

This value is always non-zero and finite.

In the case of geodesic flows this has a simple geometric interpretation.

Theorem 4.2 (Manning's volume entropy)**.** *Let $\widetilde{V}$ be the universal cover for V (with the lifted metric $\widetilde{\rho}$). Fix any point $x_0 \in \widetilde{M}$ we let $B(x_0, R) := \{x \in \widetilde{M} : d(x, x_0) < R\}$ and then*

$$h(\phi) = \lim_{R \to +\infty} \frac{1}{R} \log \mathrm{Area}_{\widetilde{\rho}}(B(x_0, R)$$

Proof. In negative curvature when we consider lifts $\widetilde{X}(\epsilon)$ to the universal cover. Then $N(T, \epsilon)$ can be used to give bounds on the area of an annulus to radius T and width approximately ϵ. However, in negative curvature this has the same rate of growth as the area $\mathrm{Area}_{\widetilde{\rho}}(B(x_0, R)$ of a ball of radius T □

The following result is classical:

Lemma 4.3. *It two flows $\phi_{1,t} : M_1 \to M_1$ and $\phi_{2,t} : M_2 \to M_2$ are topologically conjugate then the have the same topological entropy, i.e., $h(\phi_1) = h(\phi_2)$.*

4.2 Entropies of measures

Let μ be a ϕ-invariant probability measure (i.e., $\mu(\phi_t B) = \mu(B)$ for any Borel sets $B \subset M$ and $\mu(M) = 1$). We can then associate the entropy $0 \leq h(\mu) \leq h(\phi)$ of the measure μ (of the time one flow $\phi_{t=1}$).

Definition 4.4 (After Katok). *Given $T > 0$, $\delta > 0$ and $\epsilon > 0$ we let $N(\delta, \epsilon, T)$ be the cardinality of the smallest finite set $X = X(\delta, \epsilon, T) \subset M$ such that*

$$\mu\left(\left\{v \in M : \exists v' \in X \text{ with } \sup_{0 \leq t \leq T} d(\phi_t v, \phi_t v') < \epsilon\right)\right) > 1 - \delta.$$

The entropy of the measure μ is then given by

$$h(\phi) := \lim_{\epsilon \to 0} \limsup_{T \to +\infty} \frac{1}{T} \log N(T, \epsilon).$$

We return to concentrating on geodesic flows. Our main example so far of an invariant measure so far is the Liouville measure:

Example 4.5 (Liouville measure). *We recall that the Liouville measure ν is the ϕ-invariant probability measure equivalent to the volume . In the particular case, that ρ_0 is a metric of constant negative curvature $\kappa = -1$ then $h(\phi, \mu) = h(\phi) = 1$.*

There is another natural ϕ-invariant probability measure:

Example 4.6 (Measure of maximal entropy). *There exists unique ϕ-invariant probability measure $\mu_{\max}$ such that $h(\phi, \mu_{\max}) = h(\phi)$. In the particular case, that ρ is a metric of constant negative curvature then $\mu_{\max}$ is equal to the Liouville measure. (Moreover, they agree only when the metric ρ has constant negative curvature).*

There is now a classical result due to Katok relating entropies for different metrics.

Let us consider two metrics ρ_1, ρ_2 on a compact surface V. Let $h(\phi_2)$ be the topological entropy for the geodesic flow for (V, ρ_1). Let $h(\phi_1, \mu_1)$ be the entropy of the geodesic flow for (V, ρ_1) with respect to the measure μ_1. We can then consider

$$\int_{\|v\|_{\rho_1}=1} \|v\|_{\rho_2} d\mu_1(v)$$

which measures the average change in the lengths of tangent vectors.

Lemma 4.7. *There is an inequality*

$$h(\phi_2) \geq \left(\int_{\|v\|_{\rho_1}=1} \|v\|_{\rho_2} d\mu_1(v) \right)^{-1} h(\phi_1, \mu_1)$$

Proof. The idea of the proof is get a lower bound on the topological entropy $h(\phi_2)$ by constructing orbit segments for $\phi_{2,t}$.

This done using ergodic theory for the geodesic flow $\phi_{1,t} : M_1 \to M_1$ and ν_1 and the function $F : M_1 \to \mathbb{R}$

$$M_1 \ni v \mapsto F(v) = \|v\|_{\rho_2} \in \mathbb{R}$$

If we assume (for simplicity) that μ_1 is ergodic then by the Birkhoff ergodic theorem then for almost every (μ_1) $v \in M_1$ and sufficiently large T:

$$\frac{1}{T} \int_0^T F(\phi_{1,t}(v)dt = \frac{1}{T} \int_0^T \|\phi_{1,t}(v)\|_{\rho_2} dt \to \left(\int_{\|v\|_{\rho_1}=1} \|v\|_{\rho_2} d\mu_1(v) \right) \text{ as } T \to +\infty.$$

Thus for large T we have "most" orbit segments of ϕ_1- length approximately T correspond to orbit segments of ϕ_2- length

$$T \left(\int_{\|v\|_{\rho_1}=1} \|v\|_{\rho_2} d\mu_1(v) \right).$$

We can use these to get a lower bound on $h(\phi_2)$.

This leads to the main result on entropy.

Theorem 4.8 (Katok Entropy Theorem)**.** *The topological entropy is minimised on metrics of constant area at metrics of constant negative curvature (i.e., If ρ_2 is a metric of negative curvature and ρ_1 is a metric of constant negative curvature with* $\text{Area}_{\rho_1}(V) = \text{Area}_{\rho_2}(V)$ *then $h(\phi_2) \geq h(\phi_1)$).*

Proof. By Koebe's Theorem we can assume that ρ_2 is conformally equivalent to a metric ρ_1 of constant negative curvature, i.e., $\rho_2 = f(x)\rho_1$, where $f : V \to \mathbb{R}^+$ is a strictly positive smooth function.

Let ν_1 be the Liouville measure for M_1 (i.e., V with ρ_1). By conformality we can write

$$\int_{\|v\|_{\rho_1}=1} \|v\|_{\rho_2} d\nu_1(v) = \int_V f(x) d\sigma_1(x) \text{ and } \int_V f(x)^2 d\sigma_1(x) = \sigma_2(V) = 1$$

where σ_1 and σ_2 are the normalised areas on V_1 and V_2. Thus

$$\int_V f(x)d\sigma_1(x) \leq \left(\int_V f(x)^2 d\sigma_1(x)\right)^{\frac{1}{2}} = 1$$

with equality if and only if $\rho = 1$.

If ρ_1 be a metric of constant negative curvature then we know by Example 4.6 that $h(\phi_1, \mu_1) = h(\phi_1)$. We can then apply Lemma 4.7. □

Remark 4.9. *There are higher dimensional analogues of the Katok's theorem due to Besson-Contreras-Gallot.*

4.3 Smoothness of entropy

Assume that we change the metric smoothly then we might expect the entropy to vary smoothly.

We need to make sense of smooth changes of metrics. We can interpret the metric as maps $\rho \in \Gamma(V, \mathcal{S}_2)$ where $\mathcal{S}_2$ are positive symmetric 2×2-matrices.

Theorem 4.10 (Katok, Knieper, Pollicott, Weiss)**.** *Given a C^∞ family $(-\epsilon, \epsilon) \ni \lambda \mapsto \rho_\lambda \in C^\infty(V, \mathcal{S}_2)$ the map*

$$(-\epsilon, \epsilon) \ni \lambda \mapsto h(\phi^{\rho_\lambda}) \in \mathbb{R}$$

is C^∞.

There is also an interpretation for the derivative:

Theorem 4.11 (Katok, Knieper,Weiss)**.** *We can write the first derivative*

$$\frac{d}{d\lambda}h(\phi^{\rho_\lambda})|_{\lambda=0} = -\frac{1}{2}\int_{M_0}\frac{d}{d\lambda}\|v\|^2_{\rho_\lambda}d\mu_{\max}(v)$$

where $\mu_{\max}$ is the unique probability measure such that $h(\phi, \mu_{\max}) = h(\phi)$.

4.4 The Anosov property and Lyapunov exponents

The negative curvature gives rise to the the Anosov property through the negative curvature. One way to see this is via the Jacobi and Riccati equations.

Let $v \in V$ and let $\gamma_v : \mathbb{R} \to V$ be the associated geodesic on V. Let us then denote by $\kappa(t) := \kappa(\gamma_v(t)) < 0$ the curvature at $\gamma_v(t) \in V$ (i.e., after time t along the (geodesic) orbit).

The expansion and contraction in E^u and E^s along the geodesic (or orbit) can be seen through these solutions to the Jacobi equations.

Definition 4.12 (Jacobi equation). *Consider solutions $J_v : \mathbb{R} \to \mathbb{R}$ on the real line to*

$$J_v''(t) + \kappa(t) J_v(t) = 0.$$

Size of solutions $|J(t)|$ either grow or contract exponentially (for E^u and E^s) depending on the initial conditions.

If we define $a_v(t) = J_v'(t)/J_v(t)$ then the Jacobi equation reduces to the Riccati equation:

Definition 4.13 (Riccati Equation). *Consider solutions $a_v : \mathbb{R} \to \mathbb{R}$ on the real line to*

$$a_v'(t) + a_v(t)^2 + \kappa(t) = 0. \tag{1}$$

These determine the rate of growth (or contraction) for E^u and E^s along the (geodesic) orbit for v.

Example 4.14 ($\kappa = -1$). *In the case of constant negative curvature $\kappa = -1$ then one sees that there are two solutions to (1):*

1. *$a_v = 1$ corresponding to an expansion e^t in E^u;*
2. *$a_v = -1$ corresponding to a contraction e^{-t} in E^u.*

We can consider the average expansion along a typical (geodesic) orbit of the positive solution. Let μ be any ϕ-invariant (ergodic) probability measure then for a.e., (μ) $v \in V$

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T a_v(t) d\mu(v) = \int_M a_v d\mu(v)$$

This is the (positive) *Lyapunov exponent.*

4.5 Ricci flow and entropy

Given a metric ρ and the associated curvature function $\kappa : V \to \mathbb{R}$ then we can also denote $\kappa : M \to \mathbb{R}$ where $v \in T_x V$. The average curvature

$$\overline{\kappa} := \int \kappa(v) d\nu(v) = -\pi(g-1)$$

where ν is the (normalised) Liouville measure on M, using the Gauss-Bonnet theorem.

Example 4.15 (Constant curvature metrics)**.** *In the case of metrics ρ_0 of constant curvature $\kappa(x) = \overline{\kappa}$ we have that the entropy is*

$$h(\mu^{\rho_0}) = \sqrt{|\overline{\kappa}|}.$$

By Katok's theorem we have that for other metrics ρ of (variable) negative curvature and the same total area we have that

$$h(\phi^{\rho}) > h(\mu^{\rho_0}) = \sqrt{|\overline{\kappa}|}$$

It is fashionable to study how families of metrics ρ^t evolve under the Ricci flow. Recall that a Riemannian metric can be thought of as $\rho = \{\|\cdot\|_{\rho,x}\}_{x\in V}$, where $\|\cdot\|_{\rho,x}$ is a norm on $T_xV = \{x\}\times\mathbb{R}^2$. With a suitable choice of coordinates we can write each norm in terms of (positive definite) 2×2 matrices $(g_{ij}(x))$ through the associated definite quadratic form

$$\|v\|^2_{\rho,x} = g(x)(v,v) := v^T(g_{ij}(x))v.$$

We can now define the flow on the space of metrics (of fixed area A).

Definition 4.16. *We can define the Ricci flow on the space of metrics (of constant area) by*

$$\frac{d}{dt}g^t_{ij}(x) = -2(\kappa^t(x) - \overline{\kappa})g^t_{ij} \text{ for } x \in V \tag{1}$$

where $\kappa^t(x)$ is the curvature of $\rho(t) := (g^t_{ij})$.

There is a connection between solutions $\rho(t) = (g^t_{ij}(x))$ to the Ricci equation and the topological entropy.

Theorem 4.17 (Manning)**.** *Starting from a metric $\rho = (g_{ij})$ with non-constant negative curvature then the topological entropy is strictly decreasing along the solution $\rho(t)$ to the Ricci equation (1).*

To prove the entropy is decreasing along the orbit ρ_t one can use the formula for the derivative of the topological entropy (along the solution to the Ricci equation):

$$\frac{d}{dt}h(\phi^{\rho^t})|_{t=0} = -\frac{1}{2}\int_M \left(\frac{d}{dt}g_{ij}|_{t=0}\right) d\mu_{\max}(v) = \int_V (\kappa - \overline{\kappa}) d\mu_{\max}(v)$$

where $\mu_{\max}$ is the measure of maximal entropy. We want to show the derivative is negative, i.e., that

$$-\int_V \kappa(v) d\mu_{\max}(v) > \overline{\kappa}$$

Step 1. By Katok's theorem $\sqrt{\overline{\kappa}} < h(\phi^\rho)$.

Step 2. The solution $a_v := a_v(0) > 0$ to the Riccati equation (1) gives the Lyapunov exponent and we have an inequality:

Lemma 4.18 (Ruelle). *We can write*

$$h(\phi) = h(\phi, \mu_{\max}) \leq \int_M a_v d\mu_{\max}(v).$$

Step 3. By the Cauchy-Schwarz inequality we can write

$$\int_M a_v d\mu_{\max}(v) \leq \left(\int_M a_v^2 d\mu_{\max}(v)\right)^{\frac{1}{2}}$$

Step 4. We can the substitute for a_v^2 from the Riccati equation and observe that

$$\begin{aligned}\int_M a_v^2 d\mu_{\max}(v) &= -\underbrace{\int_M \frac{da}{ds} d\mu_{\max}(v)}_{=0} - \int_M \kappa(x) d\mu_{\max}(v) \\ &= -\int_M \kappa(x) d\mu_{\max}(v).\end{aligned}$$

Comparing the above inequalities the result follows.

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