

# Large deviations, Gibbs measures and closed geodesics for rank one geodesic flows

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## Abstract

We show large deviation results for closed orbits for rank one geodesic flows on surfaces weighted by potentials with unique equilibrium states for rank one geodesic flows. This generalizes the equidistribution results in [2], based on the pressure estimates in [4]. It further generalise earlier work of the author for measures of maximal entropy [8] and equilibrium states for hyperbolic flows [9].

## 1 Introduction

Let  $V$  be a compact surface with non positive curvature. Let  $\phi_t : SV \rightarrow SV$  be the geodesic flow on the unit tangent bundle  $SV$  of  $V$ . Let  $\mathcal{M}$  denote the space of  $\phi$ -invariant probability measures.

**Definition 1.1.** *Given a continuous function  $F : SV \rightarrow \mathbb{R}$  we can define the Pressure*

$$P(F) = \sup \left\{ h(\phi, \mu) + \int F d\mu : \mu \in \mathcal{M} \right\}.$$

*Any measure realising this supremum is called an equilibrium measure for  $\mu$ .*

**Definition 1.2.** *We say that a geodesic  $\gamma$  is singular if there is no non-trivial Jacobi field. Let  $\mathcal{S}$  be the collection of singular closed geodesics and let  $\mathcal{N}$  be the complimentary collection of non-singular closed geodesics.*

It shown in [2] that there is a unique equilibrium state  $\mu_F$  if  $F : SM \rightarrow \mathbb{R}$  is Hölder and

$$P(F) > \sup \left\{ h(\phi, \mu) + \int F d\mu : \mu \in \mathcal{M} \text{ and } \mu(\overline{\mathcal{S}}) = 1 \right\}. \quad (1.1)$$

A (directed) closed geodesic  $\gamma$  of length  $l(\gamma)$  corresponds to a closed orbit for the geodesic flow of least period  $\lambda(\gamma)$ . If  $\gamma$  is singular then there will be a family of closed geodesics of the same length whose union forms an annulus. On the other hand, two distinct non-singular geodesics must lie in different free homotopy classes. In the special case of surfaces of strictly negative curvature all of the closed geodesics are necessarily non-singular.

Let  $\lambda(\gamma) = \int_0^{\lambda(\gamma)} F(\phi_t x_\gamma) dt$ , where  $x_\gamma$  is a unit tangent vector to  $\gamma$ , be the integral of  $F$  around the closed orbit. Let  $\delta_\gamma$  be the unique invariant probability measure supported on the closed orbit associated to  $\gamma$ .

**Theorem 1.3.** *If  $F$  satisfies (1.1) then for any open neighbourhood  $\mu \in \mathcal{U} \subset \mathcal{M}$  then there exists  $C > 0$  and  $\eta > 0$  such that*

$$\frac{\sum_{\gamma : \mu_\gamma \notin \mathcal{U}, \lambda(\gamma) \leq T} e^{\lambda_F(\gamma)}}{\sum_{\gamma : \lambda(\gamma) \leq T} e^{\lambda_F(\gamma)}} \leq C e^{-\eta T}$$

for sufficiently large  $T$ .

In the special case of surfaces of strictly negative curvature this recovers a previous result of the author [9].

**Example 1.4.** *If  $F = 0$  then there is a unique measure of maximal entropy,  $\mu_0$  by a result of Knieper [5]. In this case Theorem 1.3 was proved in [8].*

For each  $T > 0$  we can define a  $\phi$ -invariant probability measure

$$\mu_T = \frac{\sum_{\gamma : \lambda(\gamma) \leq T} e^{\lambda_F(\gamma)} \delta_\gamma}{\sum_{\gamma : \lambda(\gamma) \leq T} e^{\lambda_F(\gamma)}}$$

supported on closed orbits associated to closed geodesics of length at most  $T$ . The following results appears in [2], but appears naturally as a corollary to Theorem 1.3.

**Corollary 1.5.** *If  $F$  has a unique equilibrium state  $\mu_F$  then*

$$\lim_{T \rightarrow +\infty} \int G d\mu_T = \int G d\mu_F.$$

*Proof.* Given  $G \in C^0(SM)$  and  $\delta > 0$  we define a weak star open set

$$\mathcal{U} = \left\{ \nu \in \mathcal{M} : \left| \int G d\mu - \int G d\mu_F \right| > \delta \right\}.$$

By the Theorem

$$\begin{aligned} & \left| \int G d\mu_T - \int G d\mu_F \right| \\ & \leq \left| \frac{\sum_{\gamma : \mu_\gamma \in \mathcal{U}, \lambda(\gamma) \leq T} e^{\lambda_F(\gamma)} \lambda_G(\gamma)}{\sum_{\gamma : \lambda(\gamma) \leq T} e^{\lambda_F(\gamma)}} - \int G d\mu_F \right| + 2\|G\|_\infty \left| \frac{\sum_{\gamma : \mu_\gamma \notin \mathcal{U}, \lambda(\gamma) \leq T} e^{\lambda_F(\gamma)}}{\sum_{\gamma : \lambda(\gamma) \leq T} e^{\lambda_F(\gamma)}} \right| \\ & \leq \delta + 2C\|G\|_\infty e^{-\eta T}. \end{aligned}$$

Since  $\delta > 0$  can be chosen arbitrarily small the result follows.  $\square$

## 2 Properties of pressure

The proof uses properties of the pressure function. The following is immediate from the definition of pressure.

**Lemma 2.1.** *The pressure function  $P : C(SV) \rightarrow \mathbb{R}$  is (Lipschitz) continuous.*

An extra ingredient in the proof of Theorem 1.3 is the following result, which is motivated by the approach in [1].

**Lemma 2.2.** *Let  $F : SV \rightarrow \mathbb{R}$  be a Hölder continuous function then*

$$P(F) = \lim_{T \rightarrow +\infty} \frac{1}{T} \log \left( \sum_{\gamma \in \mathcal{N} : \lambda(\gamma) \leq T} \exp(\lambda_F(\gamma)) \right)$$

*Proof.* It follows from the fact that non-singular closed geodesics lie in different free homotopy classes that

$$P(F) \geq \limsup_{T \rightarrow +\infty} \frac{1}{T} \log \left( \sum_{\gamma \in \mathcal{N} : \lambda(\gamma) \leq T} \exp(\lambda_F(\gamma)) \right).$$

Moreover, for any  $\epsilon > 0$  we can consider the restriction of  $\phi$  to a suitable hyperbolic set  $\Lambda_\epsilon$  such that  $P(F|\Lambda_\epsilon) > P(F) - \epsilon$  [3]. Since

$$\begin{aligned} P(F|\Lambda_\epsilon) &= \lim_{T \rightarrow +\infty} \frac{1}{T} \log \left( \sum_{\gamma \in \Lambda_\epsilon : \lambda(\gamma) \leq T} \exp(\lambda_F(\gamma)) \right) \\ &\leq \liminf_{T \rightarrow +\infty} \frac{1}{T} \log \left( \sum_{\gamma \in \mathcal{N} : \lambda(\gamma) \leq T} \exp(\lambda_F(\gamma)) \right) \end{aligned}$$

by standard properties of hyperbolic flows [7] the result follows by letting  $\epsilon$  tend to zero and comparing the two inequalities above.  $\square$

It is not clear to the author how to extend this result to higher dimensions.

### 3 Proof of Theorem 1.3

We introduce some notation.

**Definition 3.1.** *Given  $F \in C^0(SM)$  let  $Q : C^0(SM) \rightarrow \mathbb{R}$  be defined by*

$$Q(f) = P(f + F) - P(F), \text{ for } f \in C^0(SM)$$

*and let  $I : \mathcal{M} \rightarrow \mathbb{R}^+$  be defined by*

$$I(\mu) = \sup_{f \in C^0(SM)} \left\{ \int f d\mu - Q(f) \right\}.$$

Observe that  $Q(\cdot)$  is (Lipschitz) continuous as a consequence of Lemma 2.1.

As an immediate consequence of the definition we gave for pressure we have the following.

**Lemma 3.2.**  $I(\mu) = P(F) - (h(\mu) + \int F d\mu)$

*Proof.* This appears in Walters' book [10].  $\square$

Let  $\mathcal{K} \subset \mathcal{M}$  be a weak star compact subset and let us denote  $\rho_{\mathcal{K}} = \inf_{\mu \in \mathcal{K}} I(\mu)$ . The following result (and proof) generalizes a result in [8] for the measure of maximal entropy. A similar result appears in [4].

**Lemma 3.3.** *If  $F$  has a unique equilibrium state  $\mu_F$  and  $\mu_F \in \mathcal{U} \subset \mathcal{M}$  is a neighbourhood then for  $\mathcal{K} := \mathcal{M} - \mathcal{U}$  we have  $\rho_{\mathcal{K}} > 0$ .*

*Proof.* The dependence of the entropy  $h(\mu)$  is upper semi-continuous by a result of Newhouse [6]. Thus if  $\rho_{\mathcal{K}} = 0$  then there exists  $\mu$  with  $I(\mu) = 0$  and thus  $\mu = \mu_F$ . But this contradicts  $\mu_F \in \mathcal{U}$ , proving the lemma.  $\square$

We now proceed to the proof of Theorem 1.3. Given  $\epsilon > 0$ , we can cover  $\mathcal{K} := \mathcal{M} - \mathcal{U}$  by sets

$$\mathcal{U}_f = \{\nu \in \mathcal{M} : I(\nu) > \rho_{\mathcal{K}} - \epsilon\}, \text{ for } f \in C(SM),$$

which are open by the continuity of  $C^0(SM) \ni f \mapsto \int f d\mu - Q(f)$ . By weak star compactness we can take a finite cover  $\mathcal{K} \subset \cup_{i=1}^m \mathcal{U}_{f_i}$ . We can then estimate

$$\begin{aligned} & \frac{\sum_{\gamma : \mu_{\gamma} \in \mathcal{U}, \lambda(\gamma) \leq T} e^{\lambda_F(\gamma)}}{\sum_{\gamma : \lambda(\gamma) \leq T} e^{\lambda_F(\gamma)}} \\ & \leq \sum_{i=1}^m \left( \frac{\sum_{\gamma : \mu_{\gamma} \in \mathcal{U}_{f_i}, \lambda(\gamma) \leq T} e^{\lambda_F(\gamma)}}{\sum_{\gamma : \lambda(\gamma) \leq T} e^{\lambda_F(\gamma)}} \right) \\ & \leq \sum_{i=1}^m \left( \frac{\sum_{\gamma : \mu_{\gamma} \in \mathcal{U}_{f_i}, \lambda(\gamma) \leq T} e^{\lambda_F(\gamma)} e^{(T \int f_i d\mu_{\gamma} - Q(f_i) - \rho + \epsilon)}}{\sum_{\gamma : \lambda(\gamma) \leq T} e^{\lambda_F(\gamma)}} \right) \\ & \leq \sum_{i=1}^m e^{(-T(Q(f_i) - \rho + \epsilon))} \left( \frac{\sum_{\gamma : \mu_{\gamma} \in \mathcal{U}_{f_i}, \lambda(\gamma) \leq T} e^{\lambda_F(\gamma)} e^{T \int f_i d\mu_{\gamma}}}{\sum_{\gamma : \lambda(\gamma) \leq T} e^{\lambda_F(\gamma)}} \right) \\ & \leq \sum_{i=1}^m e^{-(Q(f_i) - \rho + \epsilon)T} \frac{\sum_{\gamma : \lambda(\gamma) \leq T} e^{\lambda_F(\gamma)} e^{\lambda_{f_i}(\gamma)}}{\sum_{\gamma : \lambda(\gamma) \leq T} e^{\lambda_F(\gamma)}}. \end{aligned}$$

From this we see by Lemma 2.2 that

$$\begin{aligned} & \limsup_{T \rightarrow +\infty} \frac{1}{T} \log \left( \frac{\sum_{\gamma : \mu_{\gamma} \in \mathcal{U}, \lambda(\gamma) \leq T} e^{\lambda_F(\gamma)}}{\sum_{\gamma : \lambda(\gamma) \leq T} e^{\lambda_F(\gamma)}} \right) \\ & \leq \max_i \{-(Q(f_i) - \rho + \epsilon) + Q(f_i)\} \\ & = -\rho + \epsilon. \end{aligned}$$

This completes the proof by choosing  $\rho > \epsilon > 0$  and then choosing  $0 < \eta < \rho - \epsilon$ .

## References

- [1] A. Amroun, On the number of periodic geodesics in rank 1 surfaces, preprint.
- [2] K. Burns, V. Climenhaga, T. Fisher, D. Thompson, Unique equilibrium states for geodesic flows in nonpositive curvature, arXiv:1703.10878
- [3] K. Burns and K. Gelfert, Lyapunov spectrum for geodesic flows of rank 1 surfaces, *Discrete Contin. Dyn. Syst.* 34 (2014), 1841-1872.
- [4] K. Gelfert and B. Schapira, Pressures for geodesic flows of rank one manifolds, *Nonlinearity* 27 (2014), no. 7, 1575C1594.
- [5] G. Knieper, The uniqueness of the measure of maximal entropy for geodesic flows on rank 1 manifolds. *Ann. of Math.* (2) 148 (1998) 29117314.
- [6] S. Newhouse, Continuity properties of entropy. *Ann. of Math.* (2) 129 (1989) 21517235.
- [7] W. Parry, Equilibrium states and weighted uniform distribution of closed orbits. *Dynamical systems* (College Park, MD, 19861787), 61717625, *Lecture Notes in Math.*, 1342, Springer, Berlin, 1988.
- [8] M. Pollicott, Distribution of closed geodesics for manifolds of non-positive curvature, *Discrete and Continuous Dynamical Systems*, 2 (1996) 153-161
- [9] M. Pollicott, Large deviations, Gibbs measures, and closed orbits for hyperbolic flows, *Mathematische Zeitschrift*, 220 (1995) 219-230
- [10] P. Walters, *Ergodic Theory*, Springer, 1982.