THE NUMBER OF REPRESENTATIONS OF AN INTEGER AS PRODUCT OF TWO RESTRICTED NUMBERS

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Abstract. In this paper we continue the study already initiated in [23] of the arithmetic function $\tau_N(n)$, which counts the number of representations of a positive integer $n \le N^2$ as product of two integers both smaller than N. In particular, we focus on producing an asymptotic estimate for all its positive integers moments. So doing we incidentally prove some special cases of the well-known Manin–Peyre's conjecture on singular toric algebraic varieties.

1. Motivations

In this paper we are concerned with the study of the function

$$\tau_N(n) = \#\{d|n : n/N \le d \le N\},$$

through understanding all its positive integer moments[1]. The function $\tau_N(n)$ played a fundamental role in the author's paper [23], where a study about the distribution of its values allowed him to completely classify maximal size product sets of random sets. From this point of view, this short article can be viewed as complementary to [23], where we exhibited some heuristics for the behaviour of $\tau_N(n)$ on average and found what might be considered as its normal order. In the following we highlight three basic motivations to undertake a further study of the function τ_N, particularly focusing on its positive integer moments.

1.1. **A variation on the localised divisor functions theme.** The function $\tau_N(n)$ is an arithmetic variation of localised divisor functions of the form

$$\tau(n; y, z) = \#\{d|n : y < d \le z\},$$

for values of $0 < y < z \le N$. Such functions, and their applications related to counting integers with a divisor in fixed intervals, have been extensively studied in literature, from works of Besicovitch [4] and Erdős [11–13] to those of Tenenbaum [25–30] and Ford [14, 15], to name just a few.

In all these works the focus was on localised divisor functions for parameters y, z as functions of N only. In the definition of $\tau_N(n)$ instead, the parameter y depends on n itself. This makes $\tau_N(n)$ of a different nature compared to the function $\tau(n; y, z)$, more arithmetic the former, more analytic the latter.

For instance, it is immediate to see that all the values of $\tau_N(n)$ are even (since $d|n$ and $n/N \le d \le N$ implies $(n/d)|n$ and again $n/N \le n/d \le N$) apart when n is a perfect square. When $n \le N$, $\tau_N(n)$ simply coincides with $\tau(n)$. Moreover, if p indicates a prime factor of n and we write $n = pk$, we then have

$$\tau_N(n) = 2\#\{d|k : k/N \le d \le N/p\}.$$

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[1] The author is not aware of any previous work in which a similar task has been carried out.

It is also true though that for several other analytical aspects it really shares some properties with the localised divisor function $\tau(n;y,N)$ for $y \approx N - N/(\log N)^{\log 4-1}$ (see e.g. Hall and Tenenbaum [17] for a more detailed discussion about this).

1.2. **Information on the uniform distribution of divisors on average.** In a previous work [23] the author exhibited a heuristic on the distribution of the values of $\tau_N(n)$. Roughly speaking, by assuming that the set $\{\log d/\log N : \ d|n\}$, of flat quotients $\log d/\log N$ over the divisors d of n, is uniformly distributed, the author deduced that

$$\tau_N(n) \approx \frac{\tau(n)}{\log N}. \tag{1.1}$$

From the work of Ford [14, 15] we know that for localised divisor functions the uniform distribution hypothesis on the quotients $\log d/\log N$ does not very well describe the behaviour of the set of divisors of individual integers n. Indeed, it is expected that many integers n possess various clusters of close divisors and large gaps between them. We refer to [14,15] for a concrete measure of the degree of propinquity of the divisors of a given integer. It derives that for a single n the heuristic (1.1) is doomed to fail. However, when considered on average over a large portion of positive integers $n \le N^2$ it indeed seems quite sharp. A way to understand whether this uniformity assumption could really well-describe the behaviour of the set of divisors of integers, at least on an average sense, could be to look at the moments of $\tau_N(n)$ and see if their asymptotic behaviour agrees with that of the moments of the function $\tau(n)/\log N$, as it would follow from the prediction (1.1). In particular, we note that for this last one we have:

$$\sum_{n\le N^2} \left(\frac{\tau(n)}{\log N}\right)^k \sim c_k N^2 (\log N)^{2^k-k-1} \quad \text{as } N \longrightarrow +\infty, \tag{1.2}$$

for a certain $c_k > 0$ (see e.g. Luca and Tóth's paper [22]).

1.3. **A special case of the Manin–Peyre's conjecture.** Some arithmetic variations of the function $\tau_N(n)$ have been introduced and studied in relation to the Manin–Peyre's conjecture or to counting the discriminants of number fields which are multiquadratic extensions of $\mathbb{Q}$. Particularly, it is worth mentioning works of Tolev [31] and de la Bretèche, Kurlberg and Shparlinski [10] , where an asymptotic estimate for the partial sum of the function which counts the number of pairs (or more generally m-tuples) of positive integers both smaller than N whose product is a perfect power has been provided.

The Manin's conjecture states that for many affine varieties X over a number field $\mathbb{K}$ we expect the number of integral points on a suitably nice open subset $U \subset X(\mathbb{K})$ (with respect to the Zariski topology) to satisfy

$$\#\{x \in U(\mathbb{K}) : H(x) \le B\} \sim CB^a(\log B)^{b-1},$$

with respect to a suitable height function $H : X(\mathbb{K}) \to \mathbb{R}$, where a, b and C are certain constants depending on the geometry of X. Franke, Manin, and Tschinkel [16] originally proved this conjecture for flag varieties. In [3], Batyrev and Tschinkel settled this conjecture for arbitrary smooth affine toric varieties. On the other hand, De la Bretéche and Browning [8, 9] have also shown that the asymptotic holds for many cases of del Pezzo surfaces.

In light of the above discussion, we can look at the partial sum of $\tau_N(n)^k$ as counting the number of positive integer points of height bounded by N on the following singular affine toric variety:

$$V := \{(X_1, X_2, \dots, X_{2k}) \in \mathbb{A}^{2k} : X_1X_2 - X_jX_{j+1} = 0, \text{ for any odd } 1 \le j \le 2k-1\},$$

with respect to the height function $H : \mathbb{A}^{2k} \to \mathbb{R}$ such that

$$H((x_1, x_2, \dots, x_{2k})) = \max_{1\le i\le 2k} \{|x_i|\}.$$

2. Some basic estimates

Clearly we have
$$\sum_{1\le n\le N^2}\tau_N(n)=N^2.$$
It is quite easy to prove an upper and lower bound for the average of the square of $\tau_N(n)$. This is the content of the following lemma.

Lemma 2.1. *There exist two positive constants $C_1\le C_2$ such that*
$$C_1N^2\log N\le\sum_{1\le n\le N^2}\tau_N(n)^2\le C_2N^2\log N.$$

Proof. We start with the lower bound. For any fixed pair of coprime positive integers m,n we are going to count the number of quadruples $(a,b,c,d)\in[N]^4$ such that
$$\frac{m}{n}=\frac{a}{c}=\frac{d}{b}.$$
Since $mc=an$ and $bm=dn$, we have that $m|a$ and $n|b$. Once chosen such values of a,b also c,d will be determined. Thus we get
$$\begin{aligned}\sum_{1\le n\le N^2}\tau_N(n)^2=E([N])&\ge\sum_{\substack{1\le m<n\le N/2\\(m,n)=1}}\sum_{\substack{1\le a\le N\\ m|a}}\sum_{\substack{1\le b\le N\\ n|b}}1\\&\gg\sum_{\substack{1\le m<n\le N/2\\(m,n)=1}}\Big(\min\{N/n-1,N/m-1\}\Big)^2\\&\gg N^2\sum_{1\le n\le N/2}\sum_{\substack{1\le m<n\\(m,n)=1}}\frac{1}{n^2}=N^2\sum_{1\le n\le N/2}\frac{\varphi(n)}{n^2}\gg N^2\log N,\end{aligned}$$
where $\varphi(n)$ is the Euler totient function and we used Landau's result [21, p. 184].

We now move on to the upper bound. If for positive integers k_1,k_2,j_1,j_2 we have $k_1j_1=k_2j_2$ then there exist positive integers a,b,c and d such that $k_1=ab$, $j_1=cd$, $k_2=ac$ and $j_2=bd$. To see this, take:
$$a:=(k_1,k_2),\ d:=(j_1,j_2),\ b:=k_1/a,\ c:=j_1/d.$$
Consequently, the sum in the lemma is bounded above by
$$T:=|\{(a,b,c,d):ab,cd,ac,db\le N\}|.$$
Given $b,c\le N$, we have $a,d\le\min\{N/b,N/c\}$. Thus we have
$$\begin{aligned}T\le\sum_{1\le b,c\le N}(\min\{N/b,N/c\})^2&\le 2\sum_{1\le b<c\le N}\frac{N^2}{c^2}+\sum_{1\le b\le N}\frac{N^2}{b^2}\\&\ll N^2\sum_{1\le c\le N}\frac{1}{c}+N^2\sum_{1\le b\le N}\frac{1}{b^2}\ll N^2\log N,\end{aligned}$$
by comparing the sums with their corresponding integrals. □

Proving an asymptotic for the second moment of τ_N is a much more complicated task. This corresponds to producing an asymptotic for the multiplicative energy of the first N numbers. This problem has been handled by Heath-Brown [19, Theorem 7], who among other things showed the existence of a positive constant D for which
$$\sum_{1\le n\le N^2}\tau_N(n)^2=(D+o(1))N^2\log N\quad(\text{as } N\longrightarrow+\infty).$$

3. The moments of τ_N

By (1.1) and (1.2), we heuristically expect an asymptotic for higher moments of the shape:

$$\sum_{n\leq N^2} \tau_N(n)^k \sim D_k N^2 (\log N)^{2^k-k-1} \quad (\text{as } N \longrightarrow +\infty),$$

for a certain $D_k > 0$.

The aim of this section is to formally derive the above asymptotic, by employing some results of de la Bretèche [5] on sums of arithmetic functions of many variables. These types of sums appear naturally when counting integer points of bounded height on some varieties. This procedure has been used, for example, in [6,7] to prove Manin's conjecture in some special cases.

We report next [5, Théorème 1, Théorème 2] in a slightly simplified and more compact form. In particular, we refer to [5] for a more thoroughly treatment of such results.

Before stating their content, we first need to introduce some notations. We say that a function $f : \mathbb{N}^m \longrightarrow \mathbb{C}$ is multiplicative if

$$f(d_1,\dots,d_m)f(e_1,\dots,e_m) = f(d_1e_1,\dots,d_me_m),$$

whenever the greatest common divisor

$$\gcd(d_1\cdots d_m, e_1\cdots e_m) = 1.$$

In that case, formally we have

$$F(\mathbf{s}) = \prod_p \Bigg(\sum_{\nu\in\mathbb{N}^m} \frac{f(p^{\nu_1},\dots,p^{\nu_m})}{p^{\nu_1 s_1+\cdots+\nu_m s_m}} \Bigg),$$

with the obvious vector notation for $\mathbf{s} \in \mathbb{N}^m$. We denote by $\mathcal{L}_m(\mathbb{C})$ the space of linear forms

$$l(X_1,\dots,X_m) \in \mathbb{C}[X_1,\dots,X_m].$$

We denote by $\{e_j\}_{j=1}^m$ the canonical basis of $\mathbb{C}^m$ and by $\{e_j^*\}_{j=1}^m$ the dual basis in $\mathcal{L}_m(\mathbb{C})$. We denote by $\mathcal{L}\mathbb{R}_m(\mathbb{C})$ the set of linear forms of $\mathcal{L}_m(\mathbb{C})$ for which their restriction to $\mathbb{R}^m$ maps to $\mathbb{R}$. We define $\mathcal{L}\mathbb{R}_m^+(\mathbb{C})$ similarly with respect to the set $\mathbb{R}^+$ of positive real numbers.

As usual, we use $||\cdot||_1$ to denote the L^1-norm and use $< \cdot >$ to denote the inner product of vectors from $\mathbb{R}^m$. We can view $\mathbb{R}^m$ as a partially ordered set using the relation $\mathbf{d} > \mathbf{e}$ if and only if this inequality holds componentwise for $\mathbf{d}, \mathbf{e} \in \mathbb{R}^m$.

We also apply the notations $\Re$ (real part) and $\Im$ (imaginary part) to vectors in the natural componentwise fashion.

Proposition 3.1. *Let f be a positive arithmetical function on $\mathbb{N}^m$ and F be the associated Dirichlet series*

$$F(\boldsymbol{s}) = \sum_{d_1=1}^{\infty}\cdots\sum_{d_m=1}^{\infty} \frac{f(d_1,\dots,d_m)}{d_1^{s_1}\cdots d_m^{s_m}}.$$

We assume that there exists an $\boldsymbol{a} \in (\mathbb{R}^+)^m$ such that F satisfies the following properties:

- *$F(\boldsymbol{s})$ is absolutely convergent on $\Re(\boldsymbol{s}) > \boldsymbol{a}$;*
- *There exists a family of n non-zero linear forms $\mathcal{L} = \{l^{(i)}\}_{i=1}^n$ of $\mathcal{L}\mathbb{R}_m^+(\mathbb{C})$ such that the function $H : \mathbb{C}^m \longrightarrow \mathbb{C}$, defined by*

  $$H(\boldsymbol{s}) = F(\boldsymbol{s}+\boldsymbol{a}) \prod_{i=1}^n l^{(i)}(\boldsymbol{s}),$$

  can be analytically continued in the domain:

  $$\mathcal{D}(\delta_1) := \{\boldsymbol{s} \in \mathbb{C}^m : \Re(l^{(i)}(\boldsymbol{s})) > -\delta_1, \forall i\}.$$

- *There exists $\delta_2 > 0$ such that, for all $\varepsilon_1, \varepsilon_2 > 0$, the following upper bound*

$$H(\boldsymbol{s}) \ll \prod_{i=1}^{n} (|\Im(l^{(i)}(\boldsymbol{s}))|+1)^{1-\delta_2 \min\{0, \Re(l^{(i)}(\boldsymbol{s}))\}}(1+||\Im(\boldsymbol{s})||_1^{\varepsilon_1}), \tag{3.1}$$

holds uniformly in the domain $D(\delta_1 - \varepsilon_2)$.

Then for any vector $\boldsymbol{b} \in (\mathbb{R}^+)^m$, there exists a polynomial $Q_{\boldsymbol{b}}(Y) \in \mathbb{R}[Y]$ of degree

$$\deg(Q_{\boldsymbol{b}}) \leq n - \operatorname{rank}(\{l^{(i)}\}_{i=1}^n)$$

and a real $\theta = \theta(\mathcal{L}, \delta_1, \delta_2, \boldsymbol{a}, \boldsymbol{b}) > 0$, such that for all $X \geq 1$, we have

$$S(X^{\boldsymbol{b}}) := \sum_{1 \leq d_1 \leq X^{\boldsymbol{b}_1}} \cdots \sum_{1 \leq d_m \leq X^{\boldsymbol{b}_m}} f(d_1, \dots, d_m) = X^{<\boldsymbol{a},\boldsymbol{b}>}(Q_{\boldsymbol{b}}(\log X) + O(X^{-\theta})).$$

If moreover $H(0, \dots, 0) \neq 0$, $l^{(i)}(\boldsymbol{a}) = 1$, for any $i = 1, \dots, n$, and

$$\sum_{j=1}^{m} \boldsymbol{b}_j e_j^* \in \operatorname{Conv}(\{l^{(i)}\}_{i=1}^n),$$

where

$$\operatorname{Conv}(\mathcal{L}) = \sum_{l \in \mathcal{L}} \mathbb{R}^+ l,$$

we have

$$Q_{\boldsymbol{b}}(\log X) = H(0, \dots, 0) X^{-<\boldsymbol{a},\boldsymbol{b}>} \iiint_{V(X^{\boldsymbol{b}})} 1 \, d\underline{y} + O((\log X)^{n - \operatorname{rank}(\{l^{(i)}\}_{i=1}^n) - 1}), \tag{3.2}$$

where we define

$$V(X^{\boldsymbol{b}}) := \{\underline{y} := (y_1, \dots, y_n) \in [1, +\infty)^n : \prod_{i=1}^{n} y_i^{l^{(i)}(e_j)} \leq X^{\boldsymbol{b}_j}, \text{ for any } 1 \leq j \leq m\}.$$

We are now ready to prove the main result of this section.

Theorem 3.2. *For any integer $k \geq 1$, there exists a constant $D_k > 0$ such that*

$$\sum_{n \leq N^2} \tau_N(n)^k = (D_k + o(1)) N^2 (\log N)^{2^k - k - 1}, \tag{3.3}$$

as N tends to infinity.

Remark 3.3. *The constant D_k in (3.3) can be evaluated explicitly, but we will not insert the related details here.*

Remark 3.4. *Our proof of Theorem 3.2 takes inspiration from that of [24, Lemma 2.3] and of [10, Theorem 2.2]. The degree of the polynomial $Q(Y)$ can be precisely computed under the assumption $\operatorname{rank}(\{l^{(i)}\}_{i=1}^n) = m$ thanks to [5, Théorème 2 (iv)]. In absence of this assumption, the volume (3.2) has to be analysed. In relation to this, a close reading of [24, Lemma 2.3] reveals that the computation of the degree of $Q(Y)$ has been carried out under the aforementioned assumption, which does not hold in that case. However, the degree of $Q(Y)$ in the proof of [24, Lemma 2.3] can be found through evaluating an integral volume similar to that in (3.2) and indeed leads to the same answer.*

Proof. To begin with, we note that the sum in question can be rewritten as:

$$= \#\{m_1 l_1 = m_2 l_2 = \cdots = m_k l_k : \ 1 \leq m_1, \dots, m_k \leq N \text{ and } 1 \leq l_1, \dots, l_k \leq N\}.$$

Hence, in light of Proposition 3.1, we define

$$f(d_1, \dots, d_{2k}) = \begin{cases} 1 & \text{if } d_1 d_2 = d_3 d_4 = \cdots = d_{2k-1} d_{2k}; \\ 0 & \text{otherwise,} \end{cases}$$

which is immediate to verify defines a multiplicative function.

We are interested in estimating the sum:

$$S(N^{(1,\dots,1)}) := \sum_{1\le d_1\le N}\cdots\sum_{1\le d_{2k}\le N} f(d_1,\dots,d_{2k}),$$

which corresponds to the sum in (3.3).

For a vector $s=(s_1,\dots,s_{2k})\in\mathbb{C}^{2k}$ of $2k$ complex numbers, we define the multiple Dirichlet series:

$$F(\mathbf{s}) = \sum_{d_1,\dots,d_{2k}\ge 1}\frac{f(d_1,\dots,d_{2k})}{d_1^{s_1}\cdots d_{2k}^{s_{2k}}}.$$

We define for the vector $\mathbf{a}$ in the statement of Proposition 3.1 the vector:

$$\mathbf{a} := (1/k,\dots,1/k)\in\mathbb{R}^{2k}. \tag{3.4}$$

Since clearly

$$|d_1^{s_1}\cdots d_{2k}^{s_{2k}}|\ge (d_1\cdots d_{2k})^{\sigma(\mathbf{s})},$$

where

$$\sigma(\mathbf{s}) := \min\{\Re(s_j):\ 1\le j\le 2k\}, \tag{3.5}$$

we have

$$\sum_{d_1,\dots,d_{2k}\ge 1}\left|\frac{f(d_1,\dots,d_{2k})}{d_1^{s_1}\cdots d_{2k}^{s_{2k}}}\right| \le \sum_{m\ge 1}\frac{\tau(m)^k}{m^{k\sigma(\mathbf{s})}} = \prod_p\Bigg(\sum_{j\ge 0}\frac{(j+1)^k}{p^{jk\sigma(\mathbf{s})}}\Bigg),$$

which proves the absolute convergence of $F(\mathbf{s})$ in the range $\sigma(\mathbf{s})>1/k$ and verifies the first assumption in Proposition 3.1 for $\mathbf{a}$ as given by (3.4).

The linear forms $\{e_j^*\}_{j=1}^m$, where we clearly set $m:=2k$, are explicitly given by:

$$e_j^*(X_1,\dots,X_{2k}) = X_j,\ \text{ for any } 1\le j\le 2k.$$

We now prove that the second and the third assumptions in Proposition 3.1 are satisfied with the $n:=2^k$ linear forms:

$$l^{(c_1,\dots,c_k)} := e_{c_1}^*+\cdots+e_{c_k}^*,\ \text{ where } 1\le c_1\le 2, 3\le c_2\le 4,\dots,2k-1\le c_k\le 2k.$$

Since f is multiplicative, in this range, we have:

$$F(\mathbf{s}) = \prod_p F_p(\mathbf{s}), \tag{3.6}$$

where

$$F_p(\mathbf{s}) := \sum_{r_1,\dots,r_{2k}\ge 0}\frac{f(p^{r_1},\dots,p^{r_{2k}})}{p^{r_1s_1+\cdots+r_{2k}s_{2k}}} = \sum_{\substack{r_1,\dots,r_{2k}\ge 0\\ r_1+r_2=r_3+r_4=\cdots=r_{2k-1}+r_{2k}}}\frac{1}{p^{r_1s_1+\cdots+r_{2k}s_{2k}}}.$$

By expanding the above Euler product we get

$$F_p(\mathbf{s}) = 1+\sum_{i_1=1,2}\sum_{i_2=3,4}\cdots\sum_{i_k=2k-1,2k}\frac{1}{p^{s_{i_1}+s_{i_2}+\cdots+s_{i_k}}} + \sum_{\substack{r_1,\dots,r_{2k}\ge 0\\ r_1+r_2=r_3+r_4=\cdots=r_{2k-1}+r_{2k}\ge 2}}\frac{1}{p^{r_1s_1+\cdots+r_{2k}s_{2k}}}$$

and, on $\sigma(\mathbf{s})>0$, where $\sigma(\mathbf{s})$ is given by (3.5), the absolute value of the third term of the right-hand side of the above equality is bounded by

$$\sum_{\substack{r_1,\dots,r_{2k}\ge 0\\ r_1+r_2=r_3+r_4=\cdots=r_{2k-1}+r_{2k}\ge 2}}\frac{1}{p^{(r_1+\cdots+r_{2k})\sigma(\mathbf{s})}} \le \sum_{r\ge 2}\frac{(r+1)^k}{p^{rk\sigma(\mathbf{s})}} \ll_k \frac{1}{p^{2k\sigma(\mathbf{s})}}.$$

Furthermore, for a given $A > 0$ and for $\sigma(\mathbf{s}) \geq A$, we have

$$\prod_{i_1=1,2}\prod_{i_2=3,4}\cdots\prod_{i_k=2k-1,2k}\left(1-\frac{1}{p^{s_{i_1}+s_{i_2}+\cdots+s_{i_k}}}\right) = 1 - \sum_{i_1=1,2}\sum_{i_2=3,4}\cdots\sum_{i_k=2k-1,2k}\frac{1}{p^{s_{i_1}+s_{i_2}+\cdots+s_{i_k}}} + O_A\left(\frac{1}{p^{2k\sigma(\mathbf{s})}}\right).$$

Therefore, we conclude that

$$F_p(\mathbf{s})\prod_{i_1=1,2}\prod_{i_2=3,4}\cdots\prod_{i_k=2k-1,2k}\left(1-\frac{1}{p^{s_{i_1}+s_{i_2}+\cdots+s_{i_k}}}\right) = 1 + O_A\left(\frac{1}{p^{2k\sigma(\mathbf{s})}}\right). \tag{3.7}$$

Taking the product over all primes and using the Euler product formula for the Riemann zeta function, defined as:

$$\zeta(s) = \prod_p \left(1-\frac{1}{p^s}\right)^{-1},$$

which holds for any complex s with $\Re(s) > 1$, we obtain from (3.6) and (3.7) that, for $\sigma(\mathbf{s}) > 1$, we have

$$F(\mathbf{s}) = \psi(\mathbf{s})\prod_{i_1=1,2}\prod_{i_2=3,4}\cdots\prod_{i_k=2k-1,2k}\zeta(s_{i_1}+s_{i_2}+\cdots+s_{i_k}), \tag{3.8}$$

where $\psi(\mathbf{s})$ is a holomorphic function on $\sigma(\mathbf{s}) \geq A$ for any fixed $A > 1/2k$.

We can rewrite (3.8) as

$$\begin{aligned} &F(\mathbf{s})\prod_{i_1=1,2}\prod_{i_2=3,4}\cdots\prod_{i_k=2k-1,2k}(s_{i_1}+s_{i_2}+\cdots+s_{i_k}-1) \\ &= \psi(\mathbf{s})\prod_{i_1=1,2}\prod_{i_2=3,4}\cdots\prod_{i_k=2k-1,2k}\zeta(s_{i_1}+s_{i_2}+\cdots+s_{i_k})(s_{i_1}+s_{i_2}+\cdots+s_{i_k}-1). \end{aligned} \tag{3.9}$$

One can check that the left-hand side of (3.9) verifies (3.1) in the range $\sigma(\mathbf{s}) \geq A$, for any $A > 1/2k$, by employing the Vinogradov-Korobov's bound for the Riemann zeta function (see e.g. [20, Theorem 8.27, Corollary 8.28]).

Translating each coordinate by $1/k$, we see that

$$H(\mathbf{s}) = F(\mathbf{s}+\mathbf{a})\prod_{i_1=1,2}\prod_{i_2=3,4}\cdots\prod_{i_k=2k-1,2k}(s_{i_1}+s_{i_2}+\cdots+s_{i_k})$$

verifies (3.1) in the range $\sigma(\mathbf{s}) \geq B$ for any $B = A - 1/k > -1/2k$. Hence, the second and the third assumptions in Proposition 3.1 are satisfied for $H(\mathbf{s})$. In particular, the first assertion in Proposition 3.1 holds with the choice $\mathbf{b} = (1,\ldots,1)$, $m = 2k$, $n = 2^k$ and it derives that

$$S(N^{(1,\ldots,1)}) = N^2(Q(\log N) + O(N^{-\theta})), \tag{3.10}$$

for a certain polynomial $Q(Y) \in \mathbb{R}[Y]$ and a positive real number θ as in the statement of Proposition 3.1.

In the remaining part of the proof we check that also the conditions in the second part of Proposition 3.1 are satisfied in order to apply the second assertion there and then explicitly compute the degree of the polynomial $Q(Y)$.

We start with observing that by previous considerations and since the Riemann zeta function has a simple pole at 1 with residue 1, we have the identity

$$H(0,\ldots,0) = \prod_p\left(1-\frac{1}{p}\right)^{2^k}\left(1+\frac{2^k}{p}+\sum_{r\geq 2}\frac{(r+1)^k}{p^r}\right) > 0.$$

Furthermore, it is clear that $l^{(c_1,\ldots,c_k)}(\mathbf{a}) = 1$, for any choice of $c_1,\ldots,c_k$.

Finally, we clearly have that the linear form $e_1^* + \cdots + e_m^*$ lies in the positive convex cone of the linear forms $l^{(c_1,\dots,c_k)}$. Indeed, for any linear form $l^{(c_1,\dots,c_k)}$ we have the "complementary" one given by $l^{(c_1',\dots,c_k')}$, where $\{c_1, c_1'\} = \{1,2\}, \dots, \{c_k, c_k'\} = \{2k-1, 2k\}$. Moreover, they generate 2^{k-1} distinct identities:

$$l^{(c_1,\dots,c_k)} + l^{(c_1',\dots,c_k')} = e_1^* + \cdots + e_{2k}^*.$$

In conclusion, we get

$$e_1^* + \cdots + e_{2k}^* = \sum_{c_1,\dots,c_k} \frac{l^{(c_1,\dots,c_k)}}{2^{k-1}}.$$

We have then proved that the remaining conditions in Proposition 3.1 are satisfied and we can infer that

$$Q(\log N) = \frac{H(0,\dots,0)}{N^2} \iiint_{V(N^{(1,\dots,1)})} 1 \, d\underline{y} + O((\log N)^{2^k - \mathrm{rank}(\{l^{(c_1,\dots,c_k)}\}_{c_1,\dots,c_k}) - 1}), \tag{3.11}$$

where we now have

$$V(N^{(1,\dots,1)}) := \{\underline{y} := (y_{c_1,\dots,c_k}) \in [1,+\infty)^{2^k} : \prod_{c_1,\dots,c_k} y_{c_1,\dots,c_k}^{l^{(c_1,\dots,c_k)}(e_j)} \leq N, \text{ for any } 1 \leq j \leq 2k\}.$$

In order to find the degree of $Q(Y)$ we first compute the rank of the $2k \times 2^k$ matrix given by the vectors $\{l^{(c_1,\dots,c_k)}\}_{c_1,\dots,c_k}$.

Claim 3.5. *We have*

$$\mathrm{rank}(\{l^{(c_1,\dots,c_k)}\}_{c_1,\dots,c_k}) = k+1.$$

We first exhibit a maximal independent subset among the vectors $\{l^{(c_1,\dots,c_k)}\}_{c_1,\dots,c_k}$.

Claim 3.6. *The following linear forms:*

$$\begin{aligned} l^{(1)} &:= e_1^* + e_3^* + e_5^* + \cdots + e_{2k-1}^* \\ l^{(2)} &:= e_2^* + e_3^* + e_5^* + \cdots + e_{2k-1}^* \\ l^{(3)} &:= e_1^* + e_4^* + e_5^* + \cdots + e_{2k-1}^* \\ l^{(4)} &:= e_1^* + e_3^* + e_6^* + \cdots + e_{2k-1}^* \\ &\vdots \\ l^{(k+1)} &:= e_1^* + e_3^* + e_5^* + \cdots + e_{2k}^* \end{aligned}$$

are independent and linearly generate all the other linear forms $l^{(c_1,\dots,c_k)}$.

Remark 3.7. *The above linear forms are defined in the following way: the j-th linear form $l^{(j)}$ is the sum of all the e_i^* for odd indices i except that when $j > 1$ the linear form e_{2j-3}^* is replaced by e_{2j-2}^*.*

Claim 3.6 would imply Claim 3.5. First of all, we show that they are linearly independent. Suppose indeed that we have a linear combination

$$a_1 l^{(1)} + \cdots + a_{k+1} l^{(k+1)} = 0, \text{ with } a_1, \dots, a_{k+1} \in \mathbb{R}.$$

It is immediate to verify that the following relations hold:

$$
\begin{aligned}
&1)\ \ 0 = a_1 + a_3 + a_4 + a_5 + \cdots + a_{k+1}\\
&2)\ \ 0 = a_2\\
&3)\ \ 0 = a_1 + a_2 + a_4 + a_5 + \cdots + a_{k+1}\\
&4)\ \ 0 = a_3\\
&5)\ \ 0 = a_1 + a_2 + a_3 + a_5 + \cdots + a_{k+1}\\
&6)\ \ 0 = a_4\\
&\vdots\\
&2k-1)\ \ 0 = a_1 + a_2 + a_3 + a_4 + a_5 + \cdots + a_k\\
&2k)\ \ 0 = a_{k+1}.
\end{aligned}
$$

Remark 3.8. *The above equations with a label given by an odd integer j are defined as the sum of all the coefficients $a_1, a_2, \ldots, a_{k+1}$, with the omission of the coefficient $a_{(j+3)/2}$.*

In particular, we deduce that

$$0 = a_2 = a_3 = \cdots = a_{k+1}$$

which, by substituting back into the first equation above, also leads to $a_1 = 0$.

We now move on to showing that they generate any other linear form $l^{(c_1,\ldots,c_k)}$. Indeed, we can identify each $l^{(c_1,\ldots,c_k)}$ with a vector $(i_1, 1-i_1, i_2, 1-i_2, \ldots, i_k, 1-i_k)$ with each $i_j \in \{0,1\}$ for any $j = 1, \ldots, k$. We are now then seeking for coefficients $a_1, \ldots, a_{k+1} \in \mathbb{R}$ for which

$$l^{(c_1,\ldots,c_k)} = a_1 l^{(1)} + \cdots + a_{k+1} l^{(k+1)},$$

or equivalently to solutions to the following system of equations:

$$
\begin{aligned}
&1)\ \ i_1 = a_1 + a_3 + a_4 + a_5 + \cdots + a_{k+1}\\
&2)\ \ 1 - i_1 = a_2\\
&3)\ \ i_2 = a_1 + a_2 + a_4 + a_5 + \cdots + a_{k+1}\\
&4)\ \ 1 - i_2 = a_3\\
&5)\ \ i_3 = a_1 + a_2 + a_3 + a_5 + \cdots + a_{k+1}\\
&6)\ \ 1 - i_3 = a_4\\
&\vdots\\
&2k-1)\ \ i_k = a_1 + a_2 + a_3 + a_4 + a_5 + \cdots + a_k\\
&2k)\ \ 1 - i_k = a_{k+1}.
\end{aligned}
$$

The above system is made of $2k$-equations numbered from the top to the bottom with the numbers between 1 to $2k$. We notice that the equations with an even label uniquely determine altogether the coefficients $a_2, a_3, \ldots, a_{k+1}$. Substituting them back into the first equation we also find the value of a_1. We are only left with making sure that all the equations with a label which is an odd number different from 1 are not in conflict with the values of the coefficients a_j we have just found. However, each of such equation states that for each number i_j, for $j = 2, \ldots, k$, we have

$$
\begin{aligned}
i_j &= a_1 + a_2 + \cdots + a_j + a_{j+2} + \cdots + a_{k+1}\\
&= (a_1 + a_3 + a_4 + a_5 + \cdots + a_{k+1}) + (a_2 - a_{j+1}) = i_1 + (a_2 - a_{j+1}),
\end{aligned}
$$

which is certainly satisfied by the equations with labels 2 and $2j$, because $a_2 = 1 - i_1$ and $a_{j+1} = 1 - i_j$. In conclusion, the above system admits a unique and well defined solution $a_1, a_2, a_3, \ldots, a_{k+1}$, thus showing the possibility of generating all the $l^{(c_1,\ldots,c_k)}$'s starting from our family of $(k+1)-$linear forms and concluding the proof of Claim 3.6.

We are only left with finding the degree of $Q(Y)$. In order to do that, we are going to produce a sharp lower bound for the integral in (3.11).

Claim 3.9. *We have*

$$\iiint_{V(N^{(1,\ldots,1)})} 1 \ d\underline{y} \gg N^2 (\log N)^{2^k - k - 1}.$$

To this aim we first relabel the linear forms $l^{(c_1,\ldots,c_k)}$ as $(l_v)_{v=0,\ldots,2^k-1}$ by identifying each k-tuple $(c_1, \ldots, c_k)$ with a new k-tuple $(a_0, \ldots, a_{k-1})$ via $c_j = a_{j-1} + 2j - 1$, with $a_j \in \{0, 1\}$, for any $j = 1, \ldots, k$, and then the k-tuple $(a_0, \ldots, a_{k-1})$ with a number $v \in \{0, \ldots, 2^k - 1\}$ via its binary expansion $v = \sum_{j=0}^{k-1} a_j 2^j$.

By changing variables $y_i = e^{t_{i-1}}$, for any $i = 1, \ldots, 2^k$, we can rewrite the integral in (3.11) as

$$\iiint_{W(N)} e^{<\underline{t},\mathbf{1}>} \ d\underline{t},$$

where we let

$$\begin{aligned} W(N) : &= \{\underline{t} := (t_0, t_1, \ldots, t_{2^k-1}) \in [0, +\infty)^{2^k} : \sum_{v=0}^{2^k-1} l_v(e_j) t_v \leq \log N, \ \text{for any } j = 1, \ldots, 2k\} \\ &= \{\underline{t} \in [0, +\infty)^{2^k} : \ M \cdot \underline{t} \leq (\log N) \cdot \mathbf{1}\}, \end{aligned}$$

with $M := (l_v(e_j))_{j,v}$ a $2k \times 2^k$ 0-1 matrix and $\mathbf{1}$ the vector with each component equal to 1. Thus $W(N)$ is a closed convex polytope with respect to the matrix M of $\text{rank}(M) = k + 1$.

The above type of integral has been already analysed in works such as [1,2], where a decomposition in terms of a linear combination of the same exponential integrals over the facets of the polytope has been given. However, such results are not very helpful to our context since unfortunately we do not have any control on the coefficients of such linear combination.

On the other hand, a similar argument to that employed in the work of Harper, Nikeghbali and Radziwiłł [18] leads to the estimate stated in Claim 3.9 (we omit the details here).

Combining the lower bound in Claim 3.9 with the asymptotic (3.11), it derives that $\deg(Q) = 2^k - k - 1$, which together with equation 3.10 concludes the proof of Theorem 3.2. □

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