Perfect Powers that are Sums of Consecutive k-th Powers

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A Diophantine Equation

$$(x+1)^k + (x+2)^k + \cdots + (x+d)^k = y^n.$$

Question

Fix $k \geq 2$ and $d \geq 2$. Determine all of the integer solutions (x, y, n).

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Remark: We can let $n = p$ be a prime.

A Brief History

Euler:

$$6^3 = 3^3 + 4^3 + 5^3.$$

Dickson's *"History of the Theory of Numbers"*:

Catalan, Cunningham, Lucas and Gennochi.

Later contributions from:

1. Pagliani (1829): parametric solutions.
2. Cassels (1985): $y^2 = x^3 + (x+1)^3 + (x+2)^3$.
3. Uchiyama (1979): independently to Cassels.
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Well–Known:

$$\sum_{i=0}^{d} i^3 = \sum_{i=1}^{d} i^3 = \left(\frac{d(d+1)}{2}\right)^2 .$$

Pagliani:

$$\sum_{i=1}^{v^3} \left(\frac{v^4 - 3v^3 - 2v^2 - 2}{6} + i\right)^3 = \left(\frac{v^5 + v^3 - 2v}{6}\right)^3 .$$

where $v \equiv 2, 4 \pmod 6$.

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Recent work

$$(x+1)^k + (x+2)^k + \cdots + (x+d)^k = y^n.$$

Theorem (M. A. Bennett, V. Patel, S. Siksek)

Let $k = 3$ and $2 \le d \le 50$. Then, any integer solution (x, y, n) must have $n = 2$ or $n = 3$.

$$291^3 + 292^3 + \cdots + 338^3 + 339^3 = 1155^3.$$

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The Result

Theorem (V. Patel, S. Siksek)

Let $k \geq 2$ be an even integer. Let $\mathcal{A}_k$ be the set of integers $d \geq 2$ such that the equation

$$x^k + (x+1)^k + \cdots (x+d-1)^k = y^n, \quad x, y, n \in \mathbb{Z}, \quad n \geq 2$$

has a solution (x, y, n). Then $\mathcal{A}_k$ has natural density zero. In other words we have

$$\lim_{X \to \infty} \frac{\#\{d \in \mathcal{A}_k : d \leq X\}}{X} = 0.$$

The Case $k = 2$

$$(x+1)^2 + (x+2)^2 + \cdots + (x+d)^2 = y^n.$$

$$dx^2 + d(d+1)x + \frac{d(d+1)(2d+1)}{6} = y^n.$$

$$d\left(x^2 + (d+1)x + \frac{(d+1)(2d+1)}{6}\right) = y^n.$$

Idea

Let q be a prime (not 2 or 3) such that $\operatorname{ord}_q(d) = 1$. Suppose that $q \nmid x^2 + (d+1)x + (d+1)(2d+1)/6$. Then we must have $n = 1$.

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The Bernoulli Polynomial!!!

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A reduction modulo q:

$$x^2 + x + 1/6 \not\equiv 0 \pmod{q}.$$

We complete the square and make a sensible change of variables.

$$Y^2 \not\equiv 12 \pmod{q}.$$

Legendre Symbols and a Density!

We want 12 to **NOT** be a square modulo q.

$$Y^2 \not\equiv 12 \pmod{q}.$$

$$\left(\frac{12}{q}\right) = \left(\frac{3}{q}\right) = -1$$

Precisely when $q \equiv 5, 7 \pmod{12}$.

Lemma

Let q be a prime such that $q \equiv 5, 7 \pmod{12}$. Suppose $q \mid d$. Then the equation $(x+1)^2 + (x+2)^2 + \cdots + (x+d)^2 = y^n$ has no integer solutions.

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Theorem (Dirichlet)

Let a and n be coprime integers. Then there exists infinitely many primes, $\{p_i\}$ such that $p_i \equiv a \pmod{n}$. Moreover,

$$\sum p_i^{-1} = \infty.$$

Legendre Symbols and a Density!

The setup:

1. Let $\mathcal{A}$ be a set of positive integers.
2. Define: $\mathcal{A}(X) = \#\{d \in \mathcal{A} : d \leq X\}$ for positive X.
3. Natural Density: $\delta(\mathcal{A}) = \lim_{X \to \infty} \mathcal{A}(X)/X$.
4. Given a prime q, define: $\mathcal{A}^{(q)} = \{d \in \mathcal{A} : \mathrm{ord}_q(d) = 1\}$.

Theorem (Niven)

Let $\{q_i\}$ be a set of primes such that $\delta(\mathcal{A}^{(q_i)}) = 0$ and $\sum q_i^{-1} = \infty$. Then $\delta(\mathcal{A}) = 0$.

Recall: If q is a prime such that $q \equiv 5, 7 \pmod{12}$, then we have no solutions.

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Result for $k = 2$

Proposition

Let $\mathcal{A}_2$ be the set of integers $d \geq 2$ such that the equation

$$(x+1)^2 + (x+2)^2 + \cdots + (x+d)^2 = y^n$$

has a solution (x, y, n). Then $\mathcal{A}_2$ has natural density zero.

Can we extend this result to any exponent k?

Answer: No.

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Bernoulli polynomials and relation to sums of consecutive powers

Definition (Bernoulli Numbers, b_k)

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} b_k \frac{x^k}{k!}.$$

$b_0 = 1, b_1 = -1/2, b_2 = 1/6, b_3 = 0, b_4 = -1/30, b_5 = 0, b_6 = 1/42.$

Lemma

$b_{2k+1} = 0$ *for* $k \geq 1$.

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Definition (Bernoulli Polynomial, B_k)

$$B_k(x) := \sum_{m=0}^{k} \binom{k}{m} b_m x^{k-m}.$$

Lemma

$$x^k + (x+1)^k + \cdots + (x+d-1)^k = \frac{1}{k+1}\left(B_{k+1}(x+d) - B_k(x)\right).$$

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Apply Taylor's Theorem and use $B'_{k+1}(x) = (k+1) \cdot B_k(x)$.

Lemma

Let $q \geq k+3$ be a prime. Let $d \geq 2$. Suppose that $q \mid d$. Then

$$x^k + (x+1)^k + \cdots + (x+(d-1))^k \equiv d \cdot B_k(x) \pmod{q^2}.$$

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Proposition (Criterion)

Let $k \geq 2$. Let $q \geq k+3$ be a prime such that the congruence $B_k(x) \equiv 0 \pmod{q}$ has no solutions. Let d be a positive integer such that $\mathrm{ord}_q(d) = 1$. Then the equation has no solutions. (i.e. $d \notin \mathcal{A}_k$).

Remark: Computationally we checked $k \leq 75{,}000$ and we could always find such a q.

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Recall: Result for $k = 2$

Proposition

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Can we extend this result to any exponent k?
Answer: No.

Odd k: A Complete Disaster

This is one of the few slides where we consider k to be odd!

$$x^k + (x+1)^k + \cdots + (x+(d-1))^k \equiv d \cdot B_k(x) \pmod{q^2}.$$

We want to find a prime q such that $B_k(x) \equiv 0 \pmod{q}$ has no solutions.
However, it is well–known that the odd degree Bernoulli polynomials have linear factors!

$$B_k(x) = x(x-1)(x-1/2)h(x) \equiv 0 \pmod{q}.$$

Hence our criterion fails for every single prime q.

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Relation to Densities?

We need to use Chebotarev's density theorem, which can be seen as "a generalisation of Dirichlet's theorem" on primes in arithmetic progression.

Proposition

Let $k \geq 2$ be even and let G be the Galois group of $B_k(x)$. Then there is an element $\mu \in G$ that acts freely on the roots of $B_k(x)$.

Assuming the proposition, we may then use Chebotarev's density theorem to find a set of primes q_i with positive Dirichlet density such that $\mathrm{Frob}_{q_i} \in G$ is conjugate to μ. Then we can apply Niven's results to deduce our Theorem.

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A Legendre Symbol analogue

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Conjecture

For any even integer k, $B_k(x)$ is irreducible over $\mathbb{Q}$.

Remark: The conjecture implies the Proposition. This then proves our Theorem.

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Tough Stuff

A sketch of an unconditional proof!

Proposition

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Theorem (von Staudt-Clausen)

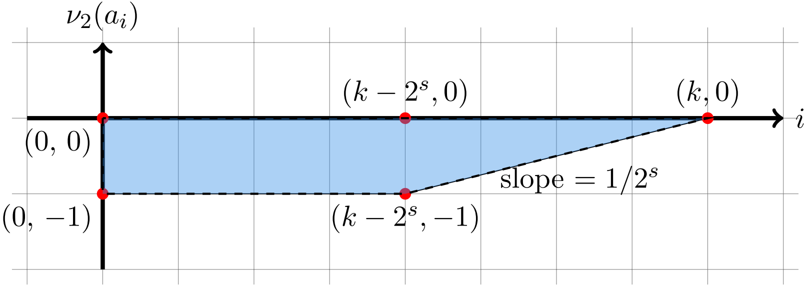
Let $n \geq 2$ be even. Then

$$b_n + \sum_{(p-1)|n} \frac{1}{p} \in \mathbb{Z}.$$

"Bad Prime 2"

The Newton Polygon of $B_k(x)$ for $k = 2^s \cdot t$, $s \geq 1$.

$$B_k(x) = \sum_{i=0}^{k} \binom{k}{k-i} b_{k-i} x^i = \sum_{i=0}^{k} a_i x^i$$



Another nice result

1. Sloping part corresponds to irreducible factor over $\mathbb{Q}_2$.
2. Root in $\mathbb{Q}_2$ must have valuation zero.
3. Root belongs to $\mathbb{Z}_2$ and is odd.
4. Symmetry $(-1)^k B_k(x) = B_k(1-x)$ gives a contradiction.

Theorem (V. Patel, S. Siksek)

Let $k \geq 2$ be an even integer. Then $B_k(x)$ has no roots in $\mathbb{Q}_2$.

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A sketch proof of the Proposition

The Setup:

- $k \geq 2$ is even.
- L is the splitting field of $B_k(x)$.
- G is the Galois group of $B_k(x)$.
- $\mathfrak{P}$ be a prime above 2.
- ν_2 on $\mathbb{Q}_2$ which we extend uniquely to $L_{\mathfrak{P}}$ (also call it ν_2).
- $H = \mathrm{Gal}(L_{\mathfrak{P}}/\mathbb{Q}_2) \subset G$ be the decomposition subgroup corresponding to $\mathfrak{P}$.

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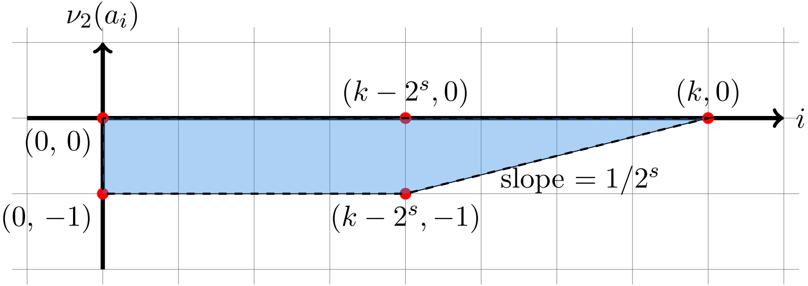
"Bad Prime = Extremely Useful Prime!"

The Newton Polygon of $B_k(x)$ for $k = 2^s \cdot t$, $s \geq 1$.



Finding μ

Lemma

Let H be a finite group acting transitively on a finite set $\{\beta_1, \ldots, \beta_n\}$. Let $H_i \subset H$ be the stabiliser of β_i and suppose $H_1 = H_2$. Let $\pi : H \to C$ be a surjective homomorphism from H onto a cyclic group C. Then there exists some $\mu \in H$ acting freely on $\{\beta_1, \ldots, \beta_n\}$ such that $\pi(\mu)$ is a generator of C.

1. Let $\mathbb{F}_{\mathfrak{P}}$ be the residue field of $\mathfrak{P}$.
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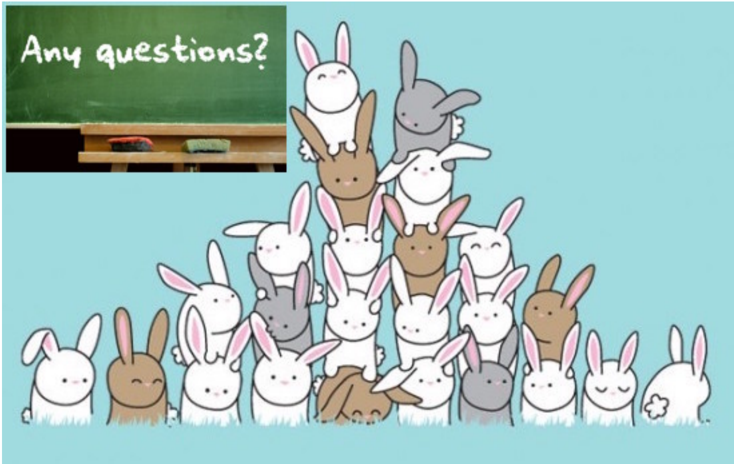
Thank you for Listening!



Solving the equations for $k = 2$

$$d\left(\left(x+\frac{d+1}{2}\right)^2+\frac{(d-1)(d+1)}{12}\right)=y^p.$$

$$X^2 + C \cdot 1^p = (1/d)y^p$$

Solving the equations for $k = 2$

d	Equation	Level	Dimension
6	$2y^p - 5 \times 7 = 3(2x+7)^2$	$2^7 \times 3^2 \times 5 \times 7$	480
11	$11^{p-1}y^p - 2 \times 5 = (x+6)^2$	$2^7 \times 5 \times 11$	160
13	$13^{p-1}y^p - 2 \times 7 = (x+7)^2$	$2^7 \times 7 \times 13$	288
22	$2 \times 11^{p-1}y^p - 7 \times 23 = (2x+23)^2$	$2^7 \times 7 \times 11 \times 23$	5,280
23	$23^{p-1}y^p - 2^2 \times 11 = (x+12)^2$	$2^3 \times 11 \times 23$	54
26	$2 \times 13^{p-1}y^p - 3^2 \times 5^2 = (2x+27)^2$	$2^7 \times 3 \times 5 \times 13$	384
33	$11^{p-1}y^p - 2^4 \times 17 = 3(x+17)^2$	$2^3 \times 3^2 \times 11 \times 17$	200
37	$37^{p-1}y^p - 2 \times 3 \times 19 = (x+19)^2$	$2^7 \times 3 \times 19 \times 37$	5,184
39	$13^{p-1}y^p - 2^2 \times 5 \times 19 = 3(x+20)^2$	$2^3 \times 3^2 \times 5 \times 13 \times 19$	1,080
46	$2 \times 23^{p-1}y^p - 3^2 \times 5 \times 47 = (2x+47)^2$	$2^7 \times 3 \times 5 \times 23 \times 47$	32,384
47	$47^{p-1}y^p - 2^3 \times 23 = (x+24)^2$	$2^5 \times 23 \times 47$	1,012
59	$59^{p-1}y^p - 2 \times 5 \times 29 = (x+30)^2$	$2^7 \times 5 \times 29 \times 59$	25,984

SOLVING THE EQUATIONS FOR $k = 4$

d	Equation	Level	Dimension
5	$y^p + 2 \times 73 = 5(X)^2$	$2^7 \times 5^2 \times 73$	5,472
6	$y^p + 7 \times 53 = 6(X)^2$	$2^8 \times 3^2 \times 7 \times 53$	12,480
7	$7^{p-1}y^p + 2^2 \times 29 = (X)^2$	$2^3 \times 7 \times 29$	42
10	$y^p + 3 \times 11 \times 149 = 10(X)^2$	$2^8 \times 5^2 \times 3 \times 11 \times 149$	449,920
13	$13^{p-1}y^p + 2 \times 7 \times 101 = (X)^2$	$2^7 \times 7 \times 13 \times 101$	28,800
14	$7^{p-1}y^p + 13 \times 293 = 2(X)^2$	$2^8 \times 7 \times 13 \times 293$	168,192
15	$y^p + 2^3 \times 7 \times 673 = 15(X)^2$	$2^5 \times 3^2 \times 5^2 \times 7 \times 673$	383,040
17	$17^{p-1}y^p + 2^3 \times 3 \times 173 = (X)^2$	$2^5 \times 3 \times 17 \times 173$	5,504
19	$19^{p-1}y^p + 2 \times 3 \times 23 \times 47 = (X)^2$	$2^7 \times 3 \times 19 \times 23 \times 47$	145,728
21	$7^{p-1}y^p + 2 \times 11 \times 1321 = 3(X)^2$	$2^7 \times 3^2 \times 7 \times 11 \times 1321$	1,584,000
26	$13^{p-1}y^p + 3^2 \times 5 \times 1013 = 2(X)^2$	$2^8 \times 3 \times 5 \times 13 \times 1013$	777,216
29	$29^{p-1}y^p + 2 \times 7 \times 2521 = (X)^2$	$2^7 \times 7 \times 29 \times 2521$	1,693,440
30	$y^p + 19 \times 29 \times 31 \times 71 = 30(X)^2$	$2^8 \times 3^2 \times 5^2 \times 19 \times 29 \times 31 \times 71$	804,384,000

Where X is a quadratic in the original variable x.