On the Equation $F_n + 2 = y^p$

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joint work with
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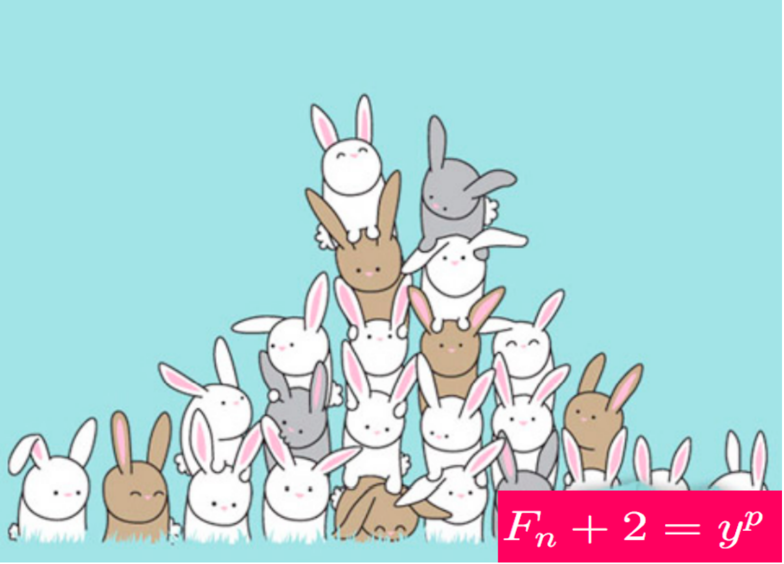
$F_n + 2 = y^p$

First Definitions ...

Definition

The Fibonacci Sequence is defined by the following recurrence relation:

$$F_{n+2} = F_{n+1} + F_n$$

with $F_0 = 0, F_1 = 1$.

The first few terms of the Fibonacci Sequence are:-

F_{-5}	F_{-4}	F_{-3}	F_{-2}	F_{-1}	F_0	F_1	F_2	F_3	F_4	F_5	F_6
					0	1	1	2	3	5	8

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5	-3	2	-1	1	0	1	1	2	3	5	8

Previous Results ...

Theorem (Bugeaud, Mignotte and Siksek)

The only perfect powers of the Fibonacci sequence are, for $n \geq 0$,

$$F_0 = 0, F_1 = F_2 = 1, F_6 = 8 \text{ and } F_{12} = 144.$$

Here we find integer solutions (n, y, p) to the equation $F_n = y^p$.

Previous Results ...

Theorem (Bugeaud, Mignotte and Siksek)

The only non-negative integer solutions (n, y, p) *to* $F_n \pm 1 = y^p$ *are*

$$\begin{aligned} F_0 + 1 &= 0 + 1 &&= 1 \\ F_4 + 1 &= 3 + 1 &&= 2^2 \\ F_6 + 1 &= 8 + 1 &&= 3^2 \\ F_1 - 1 &= 1 - 1 &&= 0 \\ F_2 - 1 &= 1 - 1 &&= 0 \\ F_3 - 1 &= 2 - 1 &&= 1 \\ F_5 - 1 &= 5 - 1 &&= 2^2. \end{aligned}$$

Previous Results ...

We can use the factorisation:

$$
\begin{aligned}
F_{4k}+1 &= F_{2k-1}L_{2k+1} &= y^p\\
F_{4k+1}+1 &= F_{2k+1}L_{2k} &= y^p\\
F_{4k+2}+1 &= F_{2k+2}L_{2k} &= y^p\\
F_{4k+3}+1 &= F_{2k+1}L_{2k+2} &= y^p
\end{aligned}
$$

For n odd,
we do have a factorisation for $F_n + 2 = y^p$.

$F_n = y^p$ ...

Finding the solutions of $F_n = y^p$.

	Method and Steps	Result
1.	Equation	$F_n = y^p$
2.	Associate an Elliptic Curve to it	$E_n := Y^2 = X^3 + L_n X^2 - X$
3.	NewForm Associated to it	NewForm Level 20 (1)
4.	Corresponding Elliptic Curve	$E := Y^2 = X^3 + X^2 - X$
5.	Congruences	Points on $E \mod m$
6.	Lower Bounds for solutions	if $n > 1$ then $n > 10^{9000}$
7.	Upper Bounds for solutions	$n < 10^{9000}$

$$a_m(E_n) \equiv a_m(E) \mod p$$

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An Overview ...

Finding the solutions of $F_n + 2 = y^p$.

	Method and Steps	Result
1.	Equation	$F_n + 2 = y^p$
2.	Associate an Elliptic Curve to it	$E_n := Y^2 = X^3 + 2\mu X^2 + 6X$
3.	NewForm Associated to it	Hilbert Newform
4.	Corresponding Elliptic Curve(s)	?
5.	Congruences	?
6.	Lower Bounds for solutions	?
7.	Upper Bounds for solutions	?

Preliminaries ....

Let $\epsilon = \frac{1+\sqrt{5}}{2}$ and $\bar{\epsilon} = \frac{1-\sqrt{5}}{2}$. By Binet's formula,

$$F_n = \frac{\epsilon^n - \bar{\epsilon}^n}{\sqrt{5}}.$$

$$\begin{aligned} F_n + 2 &= y^p \\ \frac{\epsilon^n - \bar{\epsilon}^n}{\sqrt{5}} + 2 &= y^p \\ \epsilon^{2n} - (\epsilon\bar{\epsilon})^n + 2\epsilon^n\sqrt{5} &= \epsilon^n\sqrt{5}y^p \\ (\epsilon^n - \sqrt{5})^2 - 1 - 5 &= \epsilon^n\sqrt{5}y^p \\ \mu^2 - 6 &= \epsilon^n\sqrt{5}y^p \end{aligned}$$

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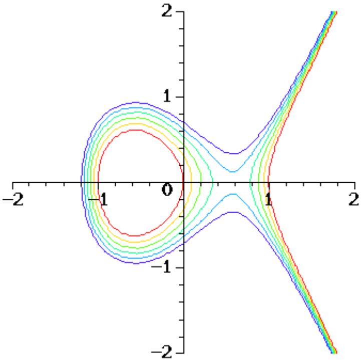
Elliptic Curves ...

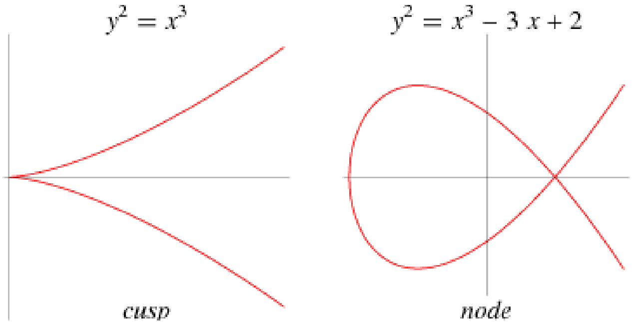


How do we find an elliptic curve?
What is an elliptic curve?

$$Y^2 = X^3 + aX + b$$

where the curve is non-singular (smooth) and $a, b \in \mathbb{R}$.

Elliptic Curves ...



The Frey Curve ...

$$\mu^2 - 6 = \epsilon^n \sqrt{5} y^p, \qquad \mu = \epsilon^n - \sqrt{5}, \qquad \epsilon = (1+\sqrt{5})/2$$

Model	Example
$E : Y^2 = X^3 + AX^2 + BX$	$E_n := Y^2 = X^3 + 2\mu X^2 + 6X$
$\Delta_E = -16 \cdot B^2(A^2 - 4B)$	$\Delta_{E_n} = 2^8 \cdot 3^2 \cdot \epsilon^n \cdot \sqrt{5} \cdot y^p$
$\mathcal{N}_E$ - Tate's Algorithm	$\mathcal{N}_{E_n} = (2)^7 \cdot (3) \cdot (\sqrt{5}) \cdot \prod_{\mathfrak{q} \mid y, \mathfrak{q} \neq (\sqrt{5})} \mathfrak{q}.$

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NewForms ...

Definition

A NewForm lives in a finite dimensional space, namely $S_k(N)$*.*

$$f(z) = q + \sum_{n \geq 2} a_n q^n, \qquad a_n \in \mathbb{C}, \;\; q = e^{2\pi i z}$$

Definition

A Hilbert NewForm is a generalisation of newforms to functions of 2 or more variables.

Ribet's Level Lowering ...

$$E_n := Y^2 = X^3 + 2\mu X^2 + 6X$$

$$\mathcal{N}_{E_n} = (2)^7 \cdot (3) \cdot (\sqrt{5}) \cdot \prod_{\mathfrak{q} | y, \mathfrak{q} \neq (\sqrt{5})} \mathfrak{q}$$

Hilbert Newform that is new with level

$$\mathcal{N} = (2)^7 \cdot (3) \cdot (\sqrt{5}).$$

There are 6144 newforms!!!!

Further Work ...

Finding the solutions of $F_n + 2 = y^p$.

	Method and Steps	Result
1.	Equation	$F_n + 2 = y^p$
2.	Associate an Elliptic Curve to it	$E_n := Y^2 = X^3 + 2\mu X^2 + 6X$
3.	NewForm Associated to it	Hilbert Newform (6144)
4.	Corresponding Elliptic Curve(s)	? E_α
5.	Congruences	? Points mod m on E_α
6.	Lower Bounds for solutions	?
7.	Upper Bounds for solutions	?

$$a_m(E_n) \equiv a_m(E_\alpha) \mod p$$

Thank you for Listening...