CLOSED GEODESICS AND PERIODS OF AUTOMORPHIC FORMS

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Abstract. We study the detailed structure of the distribution of Eichler-Shimura periods of an automorphic form on a compact hyperbolic surface. We show that these periods do not cluster around the asymptotic period over a homology class discovered by Zelditch.

0. Introduction and Results

Let $M = \mathbb{H}^2/\Gamma$ be a compact hyperbolic surface, where Γ is a discrete subgroup of $PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\{\pm I\}$. Such a surface has a countable infinity of closed geodesics, one corresponding to each non-zero conjugacy class in $\Gamma \cong \pi_1 M$. We shall denote a typical prime closed geodesic by γ, its length by $l(\gamma)$, and its homology class by $[\gamma] \in H_1(M,\mathbb{Z})$. We shall say that Γ is symmetric if it is normalized by $\epsilon = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, i.e., if $\epsilon\Gamma\epsilon = \Gamma$.

An interesting and much studied problem is to understand the distribution of the closed geodesics on M. For example, one may study the asymptotic behaviour of the prime geodesic counting function $\pi(T) := \#\{\gamma : l(\gamma) \leq T\}$. In the 1940's Delsarte [4] showed that $\pi(T) \sim e^T/T$, i.e., the ratio of the two sides converge to 1, as $T \to \infty$; since then more precise results have been obtained. A more refined problem is to fix a homology class $\alpha \in H_1(M,\mathbb{Z})$ and to study $\pi(T,\alpha) := \#\{\gamma : l(\gamma) \leq T, [\gamma] = \alpha\}$. It is known that $\pi(T,\alpha) \sim C_0 e^T/T^{\mathfrak{g}+1}$, where $\mathfrak{g} \geq 2$ denotes the genus of M, with an explicit formula for C_0 [11], [16]. A related problem is to count closed geodesics subject to a constraint on the periods $\int_\gamma \omega$, where ω is a harmonic 1-form. This is an example of the type of problem considered in [2], [13], [20]. Here there are additional features depending on whether or not the periods lie in a discrete subgroup of $\mathbb{R}$.

A natural generalization is to consider (holomorphic) m-forms. These correspond exactly to automorphic forms $f : \mathbb{H}^2 \to \mathbb{C}$ of weight $2m$ with respect to Γ. In [22], Shimura introduced a period of a weight $2m$ automorphic form f over a closed geodesic γ, which we shall denote by $r_m(f,\gamma)$. Zelditch showed that f has an "asymptotic period" $\epsilon(f)$ over a fixed homology class α. More precisely, writing $\mathcal{C}(T,\alpha) = \{\gamma : l(\gamma) \leq T, [\gamma] = \alpha\}$,

$$\lim_{T\to\infty} \frac{1}{\pi(T,\alpha)} \sum_{\gamma\in\mathcal{C}(T,\alpha)} r_m(f,\gamma) = \epsilon(f) \tag{1.1}$$

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and $\epsilon(f)$ is independent of the choice of α [23]. (In particular, if m is odd then $\epsilon(f) = 0$.) In this note we show that one cannot do better than averaging in this result, in the sense that the closed geodesics for which $r_m(f,\gamma)$ is close to $\epsilon(f)$ have density zero in the set $\{\gamma : [\gamma] = \alpha\}$. More precisely, we shall prove Theorem 1 below.

For a weight $2m$ automorphic form f, let $T(f)$ denote the closed subgroup of $\mathbb{C}$ generated by the periods $r_m(f,\gamma)$. As we shall see, $T(f)$ spans $\mathbb{C}$.

Theorem 1. *Let f be a non-zero automorphic form of weight $2m$, $m > 1$. Then, for any $\eta \in T(f)$ and $\delta > 0$, we have*

$$\#\{\gamma : l(\gamma) \leq T, [\gamma] = \alpha, |r_m(f,\gamma) - \eta| \leq \delta\} \sim C\frac{e^T}{T^{\mathfrak{g}+2}},$$

where $C > 0$ is a constant independent of α and η.

The most important special cases of this result are when $T(f)$ is a lattice or when $T(f) = \mathbb{C}$. In these cases we can make the slightly more precise statements below. In each case $\beta_f : \mathbb{R}^{2\mathfrak{g}+2} \to \mathbb{R}$ is a certain "thermodynamic" function (depending only on f) which we shall define later.

Special cases.

(1) If $T(f)$ is a lattice in $\mathbb{C}$ then, for any $\eta \in T(f)$, we have

$$\#\{\gamma : l(\gamma) \leq T, [\gamma] = \alpha, r_m(f,\gamma) = \eta\} \sim \frac{|\mathbb{C}/T(f)|}{(2\pi)^{\mathfrak{g}+1}\sqrt{\det \nabla^2\beta_f(0)}}\frac{e^T}{T^{\mathfrak{g}+2}},$$

where $|\mathbb{C}/T(f)|$ denotes the area of a fundamental domain for $T(f)$.

(2) If $T(f) = \mathbb{C}$ then, for any $\eta \in \mathbb{C}$ and $\delta > 0$, we have

$$\#\{\gamma : l(\gamma) \leq T, [\gamma] = \alpha, |r_m(f,\gamma) - \eta| \leq \delta\} \sim \frac{\pi\delta^2}{(2\pi)^{\mathfrak{g}+1}\sqrt{\det \nabla^2\beta_f(0)}}\frac{e^T}{T^{\mathfrak{g}+2}}.$$

Corollary. *For any $\delta > 0$, we have*

$$\lim_{T\to\infty} \frac{1}{\pi(T,\alpha)}\#\{\gamma : l(\gamma) \leq T, [\gamma] = \alpha, |r_m(f,\gamma) - \epsilon(f)| \leq \delta\} = 0,$$

i.e., the closed geodesics with period close to $\epsilon(f)$ have zero density in $\{\gamma : [\gamma] = \alpha\}$.

Remarks.

(i) The restriction to a fixed homology class α in (1.1) is crucial, even though the result is independent of α. Without this restriction, the averages vanish.

(ii) The asymptotic period $\epsilon(f)$ may be interpreted as the trace of an associated pseudodifferential operator acting on harmonic 1-forms [23].

(iii) The period $r_m(f,\gamma)$ can be interpreted as the pairing between certain cohomology and homology theories [10].

With the additional assumption that Γ is symmetric, we may prove the following rather more precise result on the distribution of the real parts of the periods. (Exactly analogous results also hold for the corresponding imaginary parts.)

Theorem 2. *Suppose that Γ is symmetric. Let f be a non-zero automorphic form of weight $2m$, $m > 1$, and let $R(f)$ denote the subgroup of $\mathbb{R}$ generated by the real part of the periods $\Re r_m(f, \gamma)$.*

(i) *If $R(f)$ is a discrete group then for $\alpha \in H_1(M, \mathbb{Z})$ and $\eta \in R(f)$, we have*

$$\#\{\gamma : l(\gamma) \le T, [\gamma] = \alpha, \Re r_m(f, \gamma) = \eta\} \sim C(f) \frac{e^T}{T^{\mathfrak{g}+3/2}},$$

where $C(f) > 0$ is independent of α and η.

(ii) *If $R(f) = \mathbb{R}$ then for $\alpha \in H_1(M, \mathbb{Z})$, $\eta \in \mathbb{R}$ and $\delta > 0$, we have*

$$\#\{\gamma : l(\gamma) \le T, [\gamma] = \alpha, |\Re r_m(f, \gamma) - \eta| \le \delta\} \sim 2\delta C(f) \frac{e^T}{T^{\mathfrak{g}+3/2}},$$

where $C(f) > 0$ is independent of α, η, and δ.

There are two principal approaches to the problem of obtaining asymptotic formulae associated to closed geodesics. The first is based on the Selberg trace formula and non-commutative harmonic analysis. The second, adopted in this paper, is based on a study of the geodesic flow on the unit tangent bundle SM, which is of Anosov type. In this context one may then employ the powerful machinery of hyperbolic dynamics and Thermodynamic Formalism. In particular, there is a well-developed theory, initiated by Lalley, of counting periodic orbits subject to constraints.

In section 1, we shall give the necessary background concerning automorphic forms and periods. In section 2, we discuss the application of dynamical ideas to the study of automorphic forms and orbit counting. In section 3, we apply Lalley's Theorem to study the detailed structure of the distribution of periods of automorphic forms in a fixed homology class.

1. Homology, Forms and Periods

Considered as an abstract group, Γ has a presentation

$$\langle a_1, \dots, a_{\mathfrak{g}}, b_1, \dots, b_{\mathfrak{g}} | \prod_{j=1}^{\mathfrak{g}} a_j b_j a_j^{-1} b_j^{-1} = 1 \rangle.$$

The first homology group $H_1(M, \mathbb{Z})$ of M is isomorphic to $\mathbb{Z}^{2\mathfrak{g}}$. A convenient basis is given by $\mathcal{B} = \{[a_1], \dots, [a_{\mathfrak{g}}], [b_1], \dots, [b_{\mathfrak{g}}]\}$, where $[\cdot] : \Gamma \to H_1(M, \mathbb{Z})$ is the Hurewicz map.

The cohomology group $H^1(M, \mathbb{R})$ may be identified with the space of harmonic 1-forms on M. Let $\{\omega_1, \dots, \omega_{2\mathfrak{g}}\}$ be the basis dual to $\mathcal{B}$. If ω is a harmonic 1-form and γ is a closed curve then it is a classical result that $\int_\gamma \omega = \langle [\omega], [\gamma] \rangle$, where $[\omega]$ denotes the cohomology class defined by ω and where $\langle \cdot, \cdot \rangle$ denotes the Kronecker pairing between $H^1(M, \mathbb{R})$ and $H_1(M, \mathbb{R})$.

The group Γ acts on $\mathbb{H}^2$ by Möbius transformations, i.e., if $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ then

$$gz = \frac{az + b}{cz + d}.$$

A holomorphic function $f : \mathbb{H}^2 \to \mathbb{C}$ is an automorphic form (with respect to Γ) of weight $2m$ if for every $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ we have

$$f(gz) = (cz + d)^{2m} f(z).$$

Since $g'(z) = (cz+d)^{-2}$, we then have that the m-differential $f(z)(dz)^m$ is invariant under Γ and hence defines a holomorphic m-form on M.

Eichler [5] and Shimura [22] showed how to define a period for f over a closed geodesic γ on M. For the moment, let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element of Γ and set $Q_\gamma(z) = cz^2+(d-a)z-b$. Choosing an arbitrary $z_0 \in \mathbb{H}^2$, define

$$r_m(f, \gamma) = c_\gamma \int_{z_0}^{\gamma z_0} f(z) Q_\gamma^{m-1}(z) dz,$$

where the integral is taken over any smooth path joining z_0 and γz_0 and c_γ is a normalization constant. In fact, we take $c_\gamma^{-1} = (-\mathrm{sgn}(\mathrm{trace}(\gamma)))^{m-1}((\mathrm{trace}(\gamma))^2 - 4)^{(m-1)/2}$. (This choice makes $r_m(f, \gamma)$ well-defined for $\gamma \in PSL(2, \mathbb{R})$ rather than $SL(2, \mathbb{R})$.) This will become important when we introduce functions on SM and is the correct scaling to study the distribution of the periods. Then $r_m(f, \gamma)$ is independent of z_0 and, more importantly, depends only on the conjugacy class of γ and hence only on the corresponding closed geodesic (also denoted by γ). An easy calculation shows that $r_m(f, -\gamma) = -r_m(f, \gamma)$ if m is odd and that $r_m(f, -\gamma) = r_m(f, \gamma)$ if m is even. The following result is due to S. Katok and shows that f is uniquely determined by the periods $r_m(f, \gamma)$.

Proposition 1 [9]. *If $r_m(f, \gamma) = 0$ for all closed geodesics γ then $f = 0$.*

Remark. If $m = 1$ then $r_1(f, \gamma)$ is just the usual pairing between the (complex) cohomology class determined by f and the homology class determined by γ.

Let $S_{2m}(\Gamma)$ denote the set of automorphic forms of weight $2m$. This is a finite dimensional complex Hilbert space with respect to the Petersson inner product

$$\langle f, g \rangle = \int_{\mathcal{F}} f(z)\overline{g(z)}(\Im z)^{2m-2} dz,$$

where $\mathcal{F}$ is a fundamental domain for the action of Γ on $\mathbb{H}^2$. For each $\gamma \in \Gamma$, we may construct a function $\Theta_{m,\gamma} \in S_{2m}(\Gamma)$, called the relative Poincaré series, by the following averaging procedure:

$$\Theta_{m,\gamma}(z) = c'_\gamma \sum_{g \in \Gamma_\gamma \backslash \Gamma} Q_\gamma^{-m}(gz),$$

where Γ_γ is the cyclic group generated by γ and c'_γ is a normalizing constant. As before, these forms only depend on the closed geodesic determined by γ. Provided c'_γ is chosen appropriately, they are related to the periods $r_m(\cdot, \gamma)$ by the formula

$$\langle f, \Theta_{m,\gamma} \rangle = r_m(f, \gamma) \tag{1.1}$$

for any $f \in S_{2m}(\Gamma)$ [7]. By Proposition 1, the forms $\Theta_{m,\gamma}$ span $S_{2m}(\Gamma)$ [9]. In particular, this implies that if $f \neq 0$ then the periods $r_m(f, \gamma)$ span $\mathbb{C}$. If Γ is symmetric then the forms $\Theta_{m,\gamma}$ span $S_{2m}(\Gamma)$ as a *real* vector space [10].

Lemma 1. *Suppose that Γ is symmetric. If $\Re r_m(f,\gamma) = 0$ for all closed geodesics γ then $f = 0$.*

Proof. By (1.1), the hypothesis is equivalent to the statement that $\Re\langle f, \Theta_{m,\gamma}\rangle = 0$ for all closed geodesics γ. Since the $\Theta_{m,\gamma}$ span $S_{2m}(\Gamma)$ as a real space, we may write $f = \sum_\gamma a_\gamma \Theta_{m,\gamma}$, with $a_\gamma \in \mathbb{R}$. Hence $\Re\langle f, f\rangle = \sum_\gamma a_\gamma \Re\langle f, \Theta_{m,\gamma}\rangle = 0$. However, $\langle f, f\rangle$ is real, so we have shown that $\langle f, f\rangle = 0$, i.e., $f = 0$.

2. Closed Geodesics and Orbit Counting

Problems concerning the geometry of M can often be studied via the dynamics of the geodesic flow $\phi_t : SM \to SM$, where $SM = PSL(2,\mathbb{R})/\Gamma$ is the unit tangent bundle of M. This is defined by $\phi_t(x\Gamma) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} x\Gamma$ and is a topologically weak-mixing flow of Anosov type with topological entropy equal to 1. There is a natural one-to-one correspondence between closed geodesics on M and periodic orbits of ϕ. with the length of the closed geodesic equal to the period of the orbit. We shall write $\tilde{\gamma}$ for the periodic orbit lying over the closed geodesic γ. The weak-mixing condition can be rephrased as the statement that the set of lengths $l(\gamma)$, with γ a closed geodesic, does not lie in a discrete subgroup of $\mathbb{R}$.

We shall now introduce some concepts associated the the thermodynamic analysis of ϕ; a good reference is [15]. For a Hölder continuous function $\psi : SM \to \mathbb{R}$, we define the pressure

$$P(\psi) = \sup_{\nu \in \mathcal{M}_\phi} \left\{ h_\nu(\phi) + \int \psi d\nu \right\},$$

where $\mathcal{M}_\phi$ denotes the set of ϕ-invariant probability measures on SM. The supremum is attained by a unique probability measure, called the equilibrium state of ψ. If $\psi_1, \dots, \psi_n : SM \to \mathbb{R}$ are Hölder continuous then the map $\mathbb{R}^n \to \mathbb{R} : (x_1, \dots, x_n) \mapsto P(\sum_{i=1}^n x_i \psi_i)$ is real-analytic unless the group generated by the vectors $\left(\int_{\tilde{\gamma}} \psi_1, \dots, \int_{\tilde{\gamma}} \psi_n\right)$ as $\tilde{\gamma}$ ranges all periodic ϕ-orbits fails to have full rank in $\mathbb{R}^n$.

The equilibrium state of 0 is called the measure of maximal entropy; we shall denote it by μ. In the present setting, μ is equal to the Liouville measure on SM: locally it is the product of the normalized area on M and $1/2\pi \times$ arc-length on the fibres. For any continuous $\psi : SM \to \mathbb{R}$, we have

$$\int \psi \, d\mu = \lim_{T\to\infty} \frac{1}{\pi(T)} \sum_{l(\gamma) \le T} \frac{1}{l(\gamma)} \int_{\tilde{\gamma}} \psi. \tag{2.1}$$

Dynamical methods have been applied in the context of automorphic forms by S. Katok [9]. An automorphic form $f : \mathbb{H}^2 \to \mathbb{C}$ may be lifted to a Γ-invariant smooth function $\tilde{f} : PSL(2,\mathbb{R}) \to \mathbb{C}$, which in turn defines a smooth function $F : SM \to \mathbb{C}$ [3],[9]. A convenient way to do this is as follows. Equipping $PSL(2,\mathbb{R})$ with local co-ordinates (z, ζ) with $z \in \mathbb{H}^2$ and $\zeta \in \mathbb{C}$ such that $|\zeta| = \Im z$, we may define $\tilde{f}(z,\zeta) = f(z)\zeta^m$ [9]. A useful property of this construction is that $\int_{\tilde{\gamma}} F = r_m(f,\gamma)$, where $\tilde{\gamma}$ is the periodic ϕ-orbit lying over the closed geodesic γ. (Here we have made use of our choice of normalization.) Furthermore, $\int F d\mu = 0$.

We shall now discuss the result of Zelditch concerning the existence of asymptotic periods. We begin be stating a result concerning the counting functions $\pi(T)$ and $\pi(T,\alpha)$. In the statement, $\mathrm{li}(e^T)$ denotes the logarithmic integral $\mathrm{li}(e^T) = \int_2^{e^T} 1/\log u \, du \sim e^T/T$.

Proposition 2.

(i) *There exists $\delta > 0$ such that*

$$\pi(T) = \mathrm{li}(e^T)\left(1 + O\left(e^{-\delta T}\right)\right), \quad \text{as } T \to \infty.$$

(ii) *For any $\alpha \in H_1(M,\mathbb{Z})$ there exists $C_1 = C_1(\alpha)$ such that*

$$\pi(T,\alpha) = \frac{e^T}{T^{\mathfrak{g}+1}}\left(\frac{(\mathfrak{g}-1)^{\mathfrak{g}}}{\mathrm{vol}(\mathcal{J}(M))} + \frac{C_1}{T} + O\left(\frac{1}{T^2}\right)\right), \quad \text{as } T \to \infty,$$

where $\mathcal{J}(M)$ denotes the Jacobian torus of M.

Remark. Part (i) is due to Huber [8] and part (ii) to Phillips and Sarnak [16]. (In fact, the expansions obtained were more precise than those given above.) The first order term for $\pi(T,\alpha)$ was also obtained by Katsuda and Sunada and the formula for C_0 is due to them [11]. More recently, analogous results have also been obtained in the case of variable negative curvature [1], [12], [17], [18].

In [23], Zelditch considered the summatory functions $\sum_{\gamma \in \mathcal{C}(T,\alpha)} r_m(f,\gamma)$, where f is an automorphic form of weight $2m$, and showed the following.

Proposition 3 [23]**.** *Let f be a non-zero automorphic form of weight $2m$. There exists $\epsilon_1(f) \in \mathbb{C}$, such that for any $\alpha \in H_1(M,\mathbb{Z})$*

$$\sum_{\gamma \in \mathcal{C}(T,\alpha)} r_m(f,\gamma) = \frac{e^T}{T^{\mathfrak{g}+2}}\left(\epsilon_1(f) + O\left(\frac{1}{T}\right)\right).$$

In particular,

$$\lim_{T\to\infty} \frac{1}{\pi(T,\alpha)} \sum_{\gamma \in \mathcal{C}(T,\alpha)} r_m(f,\gamma) = \epsilon(f),$$

where $\epsilon(f) = \epsilon_1(f)\mathrm{vol}(\mathcal{J}(M))/(\mathfrak{g}-1)^{\mathfrak{g}}$.

Remark. Part (i) of Proposition 2 generalizes to the statement that, for smooth functions G on SM,

$$\sum_{l(\tilde{\gamma})} \int_{\tilde{\gamma}} G = \left(\int G d\mu\right) e^T + O\left(e^{(1-\epsilon)T}\right),$$

for some $\epsilon > 0$. Since $\int F d\mu = 0$, we immediately obtain that, removing the homological restriction, the average $(1/\pi(T)) \sum_{l(\gamma)\le T} r_m(f,\gamma)$ converges to zero at an exponential rate.

3. Lalley's Theorem and the Distribution of Periods

We shall prove Theorem 1 by applying a result originally due to Lalley [13]. Alternative proofs have also been given by Sharp [20] and Babillot and Ledrappier [2]. It is to the latter paper that we shall refer.

For a family of Hölder continuous functions $\psi_1, \dots, \psi_n : SM \to \mathbb{R}$, let $A(\psi_1, \dots, \psi_n)$ denote the subgroup of $\mathbb{R}^n$ generated by the set

$$\left\{ \left(\int_{\tilde{\gamma}} \psi_1, \dots, \int_{\tilde{\gamma}} \psi_n \right) : \tilde{\gamma} \text{ is a periodic } \phi\text{-orbit} \right\}$$

and let $\widetilde{A}(\psi_1, \dots, \psi_n)$ denote the subgroup of $\mathbb{R}^{n+1}$ generated by the set

$$\left\{ \left(l(\gamma), \int_{\tilde{\gamma}} \psi_1, \dots, \int_{\tilde{\gamma}} \psi_n \right) : \tilde{\gamma} \text{ is a periodic } \phi\text{-orbit} \right\}.$$

For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, write $B(x) = P(\sum_{i=1}^n x_i \psi_i)$ and m_x for the equilibrium state of $\sum_{i=1}^n x_i \psi_i$. In particular, $m_0 = \mu$. The following result is readily obtained from Theorem 1.2 of [2].

Proposition 4 (Lalley's Theorem). *Let $\psi_1, \dots, \psi_n : SM \to \mathbb{R}$ be a Hölder continuous family such that $A(\psi_1, \dots, \psi_n)$ has full rank in $\mathbb{R}^n$ and such that $\widetilde{A}(\psi_1, \dots, \psi_n) = \mathbb{R} \times A(\psi_1, \dots, \psi_n)$. Let $0 \le p \le n$ denote the maximal rank of a lattice subgroup of $A(\psi_1, \dots, \psi_n)$. Then there exists a unique $\xi \in \mathbb{R}^n$ such that $\left(\int \psi_1 dm_\xi, \dots, \int \psi_n dm_\xi\right) = 0$. Furthermore, for any compactly supported function $\Delta : A \to \mathbb{R}$, we have*

$$\sum_{l(\gamma) \le T} \Delta \left(\int_{\tilde{\gamma}} \psi_1, \dots, \int_{\tilde{\gamma}} \psi_n \right) \sim \frac{|E|}{(2\pi)^{n/2} \sqrt{\det \nabla^2 B(\xi)}} \int_A \Delta \; d\mathrm{Haar} \; \frac{e^{h_{m_\xi}(\phi) T}}{T^{1+n/2}},$$

where $|E|$ is the p-dimensional volume of a p-dimensional fundamental domain for $A(\psi_1, \dots, \psi_n)$.

We shall show that the hypotheses of Proposition 4 are satisfied in the present setting. Recall that $\{\omega_1, \dots, \omega_{2\mathfrak{g}}\}$ is a basis for $H^1(M, \mathbb{R})$ consisting of harmonic 1-forms. For each $j = 1, \dots, 2\mathfrak{g}$, define a smooth function $G_j : SM \to \mathbb{R}$ by $G_j(x, v) = \omega_j(v)$. Then, in particular,

$$[\gamma] = \left(\int_{\tilde{\gamma}} G_1, \dots, \int_{\tilde{\gamma}} G_{2\mathfrak{g}} \right).$$

(Here we are using the basis $\mathcal{B}$ to identify $H_1(M, \mathbb{Z})$ with $\mathbb{Z}^{2\mathfrak{g}}$.)

For the subsequent analysis, it is appropriate to regard $\mathbb{C}$ as a 2-dimensional real space. Hence we shall slightly modify our notation and let $T(f)$ denote the subgroup of $\mathbb{R}^2$ generated by the pairs $(\Re r_m(f, \gamma), \Im r_m(f, \gamma))$. Set $A = A(G_1, \dots, G_{2\mathfrak{g}}, \Re F, \Im F)$ and set $\widetilde{A} = \widetilde{A}(G_1, \dots, G_{2\mathfrak{g}}, \Re F, \Im F)$.

Lemma 2. *A has full rank in $\mathbb{R}^{2\mathfrak{g}+2}$, i.e., $A = \mathbb{Z}^{2\mathfrak{g}} \times T(f)$.*

Proof. If A does not have full rank then we can find $a \in \mathbb{R}^{2\mathfrak{g}}$ and $b \in \mathbb{R}^2$, not both zero, such that

$$\langle a, [\gamma]\rangle + \langle b, r_m(f,\gamma)\rangle = 0$$

for all closed geodesics γ. However, by Proposition 3, we have that

$$\lim_{T\to\infty} \frac{1}{\pi(T,\alpha)} \sum_{\gamma\in\mathcal{C}(T,\alpha)} (\langle a, [\gamma]\rangle + \langle b, r_m(f,\gamma)\rangle) = \langle a, \alpha\rangle + \langle b, \epsilon(f)\rangle,$$

so that $\langle a, \alpha\rangle + \langle b, \epsilon(f)\rangle = 0$ for any $\alpha \in H_1(M,\mathbb{Z})$. Since $\epsilon(f)$ is independent of α, we conclude that $a = 0$. However, the periods $r_m(f,\gamma)$ span $\mathbb{C}$ and so we also have that $b = 0$. This proves the result.

Lemma 3. *We have $\widetilde{A} = \mathbb{R} \times A$.*

Proof. The statement is equivalent to the assertion that that if $\tilde{\chi}$ is a character of $\mathbb{R} \times A$ and

$$\tilde{\chi}\left(l(\tilde{\gamma}), \int_{\tilde{\gamma}} G_1, \ldots, \int_{\tilde{\gamma}} G_{2\mathfrak{g}}, \int_{\tilde{\gamma}} \Re F, \int_{\tilde{\gamma}} \Im F\right) = 1 \tag{3.1}$$

for all periodic orbits $\tilde{\gamma}$ then $\tilde{\chi}$ is the trivial character.

Since $\widehat{\mathbb{R} \times A} = \mathbb{R} \times \widehat{\mathbb{Z}^{2\mathfrak{g}}} \times \widehat{T(f)}$, we may write (3.1) in the form

$$e^{itl(\tilde{\gamma})}\chi([\gamma])\rho\left(\int_{\tilde{\gamma}} \Re F, \int_{\tilde{\gamma}} \Im F\right) = 1,$$

where $t \in \mathbb{R}$, $\chi \in \widehat{\mathbb{Z}^{2\mathfrak{g}}}$, and $\rho \in \widehat{T(f)}$. Furthermore, χ takes the form $\chi([\gamma]) = e^{2\pi i\langle[\omega],[\gamma]\rangle}$, for some harmonic 1-form $\omega = \xi_1\omega_1 + \cdots + \xi_{2\mathfrak{g}}\omega_{2\mathfrak{g}}$ and ρ takes the form $e^{i\langle u,(\int_{\tilde{\gamma}}\Re F, \int_{\tilde{\gamma}}\Im F)\rangle}$, for some $u \in \mathbb{R}^2$.

If (3.1) holds for all $\tilde{\gamma}$ then, by a result of Livsic [14] (see also [6]), there exists a function $\psi : SM \to S^1$, continuously differentiable along flow lines, such that

$$\frac{1}{2\pi i}\frac{\psi'}{\psi} = \frac{t}{2\pi} + \sum_{j=1}^{2\mathfrak{g}} \xi_j G_j + \frac{1}{2\pi}\langle u, (\Re F, \Im F)\rangle, \tag{3.2}$$

where

$$\psi'(x) = \lim_{t\to 0} \frac{\psi(\phi_t x) - \psi(x)}{t}$$

(cf. the discussion in [21]). Now, the homotopy class of ψ determines a (Bruschlinsky) cohomology class and

$$\frac{1}{2\pi i}\int \frac{\psi'}{\psi}\, d\mu$$

is the asymptotic cycle associated to μ evaluated on that cohomology class [19]. Hence, integrating (3.2) with respect to μ yields that $t = 0$. Thus (3.1) reduces to

$$\chi([\gamma])\rho\left(\int_{\tilde{\gamma}} \Re F, \int_{\tilde{\gamma}} \Im F\right) = 1$$

for all periodic orbits $\tilde{\gamma}$, i.e., $\chi \cdot \rho$ is trivial on a set which generates $A = \mathbb{Z}^{2\mathfrak{g}} \times T(f)$. Hence $\chi \cdot \rho$ is trivial on A, so we have shown that $\tilde{\chi}$ is trivial on $\widetilde{A}$.

Remark. In the case where m is odd, we may use the identity $r_m(f, -\gamma) = -r_m(f, \gamma)$ to give a simpler argument, avoiding the use of Livsic's Theorem.

Thus the hypotheses of Proposition 4 are satisfied: we shall apply it with the choice $\Delta = \chi_{\{\alpha\}} \cdot \chi_{\{z\,:\,|z-\eta|\leq\delta\}}\big|_A$. To complete the proof of Theorem 1, we note that $\int G_j \, d\mu = 0$, $j = 1, \ldots, 2\mathfrak{g}$, and $\int \Re F d\mu = 0$, $\int \Im F d\mu = 0$, so that $\xi = 0$. Thus the required exponential growth rate is given by $h_\mu(\phi) = 1$. The explicit values for the constants given in the special cases where $T(f)$ is a lattice or $T(f) = \mathbb{C}$ also follow from Proposition 4 if we define $\beta_f : \mathbb{R}^{2\mathfrak{g}+2} \to \mathbb{R}$ by

$$\beta_f(x_1, \ldots, x_{2\mathfrak{g}+2}) = P\left(\sum_{i=1}^{2\mathfrak{g}} x_i G_i + x_{2\mathfrak{g}+1} \Re F + x_{2\mathfrak{g}+2} \Im F\right).$$

Finally, we shall outline the modifications required to prove Theorem 2. In this case we shall apply Lalley's Theorem to the $2\mathfrak{g} + 1$ functions $G_1, \ldots, G_{2\mathfrak{g}}, \Re F : SM \to \mathbb{R}$.

Let $R(f)$ denote the subgroup of $\mathbb{R}$ generated by the real parts of the periods $\Re r_m(f, \gamma)$. Clearly either $R(f) = \mathbb{R}$ or $R(f)$ is a discrete subgroup of $\mathbb{R}$. Set $B = A(G_1, \ldots, G_{2\mathfrak{g}}, \Re F)$ and $\widetilde{B} = A(G_1, \ldots, G_{2\mathfrak{g}}, \Re F)$.

Lemma 4. *B has full rank in $\mathbb{R}^{2\mathfrak{g}+1}$, i.e., $B = \mathbb{Z}^{2\mathfrak{g}} \times R(f)$.*

Proof. If B does not have full rank then we can find $a \in \mathbb{R}^{2\mathfrak{g}}$ and $b \in \mathbb{R}$, not both zero, such that

$$\langle a, [\gamma] \rangle + b \Re r_m(f, \gamma) = 0$$

for all closed geodesics γ. As in the proof of Lemma 2, we can use Proposition 3 to deduce that $\langle a, \alpha \rangle + b \Re \epsilon(f) = 0$ for any $\alpha \in H_1(M, \mathbb{Z})$. Since $\epsilon(f)$ is independent of α, we conclude that $a = 0$. By Lemma 1, the terms $\Re r_m(f, \gamma)$ are not all zero and so we also have that $b = 0$. This shows that B has full rank.

The next result is proved in the same way as Lemma 3.

Lemma 5. *$\widetilde{B} = \mathbb{R} \times \mathbb{Z}^{2\mathfrak{g}} \times R(f)$.*

Theorem 2 now follows from Proposition 4, as before.

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