A CENTRAL LIMIT THEOREM FOR PERIODIC ORBITS OF HYPERBOLIC FLOWS

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Abstract. We consider a counting problem in the setting of hyperbolic dynamics. Let $\phi_t : \Lambda \to \Lambda$ be a weak mixing hyperbolic flow. We count the proportion of prime periodic orbits of ϕ_t, with length less than T, that satisfy an averaging condition related to a Hölder continuous function $f : \Lambda \to \mathbb{R}$. We show, assuming an approximability condition on ϕ, that as $T \to \infty$, we obtain a central limit theorem.

1. Introduction

Let $\phi_t : \Lambda \to \Lambda$ be a hyperbolic flow. By a celebrated result of Ratner [16], a central limit theorem holds for Hölder observables with respect to the equilibrium state of a Hölder continuous function and, in particular, with respect to the measure of maximal entropy μ. More precisely, let $f : \Lambda \to \mathbb{R}$ be a Hölder continuous function and write

$$\sigma_f^2 := \lim_{T\to\infty} \int_\Lambda \left(\int_0^T f(\phi_t(x))dt - T \int f \, d\mu \right)^2 d\mu(x);$$

Ratner showed that if $\sigma_f^2 > 0$ then

$$\mu \left(\left\{ x \in \Lambda : \frac{\int_0^T f(\phi_t x)\, dt - T \int f \, d\mu}{\sqrt{T}} \leq y \right\} \right) \to \frac{1}{\sqrt{2\pi\sigma_f^2}} \int_{-\infty}^y e^{-u^2/2\sigma_f^2} \, du,$$

as $T \to \infty$. Furthermore, she showed that $\sigma_f^2 > 0$ if and only if f is not cohomologous to a constant, where we say that two functions f and g are cohomologous if $f - g = u'$, with $u : \Lambda \to \mathbb{R}$ is continuously differentiable along flow lines and

$$u'(x) := \lim_{t\to 0} \frac{u(\phi_t x) - u(x)}{t}.$$

In this paper we shall be interested in a periodic orbit version of the above result. (We restrict to the case where the flow is weak-mixing. If the flow is not weak-mixing then, after introducing a symbolic model for the dynamics, we may reduce to the case of a constant suspension flow over a subshift of finite type, in which case the desired periodic orbit result follows from section 6 of [3].) First let us introduce some terminology. Let $\mathcal{P}$ denote the set of prime periodic ϕ-orbits. For $\gamma \in \mathcal{P}$, we shall write $l(\gamma)$ for its least period. We then write

$$\mathcal{P}(T) = \{\gamma \in \mathcal{P} : l(\gamma) \leq T\}$$

and, for $\Delta > 0$,

$$\mathcal{P}(T, \Delta) = \{\gamma \in \mathcal{P} : T < l(\gamma) \leq T + \Delta\}.$$

We also write $\pi(T) = \#\mathcal{P}(T)$ and $\pi(T, \Delta) = \#\mathcal{P}(T, \Delta)$.

For a function $f : \Lambda \to \mathbb{R}$, we write

$$l_f(\gamma) = \int_0^{l(\gamma)} f(\phi_t(x)) \, dt,$$

where x is any point on γ, and call this the f-weight of γ. We say that $f : \Lambda \to \mathbb{R}$ has *integer periods* if

$$\{l_f(\gamma) : \gamma \in \mathcal{P}\} \subset \mathbb{Z}$$

and that $f : \Lambda \to \mathbb{R}$ is *flow independent* if, for $a, b \in \mathbb{R}$, $a + bf$ has integer periods only if $a = b = 0$.

The periodic orbits of ϕ_t are equidistributed with respect to the measure of maximal entropy, in the sense that, for any $\Delta > 0$,

$$\lim_{T\to\infty} \frac{1}{\pi(T,\Delta)} \sum_{\gamma\in\mathcal{P}(T,\Delta)} \frac{l_f(\gamma)}{l(\gamma)} = \int f \, d\mu$$

[2], [11], and one can formulate a periodic orbit version of the central limit theorem to quantify deviations from this equidistribution. Such a result was first obtained by Lalley [10] but it only holds under the assumption that f is flow independent, which is strictly stronger than $\sigma_f^2 > 0$. (There is also a C^∞ condition on f in Lalley's work but this is easy to remove.) Furthermore, Lalley obtained his central limit theorem as a consequence of a local limit theorem, the proof of which requires considerable analytic machinery. (See Remark 1.2 below.) It is therefore interesting to obtain a short and direct proof which holds for all Hölder continuous f with $\sigma_f^2 > 0$. This is the purpose of the current paper. Our proof applies whenever ϕ_t is a transitive Anosov flow with stable and unstable foliations which are not jointly integrable or, for general hyperbolic flows, whenever ϕ_t satisfies a mild Diophantine condition on the periods of its periodic orbits. These conditions allow use to apply the work of Dolgopyat [4] to give bounds on iterates of a family of so-called transfer operators and hence extensions and bounds on the complex generating functions we need to study.

Recall that a real number β is Diophantine if there exists $c > 0$ and $\alpha > 1$ such that $|q\beta - p| \geq cq^{-\alpha}$ for all integers p, q with $q > 0$. We say that ϕ_t satisfies the *approximability condition* if it has three closed orbits γ_1, γ_2 and γ_3 such that

$$\frac{l(\gamma_1) - l(\gamma_2)}{l(\gamma_2) - l(\gamma_3)}$$

is Diophantine.

Our main result is the following. By replacing f with $f - \int f \, d\mu$, it is natural to assume that $\int f \, d\mu = 0$.

Theorem 1.1. *Suppose that $\phi_t : \Lambda \to \Lambda$ is either a transitive Anosov flow with stable and unstable foliations which are not jointly integrable or a hyperbolic flow satisfying the approximability condition. Let $f : \Lambda \to \mathbb{R}$ be a Hölder continuous function satisfying $\int f \, d\mu = 0$ that is not a coboundary. Then, for each fixed $\Delta > 0$,*

$$\frac{1}{\pi(T,\Delta)} \# \left\{ \gamma \in \mathcal{P}(T,\Delta) : \frac{l_f(\gamma)}{\sqrt{T}} \leq y \right\} \to \frac{1}{\sqrt{2\pi}\sigma_f} \int_{-\infty}^{y} e^{-t^2/2\sigma_f^2} \, dt,$$

for each $y \in \mathbb{R}$, as $T \to \infty$.

Remarks 1.2. (i) The requirement that the flow has non-jointly integrable stable and unstable foliations and the approximability condition each imply that the flow is weak-mixing.

(ii) It is interesting to note that Lalley's periodic orbit version of the local limit theorem for hyperbolic flows predates the measure version. In fact, Waddington states a measure local limit theorem for hyperbolic flows in [21], based on the ideas in [10] and [17], although his proof contains some technical gaps. (In particular, the passage to functions defined on a one-sided shift on page 459 of [21] needs further justification.) For semiflows satisfying some abstract conditions (which hold, for example, for a suspension semiflow over the map $x \mapsto kx$ mod 1 on the circle

$\mathbb{R}/\mathbb{Z}$, for an integer $k \geq 2$), a local limit limit theorem was obtained by Iwata [9]. Dolgopyat and Nándori recently gave a proof of the local limit theorem using a different approach [5].

(iii) The approximability condition is not robust under perturbation of the flow. However, Field, Melbourne and Török [7] have given conditions which hold for an open dense set of flows. More precisely, if for $r \geq 2$, $\mathcal{A}_r(M)$ denotes the set of C^r Axiom A flows on a compact manifold M then $\mathcal{A}_r(M)$ contains a C^2-open, C^r-dense subset which satisfies the conditions for every non-trivial basic set.

(iv) The same result holds if, instead of restricting our counting to *prime* orbits $\gamma \in \mathcal{P}(T, \Delta)$, we count over all periodic orbits γ that have length $T < l(\gamma) \leq T + \Delta$. This follows since the number of non-prime periodic orbits grows at a slower rate than the number of prime orbits. More precisely, the exponential growth rate of the prime periodic orbits is given by the topological entropy, but the exponential growth rates for non-prime periodic orbits is half this.

In the next section, we define hyperbolic and Anosov flows and discuss some of their basic properties, including the information we will need about entropy and pressure. In section 3, we mention how our central limit theorem will follow from the pointwise convergence of a family of Fourier transforms and introduce a dynamical L-function whose analytic properties will be key for our analysis. The work of Dolgopyat [4] is crucial here. In section 4, we carry out some calculations using contour integration to obtain an asymptotic formula for a summatory function related to one we require but containing extra terms. In section 5, we remove these extra terms and complete the proof of Theorem 1.1.

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2. Hyperbolic flows and their periodic orbits

We begin with the definition of a hyperbolic flow. Let $\phi_t : M \to M$ be a C^1 flow on a smooth manifold M and let $\Lambda \subset M$ be a compact flow invariant subset. We say that $\phi_t : \Lambda \to \Lambda$ is a hyperbolic flow if the following conditions are satisfied.

(1) There is a splitting of the tangent bundle $T_\Lambda M = E^0 \oplus E^s \oplus E^u$ such that
   (a) there exist $C, \lambda > 0$ with $\|D\phi_t|_{E^s}\|, \|D\phi_{-t}|_{E^u}\| \leq Ce^{-\lambda t}$, for $t \geq 0$,
   (b) E^0 is one-dimensional and tangent to the flow.
(2) The periodic orbits of Λ are dense and Λ is not a single orbit.
(3) Λ contains a dense orbit.
(4) There exists an open set $U \supset \Lambda$ such that $\Lambda = \bigcap_{t=-\infty}^{\infty} \phi_t(U)$.

If (1) holds with $\Lambda = M$ then we say that $\phi_t : M \to M$ is an Anosov flow. In this case (2) is automatically satisfied (this is the Anosov closing lemma [1]) and (4) is trivially satisfied. If, in addition, (3) holds then we say that ϕ_t is a transitive Anosov flow.

We say that ϕ_t is topologically weak-mixing if it does not admit a non-trivial eigenfrequency corresponding to a continuous function, i.e if the only $G \in C(\Lambda, \mathbb{C})$ and $a \in \mathbb{R}$ such that $G \circ \phi_t = e^{iat}G$ for all $t \in \mathbb{R}$ are the constant functions and $a = 0$. It is known that ϕ_t is topologically weak-mixing if and only if $\{l(\gamma) : \gamma \in \mathcal{P}\}$ does not lie in a discrete subgroup of $\mathbb{R}$.

Proposition 2.1 ([14]). *Under the hypotheses of Theorem 1.1, there exists $\eta > 0$ such that*

(i)
$$\pi(T) = \frac{e^{hT}}{hT}\left(1 + O\left(\frac{1}{T^\eta}\right)\right),$$

(ii)
$$\sum_{l(\gamma)\le T} l(\gamma) = \frac{e^{hT}}{h}\left(1 + O\left(\frac{1}{T^\eta}\right)\right).$$

We now recall some of the thermodynamic formalism associated to the flow ϕ_t. This is standard material which may be found in, for example, [18]. Let $\mathcal{M}(\phi)$ denote the set of ϕ_t-invariant Borel probability measures on M. We define the pressure of a Hölder continuous function $f : M \to \mathbb{R}$ to be
$$P(f) = \sup_{m\in\mathcal{M}(\phi)} \left\{h_m(\phi) + \int f\, dm\right\},$$
where $h_m(\phi)$ denotes the entropy of ϕ_t with respect to m. The supremum is attained for a unique measure $m_f \in \mathcal{M}(\phi)$, which we call the equilibrium state of f. When $f = 0$, we call m_0 the measure of maximal entropy for ϕ_t and write $\mu = m_0$. We have $P(0) = h$, the topological entropy of ϕ. For $s \in \mathbb{R}$, the function $s \mapsto P(sf)$ is real analytic, furthermore
$$\left.\frac{dP(sf)}{ds}\right|_{t=0} = \int f\, d\mu$$
and
$$\left.\frac{d^2P(sf)}{ds^2}\right|_{t=0} = \sigma_f^2.$$
Recall that $\sigma_f^2 > 0$ unless f is cohomologous to a constant. We may also extend $P(sf)$ to an analytic function for complex values of s in a sufficiently small neighbourhood of the real line. In particular, $s(t) := P(itf)$ is defined and real analytic for $|t| < \delta$, for some $\delta > 0$.

The following lemma is a consequence the above discussion.

Lemma 2.2. *If $\int f\, d\mu = 0$ and f is not a coboundary then, for $|t| < \delta$,*
$$s(t) = h - \frac{\sigma_f^2 t^2}{2} + O(t^3),$$
with $\sigma_f^2 > 0$.

A simple calculation then gives
$$\lim_{T\to\infty} e^{(h-s(t/\sqrt{T}))T} = e^{-\sigma_f^2 t^2/2}. \tag{2.1}$$

3. Fourier Transforms and L-functions

Let $f : \Lambda \to \mathbb{R}$ be a Hölder continuous function satisfying $\int f\, d\mu = 0$. By Lévy's Continuity Theorem [6], to prove Theorem 1.1 it is enough to to show that the Fourier transforms of the distributions
$$\frac{1}{\pi(T,\Delta)}\#\left\{\gamma \in \mathcal{P}(T,\Delta) : \frac{l_f(\gamma)}{\sqrt{T}} \le y\right\}$$
converge pointwise to the Fourier transform of the normal distribution $N(0,\sigma_f^2)$. In other words, we need to show that, for all $t \in \mathbb{R}$,
$$\frac{1}{\pi(T,\Delta)} \sum_{\gamma\in\mathcal{P}(T,\Delta)} e^{itl_f(\gamma)/\sqrt{T}} \to e^{-\sigma_f^2 t^2/2},$$
as $T \to \infty$. To do this, we will use the periodic orbit data $l(\gamma)$ and $l_f(\gamma)$ to build a family of dynamical L-functions $L(s,t)$. Here s is a complex variable (associated

to the lengths $l(\gamma)$) and t is a real variable (associated to the f-weights $l_f(\gamma)$). We define

$$L(s,t) = \prod_{\gamma\in\mathcal{P}} \left(1 - e^{-sl(\gamma)+itl_f(\gamma)}\right)^{-1} = \exp\left\{\sum_{\gamma\in\mathcal{P}}\sum_{m=1}^{\infty}\frac{1}{m}e^{-sml(\gamma)+itl_f(\gamma)}\right\}.$$

Then $L(s,t)$ is non-zero and analytic in the region $\mathrm{Re}(s) > h$ and for all $t \in \mathbb{R}$ [12]. In order to prove Theorem 1.1, we will need to extend $L(s,t)$ to a neighbourhood of $\mathrm{Re}(s) = h$.

In fact, it will be convenient to work with the logarithmic derivative (with respect to s) $L'(s,t)/L(s,t)$. Write $\mathcal{Q}$ for a set of all (not necessarily prime) periodic orbits of ϕ_t and, for $\gamma' \in \mathcal{Q}$, if $\gamma' = \gamma^n$, $n \geq 1$, with $\gamma \in \mathcal{P}$, write $\Lambda(\gamma') = l(\gamma)$. Then we have

$$\frac{L'(s,t)}{L(s,t)} = -\sum_{\gamma'\in\mathcal{Q}} \Lambda(\gamma')e^{-sl(\gamma')+itl_f(\gamma')},$$

whenever the series converges.

Our proof relies heavily on the following proposition.

Proposition 3.1. *There exists $C > 0$ and $\epsilon > 0$ such that, for any fixed $t \in (-\delta,\delta)$,*

$$\frac{L'(s,t)}{L(s,t)} + \frac{1}{s - s(t)}$$

is analytic in $\mathrm{Re}(s) > h - C\min\{1, |\mathrm{Im}(s)|^{-\epsilon}\}$. Furthermore, there exists $\beta > 0$, independent of $t \in (-\delta,\delta)$, such that for $\mathrm{Re}(s) > h - C\min\{1, |\mathrm{Im}(s)|^{-\epsilon}\}$,

$$\left|\frac{L'(s,t)}{L(s,t)}\right| = O\left(\max\{|\mathrm{Im}(s)|^{\beta}, 1\}\right).$$

Proof. It is a standard part of the theory of dynamical zeta functions that $L(s,t)$ has a simple pole at $s = s(t)$ and, apart from this pole, is analytic and non-zero for s close to $s(t)$ [12]. Hence, $L'(s,t)/L(s,t)$ has a simple pole at $s = s(t)$ with residue -1.

The extension to a larger domain and the bound rely on the work of Dolgopyat on bounds for iterates of transfer operators [4]. In the case where $t = 0$, the extension and bound were established [14]. For t non-zero but small, one may modify the approach in [4] to get the required results. (See for example [13] where similar calculations are carried out. The recent paper [20], as well as containing important new results, gives a detailed account of the history of this problem.) □

4. Contour Integration

The rest of the proof follows similar lines to the method used in section 2 of [15]. We need the following standard identity (see [8], page 31), which holds for any $d > 0$, $k \geq 1$,

$$\frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} \frac{x^s}{s(s+1)\cdots(s+k)}ds = \begin{cases} 0 & 0 < x < 1 \\ \frac{1}{k!}(1-1/x)^k & x \geq 1. \end{cases} \tag{4.1}$$

Applying (4.1) term-by-term to $-L'(s,t)/L(s,t)$ gives

$$\sum_{e^{l(\gamma')}\leq x} \Lambda(\gamma')e^{itl_f(\gamma')}(x - e^{l(\gamma')})^k = \frac{k!}{2\pi i}\int_{d-i\infty}^{d+i\infty}\left(-\frac{L'(s,t)}{L(s,t)}\right)\frac{x^{s+k}}{s(s+1)\cdots(s+k)}ds,$$

where γ' runs over the elements of $\mathcal{Q}$.

Lemma 4.1. *For any fixed $t \in (-\delta,\delta)$, there is $k \geq 1$ and $\alpha > 0$ such that*

$$\sum_{e^{l(\gamma')}\leq x} \Lambda(\gamma')e^{itl_f(\gamma')}(x - e^{l(\gamma')})^k = \frac{k!}{s(t)(s(t)+1)\cdots(s(t)+k)}x^{s(t)+k} + O\left(\frac{x^{h+k}}{(\log x)^{\alpha}}\right).$$

The implied constant in the above error term is independent of t.

Proof. We first note that for $s = \varsigma + i\tau$, with $\varsigma > h$, we have the trivial estimate

$$\left|\frac{L'(s,t)}{L(s,t)}\right| \le \left|\frac{L'(\varsigma,0)}{L(\varsigma,0)}\right| = O\left(\frac{1}{\varsigma - h}\right). \tag{4.2}$$

We will prove the result using contour integration. Choose numbers $r_1, r_2 > 0$ such that, for $|t| < \delta$, $s(t)$ lies in the interior of the rectangle with vertices at $r_1 + ir_2$, $h + ir_2$, $h - ir_2$ and $r_1 - ir_2$. where $r_1, r_2 > 0$. Furthermore, decreasing δ if necessary, we may assume that this rectangle lies within the region of analyticity described in Proposition 3.1. Set $c = h - CR^{-\epsilon}/2$, $d = h + (\log x)^{-1}$ and $R = (\log x)^{\kappa}$ where $0 < \kappa < 1/\epsilon$. Take $k > \beta$, where β is the same as in Proposition 3.1. By the Residue Theorem we may write

$$\sum_{e^{l(\gamma')} \le x} \Lambda(\gamma') e^{itl_f(\gamma')} (x - e^{l(\gamma')})^k = \frac{k!}{s(t)(s(t)+1)\cdots(s(t)+k)} x^{s(t)+k} + \frac{k!}{2\pi i} \int_\Gamma \left(\frac{L'(s,t)}{L(s,t)}\right) \frac{x^{s+k}}{s(s+1)\cdots(s+k)} ds,$$

where Γ is the contour consisting of the straight lines connecting the points $d + i\infty$, $d + iR$, $c + iR$, $c + ir_2$, $r_1 + ir_2$, $r_1 - ir_2$, $c - ir_2$, $c - iR$, $d - iR$ and $d - i\infty$.

Using (4.2), we have

$$\left|\int_{d\pm iR}^{d\pm i\infty} \frac{L'(s,t)}{L(s,t)} \frac{x^{s+k}}{s(s+1)\cdots(s+k)} ds\right| = O\left(x^{d+k} \int_R^\infty \frac{1}{u^{k+1}} du\right) = O\left(\frac{x^{h+k}}{R^k}\right).$$

Using the bound from Proposition 3.1, we have

$$\left|\int_{c\pm iR}^{d\pm iR} \frac{L'(s,t)}{L(s,t)} \frac{x^{s+k}}{s(s+1)\cdots(s+k)} ds\right| = O\left(R^{\beta-k-1} x^{d+k}\right),$$

$$\left|\int_{c\pm ir_2}^{c\pm iR} \frac{L'(s,t)}{L(s,t)} \frac{x^{s+k}}{s(s+1)\cdots(s+k)} ds\right| = O\left(x^{c+k}\right),$$

$$\left|\int_{c\pm ir_2}^{r_1\pm ir_2} \frac{L'(s,t)}{L(s,t)} \frac{x^{s+k}}{s(s+1)\cdots(s+k)} ds\right| = O\left(x^{c+k}\right),$$

$$\left|\int_{r_1+ir_2}^{r_1-ir_2} \frac{L'(s,t)}{L(s,t)} \frac{x^{s+k}}{s(s+1)\cdots(s+k)} ds\right| = O\left(x^{r_1+k}\right).$$

From our choice of c, d, κ, R and k, we see that $O(x^{h+k}/R^k)$ and $O(R^{\beta-k-1}x^{d+k})$ are $O(x^{h+k}/(\log x)^\alpha)$, for some $\alpha > 0$, while $O(x^{c+k})$ is $O(x^{h+k}a(x))$, where $a(x)$ tends to zero faster than $(\log x)^{-\eta}$, for any $\eta > 0$. The final term has a power saving compared to x^{h+k}. Thus the result follows. □

We claim that the previous lemma holds if we alter the sum so that it is taken over prime orbits. We define

$$S_k(x) = \sum_{e^{l(\gamma)} \le x} \Lambda(\gamma) e^{itl_f(\gamma)} (x - e^{l(\gamma)})^k,$$

where the summation is over $\gamma \in \mathcal{P}$.

Corollary 4.2. *For any fixed $t \in (-\delta, \delta)$, there is $k \ge 1$ and $\alpha > 0$ such that*

$$S_k(x) = \frac{k!}{s(t)(s(t)+1)\cdots(s(t)+k)} x^{s(t)+k} + O\left(\frac{x^{h+k}}{(\log x)^\alpha}\right).$$

The implied constant in the above error term is independent of t.

Proof. Let l_0 be the shortest length of any periodic orbit for ϕ. We write

$$\sum_{e^{l(\gamma')}\le x} \Lambda(\gamma')e^{itl_F(\gamma')}(x-e^{l(\gamma')})^k = S_k(x) + \sum_{n=2}^{\infty}\sum_{e^{nl(\gamma)}\le x} l(\gamma)e^{itnl_f(\gamma)}(x-e^{nl(\gamma)})^k.$$

Note that in the last sum over n in the above expression, the terms are non-zero only for $n \le (\log x)/l_0$. Hence,

$$\sum_{n=2}^{\infty}\sum_{e^{nl(\gamma)}\le x} l(\gamma)e^{itnl_f(\gamma)}(x-e^{nl(\gamma)})^k = O(\log x\cdot x^{h/2}\cdot \log x\cdot x^k) = O(x^{k+h/2}(\log x)^2).$$

This implies the claim. □

5. Auxiliary Calculations

We now wish to remove the terms $(x-e^{l(\gamma)})^k$ from $S_k(x)$. We will first show that the estimate for $S_k(x)$ in Corollary 4.2 implies a similar estimate for $S_{k-1}(x)$, though with the exponent of $\log x$ in the error term reduced.

Decreasing α if necessary, we suppose that $0 < \alpha < 2\eta$, where η is as in Proposition 2.1. Set $\epsilon = (\log x)^{-\alpha/2}$. We will estimate the difference

$$D(x,\epsilon) := S_k(x(1+\epsilon)) - S_k(x)$$

in two ways. Applying Corollary 4.2, we have

$$\begin{aligned} D(x,\epsilon) &= \frac{k!}{s(t)(s(t)+1)\cdots(s(t)+k-1)}x^{s(t)+k}\epsilon + O\left(x^{s(t)+k}\epsilon^2\right) + O\left(\frac{x^{h+k}}{(\log x)^\alpha}\right) \\ &= \frac{k!}{s(t)(s(t)+1)\cdots(s(t)+k-1)}x^{s(t)+k}\epsilon + O\left(\frac{x^{h+k}}{(\log x)^\alpha}\right). \qquad (5.1)\end{aligned}$$

On the other hand, we have

$$\begin{aligned} D(x,\epsilon) = &\sum_{x\le e^{l(\gamma)}\le x(1+\epsilon)} l(\gamma)e^{itl_f(\gamma)}(x-e^{l(\gamma)})^k + kx\epsilon\sum_{e^{l(\gamma)}\le x(1+\epsilon)} l(\gamma)e^{itl_f(\gamma)}(x-e^{l(\gamma)})^{k-1} \\ &+\sum_{j=2}^{k}(x\epsilon)^j\binom{k}{j}\sum_{e^{l(\gamma)}\le x(1+\epsilon)} l(\gamma)e^{itl_f(\gamma)}(x-e^{l(\gamma)})^{k-j}. \qquad (5.2)\end{aligned}$$

Rewriting part (ii) of Proposition 2.1, we have

$$\sum_{e^{l(\gamma)}\le x} l(\gamma) \sim \frac{x^h}{h}$$

and, since $\alpha/2 < \eta$,

$$\sum_{x\le e^{l(\gamma)}\le x(1+\epsilon)} l(\gamma) \sim \epsilon x^h.$$

Hence, the first term on the Right Hand Side of (5.2) is $O(x^{h+k}\epsilon^k)$ and the jth term in the final summation is $O(x^{h+k}\epsilon^j)$. Dividing by $kx\epsilon$ and comparing with (5.1) gives

$$\begin{aligned} S_{k-1}(x) &= \frac{(k-1)!}{s(t)(s(t)+1)\cdots(s(t)+k-1)}x^{s(t)+k-1} + O\left(\frac{x^{h+k-1}}{\epsilon(\log x)^\alpha}, \epsilon x^{h+k-1}\right) \\ &= \frac{(k-1)!}{s(t)(s(t)+1)\cdots(s(t)+k-1)}x^{s(t)+k-1} + O\left(\frac{x^{h+k-1}}{(\log x)^{\alpha/2}}\right). \end{aligned}$$

Proceeding inductively, we obtain the following (where the new value of α is the original α divided by 2^k.)

Lemma 5.1. *For some $\alpha > 0$, we have*

$$\sum_{e^{l(\gamma)} \le x} l(\gamma) e^{itl_F(\gamma)} = \frac{x^{s(t)}}{s(t)} + O\left(\frac{x^h}{(\log x)^\alpha}\right).$$

We note that the constant associated to the above error term is independent of $t \in (-\delta, \delta)$.

6. Proof of Theorem 1.1

We will now complete the proof of Theorem 1.1. A simple calculation using Lemma 5.1 and the limit (2.1) gives the following.

Lemma 6.1. *For any $t \in \mathbb{R}$,*

$$\sum_{l(\gamma) \le T} l(\gamma) e^{itl_f(\gamma)/\sqrt{T}} \sim \frac{e^{s(t/\sqrt{T})T}}{s(t/\sqrt{T})}.$$

This lemma, together with part (ii) of Proposition 2.1 and (2.1) again, implies that

$$\lim_{T \to \infty} \frac{\sum_{l(\gamma) \le T} l(\gamma) e^{itl_f(\gamma)/\sqrt{T}}}{\sum_{l(\gamma) \le T} l(\gamma)} \to e^{-\sigma_f^2 t^2/2},$$

provided we now assume that $\sigma_f^2 > 0$, i.e. that f is not a coboundary.

We now need to remove the terms $l(\gamma)$. From Proposition 2.1, we have that

$$\sum_{l(\gamma) \le T} l(\gamma) \sim T\pi(T), \tag{6.1}$$

as $T \to \infty$. We also have the following.

Lemma 6.2. *For any $t \in \mathbb{R}$,*

$$\sum_{l(\gamma) \le T} e^{itl_f(\gamma)/\sqrt{l(\gamma)}} \sim \frac{1}{T} \sum_{l(\gamma) \le T} l(\gamma) e^{itl_f(\gamma)/\sqrt{T}}.$$

Proof. Let $\varphi(T) = \sum_{l(\gamma) \le T} l(\gamma) e^{itl_f(\gamma)/\sqrt{T}}$. Using the Stiltjes integral, we have that

$$\begin{aligned}
\sum_{l(\gamma) \le T} e^{itl_f(\gamma)/\sqrt{l(\gamma)}} &= \int_1^T \frac{1}{u}\, d\varphi(u) + O(1) \\
&= \left[\frac{\varphi(u)}{u}\right]_1^T + \int_1^T \frac{\varphi(u)}{u^2}\, du + O(1) \\
&= \frac{\varphi(T)}{T} + O(1) + O\left(\int_1^T \frac{e^{hu}}{u^2}\, du\right).
\end{aligned}$$

Integration by parts yields the estimate

$$\int_1^T \frac{e^{hu}}{u^2}\, du = \left[\frac{e^{hu}}{hu^2}\right]_1^T + 2\int_1^T \frac{e^{hu}}{u^3}\, du = O\left(\frac{e^{hT}}{T^2}\right)$$

and the result follows. □

Combining Lemma 6.2 and (6.1) gives us the following.

Proposition 6.3.

$$\lim_{T \to \infty} \frac{1}{\#\pi(T)} \sum_{l(\gamma) \le T} e^{itl_f(\gamma)/\sqrt{l(\gamma)}} = \lim_{T \to \infty} \frac{\sum_{l(\gamma) \le T} l(\gamma) e^{itl_f(\gamma)/\sqrt{T}}}{\sum_{l(\gamma) \le T} l(\gamma)} = e^{-\sigma_f^2 t^2/2}.$$

We now complete the proof of Theorem 1.1.

Proof of Theorem 1.1. First note that it follows from the previous calculations (and $\pi(T,\Delta) = \pi(T+\Delta) - \pi(T)$) that

$$\lim_{T\to\infty} \frac{1}{\pi(T,\Delta)} \sum_{T<l(\gamma)\le T+\Delta} e^{itl_f(\gamma)/\sqrt{l(\gamma)}} = e^{-\sigma_f^2 t^2/2}.$$

Then note that, for a fixed t,

$$\sum_{T<l(\gamma)\le T+\Delta} \left| e^{itl_f(\gamma)/\sqrt{l(\gamma)}} - e^{itl_f(\gamma)/\sqrt{T}} \right| = O\left(\frac{\pi(T,\Delta)}{\sqrt{T}}\right).$$

This gives use the required convergence,

$$\lim_{T\to\infty} \frac{1}{\pi(T,\Delta)} \sum_{T<l(\gamma)\le T+\Delta} e^{itl_f(\gamma)/\sqrt{T}} = e^{-\sigma_f^2 t^2/2}.$$

□

Remark 6.4. In view of Proposition 6.3, Theorem 1.1 may be reformulated as

$$\frac{1}{\pi(T)} \# \left\{ \gamma \in \mathcal{P}(T) : \frac{l_f(\gamma)}{\sqrt{l(\gamma)}} \le y \right\} \to \frac{1}{\sqrt{2\pi}\sigma_f} \int_{-\infty}^{y} e^{-t^2/2\sigma_f^2} dt,$$

for each $y \in \mathbb{R}$, as $T \to \infty$.

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