CORRIGENDUM TO "ANOSOV FLOWS, GROWTH RATES ON COVERS AND GROUP EXTENSIONS OF SUBSHIFTS"

RHIANNON DOUGALL AND RICHARD SHARP

Abstract. This note corrects an error in our paper *Anosov flows, growth rates on covers and group extensions of subshifts*, Inventiones mathematicae 223, 445–483, 2021. This leaves our main results, Theorem 1.1, Corollary 1.2, Theorem 1.3 and Theorem 5.1, unchanged. We also fill a gap in the arguments presented in Section 9; this requires a small modification to the results in this section.

1. Correction to the proof of Theorem 5.1

In [1], the authors mistakenly claim in Lemma 5.2.(i) that

There exists $C > 0$ so that for any (η, g) and (ξ, h) with $T_\psi^k(\eta, g) = (\xi, h)$ we have

$$\nu_{\eta,g}(v) \geq C^k \nu_{\xi,h}(v).$$

This is in error as the proof actually shows the opposite inequality. Most significantly, the inequality was then used in the proof Lemma 5.3 (which is crucial to the main result Theorem 5.1). Less significantly, the proof of Theorem 5.1 is written with ξ, η not restricted to the same cylinder, and the inequality in this case was not justified. Both of these errors are corrected by using transitivity of the system. In the beginning of the paper the authors set up the notation for a skew product with ψ but then proceed to write all the proofs (and the definition of the transfer operator) with respect to a skew product with ψ^{-1} – this easily seen to not change the validity of any of the statements. We give a revision of Lemma 5.2 to take into account the case where ξ, η belong to different cylinders. We give a revision of Lemma 5.3. that does not depend on the mistaken claim. In addition show that the original statement claimed in Lemma 5.2(i) does in fact hold under transitivity (although it is no longer required for the proof of Theorem 5.1).

We remind the reader that we define the skew product T_ψ with right multiplication (i.e. $T_\psi(\eta, g) = (\sigma\eta, g\psi(\eta)))$, and so for the skew product with ψ we have $T_\psi^k(\eta, g) = (\xi, h)$ if and only if $\sigma^k\eta = \xi$ and $g\psi(\eta)\psi(\sigma\eta)\cdots\psi(\sigma^k\eta) = h$. In this way for any letters j_1, j_2 we have $T_\psi^p([j_1] \times \{e\}) \cap [j_2] \times \{h\} \neq \varnothing$ if and only if $T_\psi^p([j_1] \times \{g\}) \cap [j_2] \times \{gh\} \neq \varnothing$. Or more precisely $T_\psi^k(\eta, e) = (\xi, h)$ if and only if $T_\psi^k(\eta, g) = (\xi, gh)$.

Lemma 5.2. (Corrected version)**.** *We have the following:*

(i) There exists $C > 0$ so that for any (η, g) and (ξ, h) with $T_\psi^k(\eta, g) = (\xi, h)$ we have

$$\nu_{\eta,g}(v) \leq C^k \nu_{\xi,h}(v).$$

(ii) For any ξ, η in the same length 1 cylinder we have

$$\nu_{\eta,g}(v) \leq C_f \nu_{\xi,g}(v).$$

Assume in addition that T_ψ is transitive. Then we also have the following:

(iii) There exists $D > 0$ so that for any ξ, η we have

$$\nu_{\xi,g}(v) \leq D\nu_{\eta,g}(v).$$

Proof. Parts (i) and (ii) are the statements proved in [1] but with corrected inequality in the statement of (i).

We show (iii). Using transitivity of T_ψ, there is $r \in \mathbb{N}$ so that for any letters j_1, j_2 there is $p \leq r$ with $T_\psi^p([j_1] \times \{e\}) \cap [j_2] \times \{e\} \neq \varnothing$, whence for any $g \in G$ we have $T_\psi^p([j_1] \times \{g\}) \cap [j_2] \times \{g\} \neq \varnothing$.

Let $\eta \in [a]$ and $\xi \in [b]$. Let $(\zeta, e) \in [a] \times \{g\}$ with $T_\psi^p(\zeta, g) \in [b] \times \{g\}$ and set $T_\psi^p(\zeta, g) = (\zeta', g)$. Then ζ, η are in the same length 1 cylinder and ζ', ξ are in the same length 1 cylinder. Part (i) tells us that $\nu_{\zeta,g}(v) \geq C^p \nu_{\zeta',g}(v)$; and then part (ii) gives

$$\frac{\nu_{\eta,g}(v)}{\nu_{\xi,g}(v)} \geq C_f^{-2} \frac{\nu_{\zeta,g}(v)}{\nu_{\zeta',g}(v)} \geq C_f^{-2} C^p.$$

Noting that p is bounded by r, which is independent of g, gives the result. □

We now give a correct proof of Lemma 5.3. and include the mistaken claim as a consequence.

Lemma 5.3. (Corrected version)**.** *Assume T_ψ is transitive. We have the following:*

(i) For any $a \in G$ there is a constant $M_a < \infty$ so that

$$\sup_{g \in G} \frac{\nu_{o,ga}(v)}{\nu_{o,g}(v)} = M_a.$$

(ii) There exists $L > 0$ so that for any (η, g) and (ξ, h) with $T_\psi^k(\eta, g) = (\xi, h)$ we have

$$\nu_{\eta,g}(v) \geq L^k \nu_{\xi,h}(v).$$

Proof. We begin with part (i). Let $a \in G$ and let $o \in \Sigma^+$. Denote b the first letter of o. so $o \in [b]$. Since T_ψ is transitive there is $(\eta, a) \in [b] \times \{a\}$ and k with $T_\psi^k(\eta, a) \in [b] \times \{e\}$. Then for any $g \in G$ we have $(\eta, ga) \in [b] \times \{ga\}$ and $T_\psi^k(\eta, ga) \in [b] \times \{g\}$. Set $(\xi, g) = T_\psi^k(\eta, ga)$. We use Lemma 5.2.(i) to say that

$$\frac{\nu_{\eta,ga}(v)}{\nu_{\xi,g}(v)} \leq C^k$$

and we use Lemma 5.2.(ii) to say that

$$\frac{\nu_{o,ga}(v)}{\nu_{o,g}(v)} = \frac{\nu_{o,ga}(v)}{\nu_{o,g}(v)} \frac{\nu_{\eta,ga}(v)}{\nu_{o,ga}(v)} \frac{\nu_{o,g}(v)}{\nu_{\xi,g}(v)} \frac{\nu_{\eta,ga}(v)}{\nu_{\xi,g}(v)} \leq C_f^2 C^k.$$

We have deduced part (i).

We now show part (ii). For brevity we will consider ψ as being defined on letters.

Let (η, g) and (ξ, h) with $T_\psi^k(\eta, g) = (\xi, h)$. Let $b_0 \cdots b_k$ be the initial k letters of η. Since $\eta \in [b_0 \cdots b_k]$ we have

$$T_\psi^k(\eta, g) = (\sigma^k \eta, g\psi(b_0)\psi(b_1) \cdots \psi(b_{k-1}))$$

and by hypothesis

$$(\sigma^k \eta, g\psi(b_0)\psi(b_1) \cdots \psi(b_{k-1})) = (\xi, h).$$

Therefore $\xi \in [b_k]$ and $g\psi(b_0)\psi(b_1) \cdots \psi(b_{k-1}) = h$. Upon setting $s_i = \psi(b_i)$ we may write $g = hs_0 \cdots s_{k-1}$. We use Lemma 5.2(iii) to say that $\nu_{\xi,h}(v) \leq D\nu_{\eta,h}(v)$ and then use Lemma 5.2.(ii) to change to some fixed o belonging to the same cylinder as η, giving

$$\frac{\nu_{\eta,g}(v)}{\nu_{\xi,h}(v)} \geq D^{-1} \frac{\nu_{\eta,g}(v)}{\nu_{\eta,h}(v)} \geq C_f^{-2} D^{-1} \frac{\nu_{o,g}(v)}{\nu_{o,h}(v)}.$$

Hence it remains to find a lower bound for $\frac{\nu_{o,g}(v)}{\nu_{o,h}(v)}$. Now we have

$$\frac{\nu_{o,g}(v)}{\nu_{o,h}(v)} = \frac{\nu_{o,g}(v)}{\nu_{o,gs_0\cdots s_{k-1}}(v)} = \frac{\nu_{o,g}(v)}{\nu_{o,gs_0}(v)}\frac{\nu_{o,gs_0}(v)}{\nu_{o,gs_0s_1}(v)}\cdots\frac{\nu_{o,gs_0\cdots s_{k-2}}(v)}{\nu_{o,gs_0\cdots s_{k-1}}(v)} \geq \frac{1}{M_{s_0}M_{s_1}\cdots M_{s_{k-1}}}.$$

As s_i belong to the bounded set of generators $S = \{\psi(B) : |B| = 1\}$ the result follows by setting

$$C_o = \min\left\{C_f^{-2}D^{-1}\frac{1}{M_s} : s \in S\right\}$$

and taking the minimum over the finitely many choices of o (M_s depends on o). $\square$

2. Corrections for Section 9

Section 9 of [1] gives large deviations and and equidistribution theorems for amenable covers. In these results, we consider periodic ϕ-orbits γ satisfying $T < l(\gamma) \leq T+\epsilon$ for arbitrary $\epsilon > 0$. This needs to be modified so that, in general, we only consider sufficiently large ϵ, since for small ϵ we cannot verify the existence of such γ (despite knowing existence for the compact weak-mixing base). More precisely, Definition 9.1 should require that there exists $\epsilon_0 > 0$ such that the convergence holds for all $\epsilon \geq \epsilon_0$ and Theorem 9.6 should also be formulated for such ϵ. Secondly, equation (9.1) of [1] asserts (but does not prove) that (when G is amenable)

$$\lim_{T\to\infty}\frac{1}{T}\log\#\{\gamma \in \mathcal{P}(\phi) : T < l(\gamma) \leq T+\epsilon,\ \langle\gamma\rangle = e\} = P(\langle\xi,\Psi\rangle,\phi).$$

The existence of the limit in (9.1) is crucial for the proof of Theorem 9.6. It is the goal of the remainder of this section to show that the limit exists.

To show the limit exists, rather that continuing directly, we proceed via symbolic dynamics. Let $\sigma : \Sigma^+ \to \Sigma^+$ be a mixing one-sided subshift of finite type with alphabet S and let $\psi : \Sigma^+ \to G$ satisfy $\psi(x) = \psi(x_0, x_1)$, where $x = (x_i)_{i=0}^\infty$. Define $T_\psi : \Sigma^+ \times G \to \Sigma^+ \times G$ by $T_\psi(x,g) = (\sigma x, g\psi(x))$; we assume this skew product is transitive. Let $r : \Sigma^+ \to \mathbb{R}^{>0}$ be Hölder continuous and define $r : \Sigma^+ \times G \to \mathbb{R}^{>0}$ by $r(x,g) = r(x)$. Let

$$\Upsilon := \{(x,g,s) : (x,g) \in \Sigma^+ \times G,\ 0 \leq s \leq r(x,g)\}/\sim,$$

where the equivalence relation $\sim$ is defined by $(x,g,r(x,g)) \sim (T_\psi(x,g),0)$. Let υ^t be the suspension semiflow on Υ, i.e. $\upsilon^t(x,g,s) = (x,g,s+t)$ mod $\sim$. For $a \in S$, set

$$N_a(T,\epsilon) := \#\{(x,e,0) \in \Upsilon : \exists t \in (T,T+\epsilon] \text{ s.t. } \upsilon^t(x,e,0) = (x,e,0),\ x_0 = a\}.$$

Lemma 1. *The limit*

$$\lim_{T\to\infty}\frac{1}{T}\log N_a(T,\epsilon)$$

exists.

This lemma follows from arguments in the proof of Theorem 1.1 of [3], using the following lemma, which appears as Lemma 4.3.1 in [2]. (A similar result for sequences was used by Sarig [4].) In [3], the base system is assumed to be mixing but this is not essential.

Lemma 2. *Let $\alpha : \mathbb{R}^+ \to \mathbb{R}$ be such that there are constants $c_1, c_2 \in \mathbb{R}$ with*

(1) $\alpha(s+t+c_2) + c_1 \geq \alpha(s) + \alpha(t)$, for all $s,t \geq 0$;
(2) for each $T > 0$, $\alpha(t)$ is bounded above on the finite interval $[0,T]$.

Then $\lim_{t\to\infty}\alpha(t)/t$ exists (in $(-\infty,+\infty]$).

Hence Lemma 1 follows from Lemma 2 with $\alpha(t) = \log N_a(t,\epsilon)$. Note that taking $\epsilon \geq \epsilon_0$ ensures that $N_a(t,\epsilon) > 0$ and thus that $\alpha(t)$ is well-defined, at least for large values of t. (Kempton does not state the second condition but it is needed in the proof, and it is clearly satisfied in our application.) Standard arguments then give the existence of the limit

$$\lim_{T\to\infty} \frac{1}{T} \log \#\{\gamma \in \mathcal{P}(\phi_X) : T < l(\gamma) \leq T+\epsilon,\, \gamma \cap W \neq \varnothing\},$$

where W is the image of $[a] \times \{e\} \times (-\eta, \eta)$ in the symbolic coding of ϕ_X. It is also straightforward to compare with orbits in

$$\Pi_X(T,\epsilon) := \#\{\gamma \in \mathcal{P}(\phi) : T < l(\gamma) \leq T+\epsilon,\, \langle\gamma\rangle = e\}.$$

We conclude that for $\epsilon \geq \epsilon_0$,

$$\lim_{T\to\infty} \frac{1}{T} \log \Pi_X(T,\epsilon),$$

exists.

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References

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Department of Mathematical Sciences, Durham University, Upper Mountjoy, Durham DH1 3LE, UK

Email address: rhiannon.dougall@durham.ac.uk

Mathematics Institute, University of Warwick, Coventry CV4 7AL, UK

Email address: R.J.Sharp@warwick.ac.uk