# DISTORTION AND ENTROPY FOR AUTOMORPHISMS OF FREE GROUPS

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Abstract. Recently, several numerical invariants have been introduced to characterize the distortion induced by automorphisms of a free group. We unify these by interpreting them in terms of an entropy function of a kind familiar in thermodynamic ergodic theory. We draw an analogy between this approach and the Manhattan curve associated to a pair of hyperbolic surfaces.

## 0. Introduction

Let $F$ be a free group on $k \geq 2$ generators and let $\mathcal{A} = \{a_1, \ldots, a_k\}$ be a free basis. We define the word length $|\cdot| = |\cdot|_{\mathcal{A}}$ (with respect to $\mathcal{A}$) by $|1| = 0$ and, for $x \neq 1$,

$$|x| = \min\{n : x = x_0 x_1 \cdots x_{n-1},\ x_i \in \mathcal{A} \cup \mathcal{A}^{-1}\},$$

where $\mathcal{A}^{-1} = \{a_1^{-1}, \ldots, a_k^{-1}\}$. Recall that any $x \neq 1$ may be written uniquely as

$$x = x_0 x_1 \cdots x_{n-1},$$

where $n = |x|$, $x_i \in \mathcal{A} \cup \mathcal{A}^{-1}$, $i = 0, \ldots, n-1$, and $x_{i+1} \neq x_i^{-1}$, $i = 0, \ldots, n-2$. We call such an expression a reduced word.

Let $\mathrm{Aut}(F)$ denote the group of automorphisms of $F$. An automorphism $\phi$ is said to be inner if it is a conjugation, i.e., $\phi(x) = y^{-1}xy$, for some $y \in F$. If an automorphism $\phi$ acts by permuting $\mathcal{A} \cup \mathcal{A}^{-1}$ then we call $\phi$ a permutation automorphism. Following the notation of [7], [12], we say that $\phi$ is simple if it is the product of an inner automorphism and a permutation automorphism.

Let $\partial F$ denote the boundary of $F$ in the sense of the theory of hyperbolic groups [5], [6]. This is a Cantor set and may be naturally identified with the one-sided shift space

$$\Sigma^+ = \{(x_n)_{n=0}^{\infty} \in (\mathcal{A} \cup \mathcal{A}^{-1})^{\mathbb{Z}^+} : x_{n+1} \neq x_n^{-1}, n \geq 0\}.$$

There is a dynamical systems associated to this space, namely the shift map $\sigma : \Sigma^+ \to \Sigma^+$, and its ergodic theory will play a key role in this paper.

Define a $\sigma$-invariant Borel probability measure $\mu_0$ on $\Sigma^+$ by

$$\mu_0([x_0, x_1, \ldots, x_{n-1}]) = \frac{1}{2k(2k-1)^{n-1}},$$

where $[x_0, x_1, \ldots, x_{n-1}] = \{(y_n)_{n=0}^{\infty} \in \Sigma^+ : y_i = x_i, i = 0, \ldots, n-1\}$. This measure allows us to describe generic behaviour of sequences in $F$, with respect to the basis $\mathcal{A}$.

In this paper, we shall be interested in various measures of the distortion of $F$ under an automorphism $\phi : F \to F$. The first of these quantifies the generic distortion or stretching. Following Kaimanovich, Kapovich and Schupp [7], define the generic stretching factor $\lambda(\phi)$ (with respect to $\mathcal{A}$) by

$$\begin{aligned}\lambda(\phi) &= \lim_{n\to+\infty} \int \frac{|\phi(x_0 x_1 \cdots x_{n-1})|}{n}\, d\mu_0((x_n)_{n=0}^{\infty}) \\ &= \lim_{n\to+\infty} \frac{|\phi(x_0 x_1 \cdots x_{n-1})|}{n} \qquad \text{for } \mu_0\text{-a.e. } (x_n)_{n=0}^{\infty}.\end{aligned}$$

**Proposition 1** [7]**.** *For any $\phi \in \mathrm{Aut}(F)$, $\lambda(\phi) \geq 1$. Furthermore, $\lambda(\phi) = 1$ if and only if $\phi$ is simple.*

We now make a further definition, which measures the proportion of elements which are not stretched by a factor greater than one. Following Myasnikov and Shpilrain [12], define the curl of $\phi$, $\mathrm{Curl}(\phi)$, by

$$\mathrm{Curl}(\phi) = \limsup_{n\to+\infty} \left( \frac{\#\{x \in F : |x| \leq n,\ |\phi(x)| \leq n\}}{\#\{x \in F : |x| \leq n\}} \right)^{1/n},$$

i.e., the growth rate of the proportion of points in the balls $\{x \in F : |x| \leq n\}$ which remain there under $\phi$.

**Proposition 2** [12]**.** *For any $\phi \in \mathrm{Aut}(F)$, $0 < \mathrm{Curl}(\phi) \leq 1$. Furthermore, $\mathrm{Curl}(\phi) = 1$ if and only if $\phi$ is simple.*

Finally, we define a set introduced by Kapovich, which captures all possible distortions induced by $\phi$. Let $\mathcal{C}(F)$ denote the set of all non-trivial conjugacy classes in $F$ and note that, for $w \in \mathcal{C}(F)$, $\phi(w) \in \mathcal{C}(F)$ is well-defined. Following Kapovich [8], we define the *conjugacy distortion spectrum* of $\phi$ to be the set

$$\mathcal{D}_\phi = \left\{ \frac{|\phi(w)|}{|w|} : w \in \mathcal{C}(F) \right\},$$

where $|w| = \min\{|x| : x \in w\}$, and let $\overline{\mathcal{D}_\phi}$ denote the closure of $\mathcal{D}_\phi$. The structure of $\mathcal{D}_\phi$ was studied by Kapovich, who showed the following.

**Proposition 3** [8]**.** *$\overline{\mathcal{D}_\phi}$ is a closed interval with rational endpoints and $\mathcal{D}_\phi = \overline{\mathcal{D}_\phi} \cap \mathbb{Q}$. If $\phi$ is simple then $\mathcal{D}_\phi = \{1\}$. If $\phi$ is not simple then $\overline{\mathcal{D}_\phi}$ has non-empty interior which contains 1 and $\lambda(\phi)$.*

These quantities may be related by the following theorem.

**Theorem.** *Suppose that $\phi$ is not simple. Then there exists a strictly concave analytic function $\mathfrak{h} : \mathrm{int}(\overline{\mathcal{D}_\phi}) \to \mathbb{R}^+$ such that, for each $\rho \in \mathrm{int}(\overline{\mathcal{D}_\phi})$, $0 < \mathfrak{h}(\rho) \leq \log(2k-1)$ and, if $\rho$ is rational, then*

$$\limsup_{n\to+\infty} \frac{1}{n} \log \# \left\{ w \in \mathcal{C}(F) \, : \, |w| = n, \ \frac{|\phi(w)|}{|w|} = \rho \right\} = \mathfrak{h}(\rho). \tag{0.1}$$

*Furthermore, $\mathfrak{h}(\rho) = \log(2k-1)$ if and only if $\rho = \lambda(\phi)$ and* $\mathrm{Curl}(\phi) = e^{\mathfrak{h}(1)}/(2k-1)$.

This will be proved as Theorems 1, 2 and 3 below.

We shall now outline the contents of the paper. In section 1, we discuss the thermodynamic formalism associated to a class of dynamical systems called subshifts of finite. In section 2, we discuss the subshift associated to a free group, the relationship between periodic orbits and conjugacy classes and how to encode the quantity $|\phi(\cdot)|$ in terms of a function on this subshift. We also introduce the function $\mathfrak{h}$ and relate it to the generic stretch. In section 3, we study the conjugacy distortion spectrum via the periodic points of the shift map, proving equation (0.1). In sections 4 and 5, we show how to obtain the relationship between $\mathrm{Curl}(\phi)$ and $\mathfrak{h}(1)$. In the final section, we recast our results in terms of a "Manhattan curve", analogous to that associated by Burger to a pair of hyperbolic surfaces.

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## 1. Thermodynamic Formalism

In this section, we shall describe the ergodic theory associated to the shift map $\sigma : \Sigma^+ \to \Sigma^+$. The theory we describe goes under the name of thermodynamic formalism; standard references are [13], [14, Appendix II], [23].

We begin with the definition of a subshift of finite type. Let $A$ be finite matrix, indexed by a set $\mathcal{I}$, with entries zero and one. We define the shift space

$$\Sigma_A^+ = \{(x_n)_{n=0}^\infty \in \mathcal{I}^{\mathbb{Z}^+} \, : \, A(x_n, x_{n+1}) = 1 \ \forall n \in \mathbb{Z}^+\}$$

and the (one-sided) subshift of finite type $\sigma : \Sigma_A^+ \to \Sigma_A^+$ by $(\sigma x)_n = x_{n+1}$. We give $\mathcal{I}$ the discrete topology, $\mathcal{I}^{\mathbb{Z}^+}$ the product topology and $\Sigma_A^+$ the subspace topology. A compatible metric is given by

$$d((x_n)_{n=0}^\infty, (y_n)_{n=0}^\infty) = \sum_{n=0}^\infty \frac{1 - \delta_{x_n y_n}}{2^n},$$

where $\delta_{ij}$ is the Kronecker symbol.

We say that $A$ is irreducible if, for each $(i,j) \in \mathcal{I}^2$, there exists $n(i,j) \geq 1$ such that $A^{n(i,j)}(i,j) > 0$ and aperiodic if there exists $n \geq 1$ such that, for each $(i,j) \in \mathcal{I}^2$, $A^n(i,j) > 0$. The latter statement is equivalent to $\sigma : \Sigma_A^+ \to \Sigma_A^+$ being topologically mixing (i.e. that there exists $n \geq 1$ such that for any two non-empty open sets $U, V \subset \Sigma_A^+$, $\sigma^{-m}(U) \cap V \neq \varnothing$, for all $m \geq n$).

If $A$ is aperiodic then it has a positive simple eigenvalue $\beta$ which is strictly maximal in modulus (i.e. every other eigenvalue has modulus strictly less that $\beta$) and the topological entropy $h(\sigma)$ of $\sigma$ is equal to $\log \beta$.

If an ordered $n$-tuple $(x_0, x_1, \ldots, x_{n-1}) \in \mathcal{I}^n$ is such that $A(x_m, x_{m+1}) = 1$, $m = 0, 1, \ldots, n-2$ then we say that $(x_0, x_1, \ldots, x_{n-1})$ is an allowed word of length $n$ in $\Sigma_A^+$; the set of these is denoted $W_A^{(n)}$. If $\sigma^n x = x$ then we say that $\{x, \sigma x, \ldots, \sigma^{n-1}x\}$ is a periodic orbit for $\sigma$. Clearly any such an $x$ is obtained by repeating a word $(x_0, x_1, \ldots, x_{n-1}) \in W_A^{(n)}$ with the additional property that $A(x_{n-1}, x_0) = 1$. Note that we regard the periodic orbits $\{x, \sigma x, \ldots, \sigma^{n-1}x\}$, $\{x, \sigma x, \ldots, \sigma^{n-1}x\}, x, \ldots, \sigma^{n-1}x$, etc., as distinct objects (even though they are identical as point sets). If $\sigma^n x = x$ but $\sigma^m x \neq x$ for $0 < m < n$ then we say that $\{x, \sigma x, \ldots, \sigma^{n-1}x\}$ is a prime periodic orbit.

It is sometimes convenient to replace $\sigma : \Sigma_A^+ \to \Sigma_A^+$ by a shift of finite type on a new space $\Sigma_{A_N}^+$, whose symbols are $W_A^{(N)}$ and for which $\mathbf{x} = (x_0, x_1, \ldots, x_{N-1})$ may be followed by $\mathbf{y} = (y_0, y_1, \ldots, y_{N-1})$ if and only if $y_n = x_{n+1}$, $n = 0, 1, \ldots, N-2$. (Since $\mathbf{x} \in W_A^{(N)}$, the latter condition automatically implies that $A(x_{N-1}, y_{N-1}) = 1$.) We continue to denote the shift map by $\sigma : \Sigma_{A_N}^+ \to \Sigma_{A_N}^+$. More formally, define a $W_A^{(N)} \times W_A^{(N)}$ zero-one matrix $A_N$ by

$$A_N(\mathbf{x}, \mathbf{y}) = \begin{cases} 1 \text{ if } y_n = x_{n+1}, n = 0, \ldots, N-2 \\ 0 \text{ otherwise.} \end{cases}$$

Then

$$\Sigma_{A_N}^+ = \{(\mathbf{x}_n)_{n=0}^{\infty} \in (W_A^{(n)})^{\mathbb{Z}^+} \,:\, A_N(\mathbf{x}_n, \mathbf{x}_{n+1}) \;\forall n \in \mathbb{Z}^+\}.$$

Note that there is a natural period preserving bijection between periodic points for $\sigma : \Sigma_A^+ \to \Sigma_A^+$ and $\sigma : \Sigma_{A_N}^+ \to \Sigma_{A_N}^+$.

Let $\mathcal{M}$ denote the space of all Borel probability measures on $\Sigma_A^+$, equipped with the weak* topology, and let $\mathcal{M}_\sigma$ denote the subspace consisting of $\sigma$-invariant probability measures. For $\mu \in \mathcal{M}_\sigma$, write $h(\mu)$ for the measure theoretic entropy of $\mu$. There is a unique measure $\mu_0 \in \mathcal{M}_\sigma$, called the measure of maximal entropy, for which

$$h(\mu_0) = \sup_{\mu \in \mathcal{M}_\sigma} h(\mu)$$

and this value coincides with the topological entropy $h(\sigma)$.

For a continuous function $f : \Sigma_A^+ \to \mathbb{R}$, we define the pressure $P(f)$ of $f$ by the formula

$$P(f) = \sup_{\mu \in \mathcal{M}_\sigma} \left( h(\mu) + \int f \, d\mu \right) \tag{1.1}$$

and call any measure for which the supremum is attained an equilibrium state for $f$. If $f$ is Hölder continuous (i.e. there exists $\alpha > 0$ and $C(f, \alpha) \geq 0$ such that $|f(x) - f(y)| \leq C(f, \alpha) d(x, y)^\alpha$, for all $x, y \in \Sigma_A^+$), then $f$ has a unique equilibrium state which we denote by $\mu_f$. The latter is fully supported and $h(\mu_f) > 0$. The equilibrium state of the zero function is the measure of maximal entropy, so this is consistent with our earlier notation. The pressure of $f$ also has the following characterization in terms of periodic points:

$$P(f) = \lim_{n \to +\infty} \frac{1}{n} \log \sum_{\sigma^n x = x} e^{f^n(x)}.$$

We say that two functions $f, g : \Sigma_A^+ \to \mathbb{R}$ are (continuously) cohomologous if there is a continuous function $u : \Sigma_A^+ \to \mathbb{R}$ such that $f = g + u \circ \sigma - u$. The cohomology class of a Hölder continuous function is determined by its values around periodic orbits. More precisely, writing $f^n = f + f \circ \sigma + \cdots + f \circ \sigma^{n-1}$, two Hölder continuous functions $f, g : \Sigma_A^+ \to \mathbb{R}$ are cohomologous if and only if $f^n(x) = g^n(x)$ whenever $\sigma^n x = x$.

If $f$ and $g$ are cohomologous then $P(f) = P(g)$ and if $c$ is a real number then $P(f+c) = P(f) + c$. Now suppose that $f$ is Hölder continuous and, for $t \in \mathbb{R}$, consider the function $t \mapsto P(tf)$ This function is convex and real analytic and

$$P'(tf) = \int f \ d\mu_{tf}.$$

Furthermore, if $f$ is not cohomologous to a constant then $P(tf)$ is strictly convex and $P''(tf) > 0$ everywhere. (If $f$ is cohomologous to a constant $c$ then $P(tf) = h(\sigma) + tc$.)

Suppose that a Hölder continuous function $r : \Sigma_A^+ \to \mathbb{R}$ is cohomologous to a strictly positive function. If $\delta > 0$ satisfies $P(-\delta r) = 0$ then, since $\mu_{-\delta r}$ attains the supremum in (1.1), we have the relation

$$\delta = \frac{h(\mu_{-\delta r})}{\int r d\mu_{-\delta r}}. \tag{1.2}$$

For any continuous function $f : \Sigma_A^+ \to \mathbb{R}$, we have

$$I_f := \left\{ \int f \ d\mu \ : \ \mu \in \mathcal{M}_\sigma \right\} = \overline{\left\{ \frac{f^n(x)}{n} \ : \ \sigma^n x = x \right\}}$$

and $I_f$ is a closed interval. If $f$ is Hölder continuous then

$$\text{int}(I_f) = \left\{ \int f \ d\mu_{tf} \ : \ t \in \mathbb{R} \right\}.$$

In particular, if $0 \in \text{int}(I_f)$ then there exists a unique $\xi \in \mathbb{R}$ such that $\int f d\mu_{\xi f} = 0$. Furthermore,

$$P(\xi f) = h(\mu_{\xi f}) = \sup \left\{ h(\mu) \ : \ \mu \in \mathcal{M}_\sigma \text{ and } \int f \ d\mu = 0 \right\}$$

and $\mu_{\xi f}$ is the only measure for which the supremum is realized.

We say that a function $f : \Sigma_A^+ \to \mathbb{R}$ is locally constant if there exists $N \geq 0$ such that if $x = (x_n)_{n=0}^\infty, y = (x_n)_{n=0}^\infty$ have $x_n = y_n$ for all $n \geq N$ then $f(x) = f(y)$. In other words, $f$ may be regarded as a function on $W_A^{(N)}$. Clearly, if $f$ is locally constant then $f$ is Hölder continuous (for any choice of exponent $\alpha > 0$).

We may also regard such a locally constant function as a function $f : \Sigma_{A_N}^+ \to \mathbb{R}$, which depend on only one co-ordinate. As we noted above, there is a natural correspondence between periodic points for $\sigma : \Sigma_A^+ \to \Sigma_A^+$ and $\sigma : \Sigma_{A_N}^+ \to \Sigma_{A_N}^+$. One easily sees that the value of $f^n(x)$ is the same for corresponding periodic orbits for the two shift maps.

If $f$ is locally constant then $P(tf)$ may be given in terms of a matrix. For $t \in \mathbb{R}$, define a $W_A^{(N)} \times W_A^{(N)}$ matrix $A_f(t)$ by

$$A_f(t)(\mathbf{x},\mathbf{y}) = A_N(\mathbf{x},\mathbf{y})e^{tf(\mathbf{x})},$$

where $\mathbf{x} = (x_0, x_1, \ldots, x_{N-1}), \mathbf{y} = (y_0, y_1, \ldots, y_{N-1}) \in W_A^{(N)}$. One can easily see that $A_f(t)$ is non-negative and aperiodic and thus has a simple positive eigenvalue which is strictly greater in modulus that all the other eigenvalues of $A_f(t)$; in fact, this eigenvalue is equal to $e^{P(tf)}$.

## 2. Shifts and Free Groups

In this section we shall consider the shift of finite type $\sigma : \Sigma^+ \to \Sigma^+$ associated to the free group $F$ and free basis $\mathcal{A} = \{a_1, \ldots, a_k\}$. Setting $\mathcal{I} = \mathcal{A} \cup \mathcal{A}^{-1}$, it is clear that $\Sigma^+ = \Sigma_A^+$, where $A(i,j) = 1$ unless $j$ is the inverse of $i$, and that $A$ is aperiodic. We may also think of $\Sigma^+$ as the space of infinite reduced words in $\mathcal{A} \cup \mathcal{A}^{-1}$ and, for a free group, this may be identified with the Gromov boundary of $F$. Furthermore, $W^{(n)} = W_A^{(n)}$ may be identified with the set $\{x \in F : |x| = n\}$.

A simple calculation shows that the topological entropy of $\sigma : \Sigma^+ \to \Sigma^+$ is $h(\sigma) = \log(2k-1)$ and the measure of maximal entropy is the measure $\mu_0$ defined in the introduction.

Recall that $\mathcal{C}(F)$ denotes the set of non-trivial conjugacy classes in $F$. A conjugacy class $w \in \mathcal{C}(F)$ contains a cyclically reduced word in $\mathcal{A} \cup \mathcal{A}^{-1}$, i.e., a reduced word $x_0x_1 \cdots x_{n-1}$ such that $x_{n-1} \neq x_0^{-1}$. The only other cyclically reduced elements of $w$ are obtained from this by cyclic permutation (and also have word length $n$) and non-cyclically reduced elements of $w$ have word length greater than $n$. Therefore it is natural to define the length of $w$ (with respect to $\mathcal{A}$) to be

$$|w| := n = \min_{x \in w} |x|.$$

It is immediate from the definition that, for $m \geq 1$,

$$|w^m| = m|w|,$$

where $w^m$ is the conjugacy class $\{x^m : x \in w\}$. Furthermore, it is clear from the preceding discussion that there is a natural bijection between $\mathcal{C}(F)$ and the set of periodic orbits of $\sigma : \Sigma^+ \to \Sigma^+$ (and between primitive conjugacy classes and prime periodic orbits), such that, if $x, \sigma x, \ldots, \sigma^{n-1}x$ ($\sigma^n x = x$) corresponds to $w \in \mathcal{C}(F)$ then $|w| = n$.

In order to represent elements of $F$ as elements of a shift space, it is convenient to augment $\Sigma^+$ by adding an extra "dummy" symbol 0. Introduce a square matrix $A_0$, with rows indexed by $\mathcal{A} \cup \mathcal{A}^{-1} \cup \{0\}$, such that

$$A_0(i,j) = \begin{cases} A(i,j) & \text{if } i,j \in \mathcal{A} \cup \mathcal{A}^{-1} \\ 1 & \text{if } i \in \mathcal{A} \cup \mathcal{A}^{-1} \cup \{0\} \text{ and } j = 0 \\ 0 & \text{if } i = 0 \text{ and } j \in \mathcal{A} \cup \mathcal{A}^{-1}. \end{cases}$$

We then define $\Sigma_0^+ = \Sigma_{A_0}^+$. In other words, an element of $\Sigma_0^+$ is either an element of $\Sigma^+$ or an element of $W^{(n)}$, for some $n \geq 0$, followed by an infinite string of 0s, which we shall denote by $\dot{0}$. (Of course, this procedure does not introduce any extra periodic points.)

Let $\phi : F \to F$ be an automorphism. We wish to encode the quantity $|\phi(\cdot)|$ in terms of a function $r : \Sigma^+ \to \mathbb{R}$. The following lemma is easily seen.

**Lemma 2.1.** *Let $S$ be any finite subset of $F$. Then there exists an integer $M \geq 1$ such that for any reduced word $y_0 y_1 \cdots y_{m-1}$, $m \geq M$, and any $s \in S$*

$$|s y_0 y_1 \cdots y_{m-1}| - m = |s y_0 y_1 \cdots y_{M-1}| - M.$$

*Proof.* Let $M = \max\{|s| : s \in S\}$ and suppose $m \geq M$. Then $s y_0 y_1 \cdots y_{M-1}$ may be written as a reduced word $s'$ such that $s' y_M \cdots y_{m-1}$ is also a reduced word and, in particular, $|(s y_0 y_1 \cdots y_{M-1}) y_M \cdots y_{m-1}| = |s y_0 y_1 \cdots y_{M-1}| + m - M$. $\square$

Noting that there exists $C > 1$ such that, for all $x \in F$,

$$C^{-1}|x| \leq |\phi(x)| \leq C|x|,$$

the following may be deduced from Lemma 2.1. (It may be compared with Proposition 4 of [16].)

**Lemma 2.2.** *There exists an integer $N \geq 1$ such that if $x_0 x_1 \cdots x_{n-1}$ is a reduced word and $n \geq N$ then*

$$|\phi(x_0 x_1 \cdots x_{n-1})| - |\phi(x_1 \cdots x_{n-1})| = |\phi(x_0 x_1 \cdots x_{N-1})| - |\phi(x_1 \cdots x_{N-1})|.$$

*Proof.* We shall apply Lemma 2.1. Let $S = \{\phi(a_1)^{\pm 1}, \ldots, \phi(a_k)^{\pm 1}\}$ and, as before, $M = \max\{|s| : s \in S\}$. For each $p \geq 1$, we have $|\phi(x_1 \cdots x_p)| \geq C^{-1} p$. Choose $N = [CM] + 1$; then $n \geq N$ implies $|\phi(x_1 \cdots x_{n-1})| \geq C^{-1}[CM] \geq M$, from which the result follows. $\square$

We now introduce a function on $\Sigma_0^+$ which the values $|\phi(x)|$, $x \in F$, may be recovered. Define $r : \Sigma_0^+ \to \mathbb{R}$ by

$$r((x_n)_{n=0}^{\infty}) = |\phi(x_0 \cdots x_{N-1})| - |\phi(x_1 \cdots x_{N-1})|.$$

The following result is immediate from the definition.

**Lemma 2.3.** *Suppose that $x_0 x_1 \cdots x_{n-1}$ is reduced word. Then*

$$|\phi(x_0 x_1 \cdots x_{n-1})| = r^n(x_0, x_1, \cdots, x_{n-1}, \dot{0}).$$

*Remark.* This construction has been used by Lalley [10], Bourdon [2] and Pollicott and Sharp [17],[18]. Apparently, it goes back to Eichler [4].

Clearly, $r$ is a locally constant function. To make this explicit, for $\mathbf{x} = (x_0, x_1, \ldots, x_{N-1}) \in W^{(N)}$, we also write

$$r(\mathbf{x}) = |\phi(x_0 \cdots x_{N-1})| - |\phi(x_1 \cdots x_{N-1})|$$

and, as above, define

$$A_r(t)(\mathbf{x},\mathbf{y}) = A_N(\mathbf{x},\mathbf{y})e^{tr(\mathbf{x})}.$$

Define a function $\mathfrak{p} : \mathbb{R} \to \mathbb{R}$ by $\mathfrak{p}(t) = P(tr)$ and recall that $e^{\mathfrak{p}(t)}$ may be characterized as the (simple) maximal eigenvalue of $A_r(t)$. We know that $\mathfrak{p}(t)$ is a convex real-analytic function of $t$. We define an associated concave function $\mathfrak{h} : \mathrm{int}(I_r) \to \mathbb{R}$ by

$$\mathfrak{h}(\rho) = \sup\left\{ h(\mu) : \mu \in \mathcal{M}_\sigma \text{ and } \int r\, d\mu = \rho \right\}.$$

Recalling that $\mathrm{int}(I_r) = \{\mathfrak{p}'(t) : t \in \mathbb{R}\}$, we have $\mathfrak{h}(\rho) = \mathfrak{p}(\xi) - \xi\rho$, where $\xi \in \mathbb{R}$ is chosen to be the unique value with $\mathfrak{p}'(\xi) = \rho$ and, in particular, $\mathfrak{h}$ is real-analytic. (In the language of convex analysis, $-\mathfrak{h}$ is the Legendre transform of $\mathfrak{p}$ [19].)

The next result relates the generic stretch $\lambda(\phi)$ to the functions $\mathfrak{p}(t)$ and $\mathfrak{h}(\rho)$.

**Theorem 1.** *Suppose that $\phi$ is not simple.*
*(i)*

$$\lambda(\phi) = \int r\, d\mu_0 = \mathfrak{p}'(0).$$

*(ii) For $\rho \in \mathrm{int}(I_r)$, $0 < \mathfrak{h}(\rho) \le 2k-1$ and $\mathfrak{h}(\rho) = 2k-1$ if and only if $\rho = \lambda(\phi)$.*

*Proof.*
(i) For $x = (x_n)_{n=0}^{\infty} \in \Sigma^+$,

$$r^n(x) = r^n(x_0, x_1, \cdots, x_{n-1}, \dot{0}) + O(1).$$

Thus, by Lemma 2.3,

$$\lim_{n\to+\infty} \frac{r^n(x)}{n} = \lim_{n\to+\infty} \frac{|\phi(x_0x_1\cdots x_{n-1})|}{n}$$

(provided the limit exists). By the ergodic theorem, the Left Hand Side exists $\mu_0$-a.e. and the limit is equal to $\int r\, d\mu_0$.

(ii) It is clear that $\mathfrak{h}(\rho) \le 2k-1$. Using the formula $\mathfrak{h}(\rho) = \mathfrak{p}(\xi) - \xi\rho$ and the definition of $\mathfrak{p}$, we see that $\mathfrak{h}(\rho) = h(\mu_{\xi r}) > 0$. By part (i), we have $\mathfrak{h}(\lambda(\phi)) = 2k-1$ while, for $\rho \ne \lambda(\phi)$, $\xi \ne 0$, so $\mathfrak{h}(\rho) < 2k-1$. □

## 3. Conjugacy Distortion Spectrum

The construction in the preceding section is particularly well suited to studying the action of $\phi$ on conjugacy classes. As shown in the next lemma, the length of the resulting conjugacy classes is simply obtained by summing $r$ around the corresponding periodic orbit. Fortunately, there is a well developed theory of periodic orbit sums which we will then able to use.

**Lemma 3.1.** *Let $\sigma^n x = x$ correspond to the conjugacy class $w$ containing the cyclically reduced word $x_0 x_1 \cdots x_{n-1}$. Then*

$$|\phi(w)| = r^n(x).$$

*Proof.* Let $x^{(m)}$ denote the reduced word obtained from the $m$-fold concatenation of $x_0 x_1 \cdots x_{n-1}$. Since $r$ is locally constant, there exists $N \geq 1$ such that

$$|r^{mn}(x) - r^{mn}(x^{(m)}, 0, 0, \dots)| \leq 2N \|r\|_\infty.$$

Noting that $r^{mn}(x) = mr^n(x)$, the above estimate gives us that

$$r^n(x) = \lim_{m \to +\infty} \frac{1}{m} r^{mn}(x^{(m)}, 0, 0, \dots) = \lim_{m \to +\infty} \frac{1}{m} |\phi(x^{(m)})|.$$

Thus it remains to show that this last quantity is equal to $|\phi(w)|$. We shall do this by proving inequalities in both directions.

First observe that $x^{(m)}$ is a cyclically reduced word in $w^m$. Therefore

$$m|\phi(w)| = |\phi(w)^m| = |\phi(w^m)| \leq |\phi(x^{(m)})|$$

and so $|\phi(w)| \leq \lim_{m \to +\infty} m^{-1} |\phi(x^{(m)})|$.

Now suppose that $v \in w$. It is clear that

$$\lim_{m \to +\infty} m^{-1} |\phi(v^m)| = \lim_{m \to +\infty} m^{-1} |\phi(x^{(m)})|.$$

On the other hand, since $\{|\phi(v^m)|\}_{m \geq 1}$ is a subadditive sequence,

$$\lim_{m \to +\infty} m^{-1} |\phi(v^m)| = \inf_{m \geq 1} m^{-1} |\phi(v^m)|$$

and so $\lim_{m \to +\infty} m^{-1} |\phi(x^{(m)})| \leq |\phi(v)|$. Hence

$$\lim_{m \to +\infty} m^{-1} |\phi(v^m)| \leq \inf_{v \in w} |\phi(v)| = |\phi(w)|. \quad \square$$

Suppose that $w \in \mathcal{C}(F)$. If $x, y \in w$ then $\phi(x), \phi(y)$ are conjugate, so the conjugacy class $\phi(w)$ is well defined. As in the introduction, we define the *conjugacy distortion spectrum* of $\phi$ to be the set

$$\mathcal{D}_\phi = \left\{ \frac{|\phi(w)|}{|w|} \, : \, w \in \mathcal{C}(F) \right\}$$

and let $\overline{\mathcal{D}_\phi}$ be the closure of $\mathcal{D}_\phi$. The structure of $\mathcal{D}_\phi$ was studied by Kapovich [8], who showed the following.

**Proposition 3.1** [8]**.**
*(i)* $\overline{\mathcal{D}_\phi}$ *is a closed interval with rational endpoints and* $\mathcal{D}_\phi = \overline{\mathcal{D}_\phi} \cap \mathbb{Q}$*. (In other words, if* $\rho \in \overline{\mathcal{D}_\phi}$ *is rational then there exists* $w \in \mathcal{C}(F)$ *such that* $|\phi(w)| = \rho|w|$*.)*
*(ii) If* $\phi$ *is not simple then* $\overline{\mathcal{D}_\phi}$ *has non-empty interior which contains* $1$ *and* $\lambda(\phi)$*.*

We shall strengthen Kapovich's result by showing that the set conjugacy classes with a given distortion has exponential growth and that the growth rate is given by the function $\mathfrak{h}$.

**Theorem 2.** *Suppose that* $\phi$ *is not simple.*
*(i)* $\overline{\mathcal{D}_\phi} = I_r$*.*
*(ii) If* $\rho \in \overline{\mathcal{D}_\phi}$ *is rational then*

$$\limsup_{n\to+\infty} \frac{1}{n} \log \# \left\{ w \in \mathcal{C}(F) \,:\, |w| = n,\ \frac{|\phi(w)|}{|w|} = \rho \right\} = \mathfrak{h}(\rho).$$

*(iii) If* $\rho \in \operatorname{int}(\overline{\mathcal{D}_\phi})$ *is rational then there exists an integer* $d(\rho) \geq 1$ *and a constant* $C(\rho) > 0$ *such that*

$$\# \left\{ w \in \mathcal{C}(F) \,:\, |w| = d(r)n,\ \frac{|\phi(w)|}{|w|} = \rho \right\} \sim C(\rho) \frac{e^{d(\rho)n\mathfrak{h}(\rho)}}{\sqrt{d(r)n}}, \quad \text{as } n \to +\infty.$$

**Corollary.** *For each* $\rho$*, the set* $\{w \in \mathcal{C}(F) \,:\, |\phi(w)|/|w| = \rho\}$ *has zero density in in* $\mathcal{C}(F)$*, i.e.,*

$$\lim_{n\to+\infty} \frac{\#\{w \in \mathcal{C}(F) \,:\, |w| = n,\ |\phi(w)|/|w| = \rho\}}{\#\{w \in \mathcal{C}(F) \,:\, |w| = n\}} = 0.$$

We shall prove this theorem by rephrasing it in terms of periodic points for the shift map $\sigma : \Sigma^+ \to \Sigma^+$ (or, more precisely, $\sigma : \Sigma^+_{A_N} \to \Sigma^+_{A_N}$). In view of the correspondence between $\mathcal{C}(F)$ and periodic points for $\sigma$ and Lemma 3.1, we have that

$$\left\{ w \in \mathcal{C}(F) : |w| = n,\ \frac{|\phi(w)|}{|w|} = \rho \right\} = \left\{ x \in \mathrm{Fix}_n \,:\, \frac{r^n(x)}{n} = \rho \right\}.$$

The theorem will follow from the main result of [15], stated in the lemma below, once we have checked the hypotheses.

**Lemma 3.2** [15]**.** *Let* $A$ *be an aperiodic zero-one matrix. Suppose that* $f : \Sigma^+_A \to \mathbb{Z}$ *is a locally constant function depending on two coordinates, i.e.,* $f(x) = f(x_0, x_1)$*, satisfying the following conditions:*

(1) *there exists* $\xi \in \mathbb{R}$ *such that*

$$\int f \, d\mu_{\xi f} = 0;$$

(2) $\bigcup_{n\geq 1}\{f^n(x) \,:\, \sigma^n x = x\}$ *is not contained in any proper subgroup of* $\mathbb{Z}$*.*

*Then there exists an integer $d(f) \geq 1$ and a constant $C(f) > 0$ such that*

$$\{x \in Fix_{dn} \,:\, f^{dn}(x) = 0\} \sim C(f) \frac{e^{d(f)nh(\mu_{\xi f})}}{\sqrt{d(f)n}}, \quad \text{as } n \to +\infty.$$

*Remark.* The requirement that $f$ depends on only two coordinates is only a technical simplification and may be easily removed.

*Proof of Theorem 2.* Write $\rho = p/q$ in lowest terms. Note that $r^n(x)/n = \rho$ if and only if $(qr - p)^n(x) = qr^n(x) - np = 0$. Thus we will apply the above result to the function $f_\rho = qr - p$. We shall show that $f_\rho$ satisfies conditions (1) and (2).

First observe that $I_{f_\rho} = qI_r - p$. We know that $p/q \in \mathrm{int}(I_r)$, so $0 \in \mathrm{int}(I_{f_\rho})$. Hence condition (1) is satisfied. Furthermore, as $I_{f_\rho}$ has interior, $f_\rho$ is not cohomologous to a constant (necessarily zero), so $\bigcup_{n\geq 1}\{f^n(x) : \sigma^n x = x\}$ generates a group $a\mathbb{Z}$, where $a \geq 1$ is an integer. Replacing $f_\rho$ by $f_\rho/a$, condition (2) is satisfied. This proves part (iii) and hence part (ii) of Theorem 2. □

*Remark.* One may read off upper and lower bounds on the number of $w \in \mathcal{C}(F)$ with $|w| = n$ and $|\phi(w)|/|w| = \rho$ from the paper [11]. One has

$$\#\left\{w \in \mathcal{C}(F) \,:\, |w| = n,\ \frac{|\phi(w)|}{|w|} = \rho\right\} \leq (n+1)^{2k(2k-1)^N+1} e^{n\mathfrak{h}(\rho)}.$$

The lower bound is more involved. Given $\epsilon > 0$, there exists $\delta > 0$ and $m \in \mathbb{N}$ such that, if $d(\rho)|n$ and $n$ is sufficiently large then

$$\#\left\{w \in \mathcal{C}(F) \,:\, |w| = n,\ \frac{|\phi(w)|}{|w|} = \rho\right\} \geq \delta n^{-m} e^{n(\mathfrak{h}(\rho)-\epsilon)}.$$

## 4. Counting Group Elements

Our aim in the next two sections is to relate the number $\mathrm{Curl}(\phi)$ to the function $\mathfrak{h}$. This is complicated by the fact that $\mathrm{Curl}(\phi)$ is defined in terms of the action of $\phi$ on elements of $F$ rather than conjugacy classes. Group elements do not have such a nice correspondence to elements of $\Sigma^+$ as occurs in the correspondence between conjugacy classes and periodic orbits for $\sigma : \Sigma^+ \to \Sigma^+$. However, if we consider the larger shift space $\Sigma_0^+$ then we may identify $F$ with a set of pre-images of a point under $\sigma : \Sigma_0^+ \to \Sigma_0^+$.

Suppose that $\phi$ is not simple. As in the previous section, write $f = f_1 = r - 1$ and define $\xi$ by

$$h(\mu_{\xi f}) = \mathfrak{h}(1) = \sup\left\{h(\mu) \,:\, \mu \in \mathcal{M}_\sigma \text{ and } \int f d\mu = 0\right\}.$$

We know $f$ is a locally constant function and may be regarded as a function on $W_{A_0}^{(N)}$. To avoid too many subscripts, we shall write $B = A_0$. As in section 1, we write

$$B_f(\xi + it)(\mathbf{x}, \mathbf{y}) = B_N(\mathbf{x}, \mathbf{y}) e^{(\xi+it)f(\mathbf{x})},$$

where $\mathbf{x} = (x_0, x_1, \ldots, x_{N-1}), \mathbf{y} = (y_0, y_1, \ldots, y_{N-1}) \in W_B^{(N)}$. (Note that the argument is now $\xi + it$.) The matrices $A_f(\xi + it)$ are defined similarly and have the same non-zero spectrum as $B_f(\xi + it)$. In particular, $B_f(\xi)$ has a simple maximal eigenvalue equal to $e^{P(\xi f)} = e^{\mathfrak{p}(\xi)-\xi} = e^{\mathfrak{h}(1)}$. For small values of $|t|$, this eigenvalue persists and $B_f(\xi+it)$ has a simple eigenvalue $\beta(t)$, depending analytically on $t$, with $\beta(0) = e^{\mathfrak{h}(1)}$.

A simple calculation shows that

$$\#\{x \in F : |x| = n\} = \sum_{\mathbf{x} \in W_A^{(N)}} \sum_{a \in \mathcal{A} \cup \mathcal{A}^{-1}} B^n(\mathbf{x}, \mathbf{a}), \tag{4.1}$$

where $\mathbf{a} = (a, 0, \ldots, 0)$.

**Lemma 4.1.**

$$\begin{aligned} \#\{x \in F : |\phi(x)| = |x| = n\} &= \sum_{|x|=n} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{(\xi+it)(|\phi(x)|-|x|)}\, dt \\ &= \sum_{\mathbf{x} \in W_A^{(N)}} \sum_{a \in \mathcal{A} \cup \mathcal{A}^{-1}} \frac{1}{2\pi} \int_{-\pi}^{\pi} B_f(\xi + it)(\mathbf{x}, \mathbf{a})\, dt. \end{aligned}$$

*Proof.* By orthogonality,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{it(|\phi(x)|-|x|)}\, dt = \begin{cases} 1 & \text{if} \quad |\phi(x)| = |x| \\ 0 & \text{otherwise.} \end{cases}$$

In the former case, the factor $e^{\xi(|\phi(x)|-|x|)} = 1$, so the first equality holds. The second follows from Lemma 2.3 and (4.1). $\square$

This may be handled as in [22, pp.899-900] (where the roles of $A$ and $B$ are switched) to show the following.

**Proposition 4.1.** *There exists an integer $d \geq 1$ and a real number $C > 0$ such that*

$$\#\{x \in F : |\phi(x)| = |x| = dn\} \sim C \frac{e^{dn\mathfrak{h}(1)}}{\sqrt{n}}, \quad \text{as } n \to +\infty.$$

**Corollary.**

$$\mathrm{Curl}(\phi) \geq \frac{e^{\mathfrak{h}(1)}}{2k-1}.$$

*Proof.* Since

$$\#\{x \in F : |\phi(x)| = |x| = dn\} \leq \#\{x \in F : |x| \leq dn,\ |\phi(x)| \leq dn\},$$

this follows from Proposition 4.1 and the definition of $\mathrm{Curl}(\phi)$. $\square$

## 5. The Formula for Curl($\phi$)

We now establish our formula for Curl($\phi$) by obtaining an upper bound. It is clear that, for each $n \geq 1$ and $0 \leq p \leq 1$,

$$\#\{x \in F : |x| \leq n, \ |\phi(x)| \leq n\} \leq \#\{x \in F : (1-p)|x| + p|\phi(x) \leq n\},$$

so

$$\mathrm{Curl}(\phi) \leq \frac{1}{2k-1} \inf_{0 \leq p \leq 1} \left( \limsup_{n \to +\infty} (\#\{x \in F : (1-p)|x| + p|\phi(x)| \leq n\})^{1/n} \right). \tag{5.1}$$

We shall show that there exists a value of $p$ for which the above lim sup is equal to $\mathfrak{h}(1)$. Combining this with the corollary to Proposition 4.1 will give the desired formula.

For $0 \leq p \leq 1$, write

$$\alpha(p) = \limsup_{n \to +\infty} \frac{1}{n} \#\{x \in F : (1-p)|x| + p|\phi(x)| \leq n\}.$$

Then $\alpha(p)$ is the abscissa of convergence of the Dirichlet series

$$\sum_{x \in F} e^{-t((1-p)|x| + p|\phi(x)|)}.$$

To analyse this series, define locally constant functions $r_p : \Sigma_0^+ \to \mathbb{R}$ by $r_p = (1-p) + pr$ and introduce matrices

$$B_p(t) := B_{r_p}(t) = B_N(\mathbf{x}, \mathbf{y}) e^{t r_p(\mathbf{x})}.$$

Then

$$\sum_{x \in F} e^{-t((1-p)|x| + p|\phi(x)|)} = \sum_{n=0}^{\infty} \sum_{\mathbf{x} \in W_A^{(N)}} \sum_{a \in \mathcal{A} \cup \mathcal{A}^{-1}} B_p^n(-t)(\mathbf{x}, \mathbf{a}),$$

so $\alpha(p)$ is the value of $t$ for which $B_p(-t)$ has spectral radius equal to one. Thus, $\alpha(p)$ is determined by the equation

$$P(-\alpha(p) r_p) = 0,$$

which is equivalent to

$$\sup_{m \in \mathcal{M}_\sigma} \left( h(m) - \alpha(p) \int r_p \, dm \right) = 0$$

or

$$\alpha(p)(1-p) = \sup_{m \in \mathcal{M}_\sigma} \left( h(m) - \alpha(p) p \int r \, dm \right). \tag{5.2}$$

The supremum in (5.2) is uniquely attained at $\mu_{-\alpha(p)pr}$, the equilibrium state of $-\alpha(p)pr$.

**Lemma 5.1.** *There exists* $0 < p^* < 1$ *such that*

$$\int r \; d\mu_{\alpha(p^*)p^*r} = 1.$$

*Proof.* First note that $R(p) := \int r \; d\mu_{\alpha(p)pr}$ depends continuously on $p$. We shall show that $R(0) > 1$ and $R(1) < 1$; hence, by the Intermediate Value Theorem, there will be a value $p^*$ with $0 < p^* < 1$ and $R(p^*) = 1$.

The first inequality is straightforward since

$$R(0) = \int r \; d\mu_0 = \lambda(\phi) > 1.$$

Since $|\phi(x)| = |x|_{\phi^{-1}(\mathcal{A})}$, one has $\alpha(1) = \log(2k-1) = h(\sigma)$. On the other hand, by equation (1.2),

$$\alpha(1) = \frac{h(\mu_{-\alpha(1)r})}{\int r d\mu_{-\alpha(1)r}} = \frac{h(\mu_{-h(\sigma)r})}{R(1)}.$$

Since $r$ is not cohomologous to a constant, $\mu_{-h(\sigma)r} \neq \mu_0$, so $h(\mu_{-h(\sigma)r}) < h(\sigma)$. Consequently, $R(1) < 1$. □

Consider (5.2) for the value $p^*$. Since the supremum is attained at $\mu_{-\alpha(p^*)p^*r}$ and $\int r d\mu_{-\alpha(p^*)p^*r} = 1$, we may rewrite (5.2) as

$$\begin{aligned}
\alpha(p^*)(1-p^*) &= \sup\left\{h(m) - \alpha(p^*)p^* \int r \; dm \, : \, m \in \mathcal{M}_\sigma, \; \int r \; dm = 1\right\} \\
&= \sup\left\{h(m) \, : \, m \in \mathcal{M}_\sigma, \; \int r \; dm = 1\right\} - \alpha(p^*)p^* \\
&= \mathfrak{h}(1) - \alpha(p^*)p^*.
\end{aligned}$$

The terms $-\alpha(p^*)p^*$ cancel on each side to give

$$\alpha(p^*) = \mathfrak{h}(1).$$

Substituting this in inequality (5.1) gives

$$\mathrm{Curl}(\phi) \leq \frac{e^{\mathfrak{h}(1)}}{2k-1}.$$

Combining this with the corollary to Proposition 4.1 gives the following theorem.

**Theorem 3.** *If* $\phi$ *is not simple then*

$$\mathrm{Curl}(\phi) = \frac{e^{\mathfrak{h}(1)}}{2k-1}.$$

## 6. A Manhattan Curve

In this final section, we draw a parallel between the functions introduced in this paper and Burger's "Manhattan Curve", associated to a pair of hyperbolic Riemann surfaces [3].

We recall Burger's definition. Let $\Sigma_1$ and $\Sigma_2$ be two compact hyperbolic Riemann surfaces, which are assumed to be homeomorphic, and let $\psi : \Sigma_1 \to \Sigma_2$ be a homeomorphism between them. In particular, $\psi$ allow us to identify free homotopy classes on the surfaces. For a free homotopy class $\gamma$, we write $l_1(\gamma)$ and $l_2(\gamma)$, respectively, for the lengths of the corresponding closed geodesics on the two surfaces.

The Manhattan curve $\mathfrak{M}(\Sigma_1, \Sigma_2)$ is defined to be the boundary of the convex set

$$\left\{ (a,b) \in \mathbb{R}^2 : \sum_{\gamma} e^{-al_1(\gamma)-bl_2(\gamma)} < +\infty \right\}.$$

This was studied in [3] and [21], where the following results were proved.

**Proposition 6.1** [3],[21]**.**

(i) $\mathfrak{M}(\Sigma_1, \Sigma_2)$ *is a straight line if and only if* $\Sigma_1$ *and* $\Sigma_2$ *are isometric.*
(ii) $\mathfrak{M}(\Sigma_1, \Sigma_2)$ *is real analytic.*
(iii) $\mathfrak{M}(\Sigma_1, \Sigma_2)$ *has asymptotes whose normals have slopes equal to the maximum and minimum geodesic stretch between* $\Sigma_1$ *and* $\Sigma_2$*.*
(iv) $\mathfrak{M}(\Sigma_1, \Sigma_2)$ *passes through* $(1,0)$*, where its normal has slope equal to the intersection* $i(\Sigma_1, \Sigma_2)$*. (It also passes through* $(0,1)$*, where its normal has slope equal to* $1/i(\Sigma_2, \Sigma_1)$*.*
(v) *There is a unique point* $(a,b) \in \mathfrak{M}(\Sigma_1, \Sigma_2)$ *where the normal has slope 1 and* $a+b$ *is equal to the correlation number of* $\Sigma_1$ *and* $\Sigma_2$*.*

See [1],[3] for the definitions of max and min geodesic stretch and intersection. The correlation number of $\Sigma_1$ and $\Sigma_2$ is defined to be the exponential growth rate of

$$\#\{\gamma : l_1(\gamma), l_2(\gamma) \in (T, T+\epsilon)\},$$

for fixed $\epsilon > 0$, as $T \to +\infty$ [20].

In our setting, let us define a Manhattan curve $\mathfrak{M}_\phi$, associated to $\phi$, to be the boundary of the set

$$\left\{ (a,b) \in \mathbb{R}^2 : \sum_{w \in \mathcal{C}(F)} e^{-a|w|-b|\phi(w)|} < +\infty \right\}.$$

By writing this in terms of periodic points for the shift map, this may be described as the set

$$\{(a,b) \in \mathbb{R}^2 : P(-a-br) = 0\}$$

or, equivalently,

$$\{(a,b) \in \mathbb{R}^2 : \mathfrak{p}(-b) = a\}.$$

Let us define $\mathfrak{q}(s)$ implicitly by

$$\mathfrak{p}(-\mathfrak{q}(s)) = s; \tag{6.1}$$

then $\mathfrak{M}_\phi$ is the graph of $\mathfrak{q}$. Since $\mathfrak{p}'(-t) = -\int r d\mu_{-tr} \neq 0$, the Implicit Function Theorem gives that $\mathfrak{q}$ is real analytic. (This parallels the analysis of [21].)

Let us examine the slope of the normal at a point $(a, b) = (s, \mathfrak{q}(s))$ on $\mathfrak{M}_\phi$. At this point, the normal has slope $-1/\mathfrak{q}'(s)$. Now, differentiating (6.1),

$$1 = \frac{d}{ds}\mathfrak{p}(-\mathfrak{q}(s)) = \left(-\int r \; d\mu_{-\mathfrak{q}(s)r}\right)\mathfrak{q}'(s),$$

so the normal to $\mathfrak{M}_\phi$ at $(s, \mathfrak{q}(s))$ has slope

$$\frac{-1}{\mathfrak{q}'(s)} = \int r \; d\mu_{-\mathfrak{q}(s)r}.$$

Therefore the set of normals to $\mathfrak{M}_\phi$ is equal to

$$\left\{\int r \; d\mu_t \, : \, t \in \mathbb{R}\right\} = \mathrm{int}(I_r) = \mathrm{int}(\overline{\mathcal{D}_\phi})$$

and it is easy to recover the following statements.

**Theorem 4.**

(i) *$\mathfrak{M}_\phi$ is a straight line if and only if $\phi$ is simple.*
(ii) *$\mathfrak{M}_\phi$ is real analytic.*
(iii) *$\mathfrak{M}_\phi$ has asymptotes whose normals have slopes equal to the* $\max \mathcal{D}_\phi$ *and* $\min \mathcal{D}_\phi$.
(iv) *$\mathfrak{M}_\phi$ passes through* $(\log(2k-1), 0)$, *where its normal has slope equal to the generic stretch $\lambda(\phi)$*
(v) *There is a unique point $(a, b) \in \mathfrak{M}_\phi$ where the normal has slope 1 and $a + b = \mathfrak{h}(1)$.*

*Proof.* The only statement which requires proof is (v). If the normal has slope 1 at $(a, b) = (s, \mathfrak{q}(s))$ then

$$\int r \; d\mu_{-\mathfrak{q}(s)r} = 1,$$

and $-\mathfrak{q}(s) = \xi$, where

$$\mathfrak{h}(1) = \mathfrak{p}(\xi) - \xi = \mathfrak{p}(-\mathfrak{q}(s)) + \mathfrak{q}(s) = s + \mathfrak{q}(s) = a + b. \quad \square$$

*Remark.* Kaimanovich, Kapovich and Schupp have shown how to identify the generic stretching factor $\lambda(\phi)$ as an intersection of currents [7], [9].

## References


1. F. Bonahon, *The geometry of Teichmüller space via geodesic currents*, Invent. math. **92** (1988), 139-162.
2. M. Bourdon, *Actions quasi-convexes d'un groupe hyperbolique, flot géodésique*, PhD thesis (Orsay) 1993.
3. M. Burger, *Intersection, the Manhattan curve, and the Patterson-Sullivan theory in rank 2*, Int. Math. Res. Notices **7** (1993), 217-225.

4. M. Eichler, *Grenzkreisgruppen und kettenbruchartige Algorithmen*, Acta Arith. **11** (1965), 169-180.
5. E. Ghys and P. de la Harpe, *Sur les groupes hyperboliques d'après Mikhael Gromov*, Birkhauser., Boston, 1990.
6. M. Gromov, *Hyperbolic groups*, Essys in group theory, M.S.R.I. publication vol. 8, MSRI, Berkeley.
7. V. Kaimanovich, I. Kapovich and P. Schupp, *The subadditive ergodic theorem and generic stretching factors for free group automorphisms*, Israel J. Math. **157** (2007), 1-46.
8. I. Kapovich, *The frequency space of a free group*, Internat. J. Algebra Comput. **15** (2005), 939-969.
9. I. Kapovich, *Currents on free groups*, Topological and asymptotic aspects of group theory, Contemp. Math., 394, Amer. Math. Soc., Providence, RI, 2006, pp. 149-176.
10. S. Lalley, *Renewal theorems in symbolic dynamics, with applications to geodesic flows, noneuclidean tessellations and their fractal limits*, Acta Math. **163** (1989), 1-55.
11. B. Marcus and S. Tuncel, *Entropy at a weight-per-symbol and embeddings of Markov chains*, Invent. math. **102** (1990), 235-266.
12. A. G. Myasnikov and V. Shpilrain, *Some metric properties of automorphisms of groups*, J. Algebra **304** (2006), 782-792.
13. W. Parry and M. Pollicott, *Zeta functions and the periodic orbit structure of hyperbolic dynamics*, Astérisque **187-188** (1990), 1-268.
14. Y. Pesin, *Dimension theory in dynamical systems. Contemporary views and applications*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1997.
15. M. Pollicott and R. Sharp, *Rates of recurrence for* $\mathbb{Z}^q$ *and* $\mathbb{R}^q$ *extensions of subshifts of finite type*, J. London Math. Soc. **49** (1994), 401-416.
16. M. Pollicott and R. Sharp, *Poincaré series and zeta functions for surface group actions on* $\mathbb{R}$*-trees*, Math. Zeit. **226** (1997), 335-347.
17. M. Pollicott and R. Sharp, *Comparison theorems and orbit counting in hyperbolic geometry*, Trans. Amer. Math. Soc. **350** (1998), 473-499.
18. M. Pollicott and R. Sharp, *Poincaré series and comparison theorems for variable negative curvature*, Topology, ergodic theory, real algebraic geometry, Amer. Math. Soc. Transl. Ser. 2, 202, Amer. Math. Soc., Providence, RI, 2001, pp. 229-240.
19. R. Rockafeller, *Convex Analysis*, Princeton University Press, Princeton, New Jersey, 1970.
20. R. Schwartz and R. Sharp, *The correlation of length spectra of two hyperbolic surfaces*, Comm. Math. Phys. **153** (1993), 423-430.
21. R. Sharp, *The Manhattan curve and the correlation of length spectra on hyperbolic surfaces*, Math. Zeit. **228** (1998), 745-750.
22. R. Sharp, *Local limit theorems for free groups*, Math. Ann. **321** (2001), 889-904.
23. P. Walters, *An introduction to ergodic theory*, Springer-Verlag, Berlin, 1982.

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