UNIFORM ESTIMATES FOR CLOSED GEODESICS AND HOMOLOGY ON FINITE AREA HYPERBOLIC SURFACES

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Abstract. In this note, we study the distribution of closed geodesics in homology on a finite area hyperbolic surface. We obtain an estimate which is uniform as the homology class varies, refining an asymptotic formula due to C. Epstein.

0. Introduction

Let M be a finite area hyperbolic surface, i.e., the quotient of the hyperbolic plane $\mathbb{H}^2$ by the free action of a group of isometries such that the fundamental domain has finite area. It is well-known that if we define $\pi(T)$ to be the number of (prime) closed geodesics on M of length at most T then $\lim_{T\to\infty} e^{-T}T\pi(T) = 1$. A more delicate problem is to estimate the number of closed geodesics lying in a prescribed homology class. Here there are striking differences depending on whether or not M is compact. The compact case has been studied in [11],[15] and [18]; here we shall concentrate on the case where M has at least one cusp.

Suppose that M has genus g and $p+1$ cusps. Then M has area $\mu(M) = 2\pi(2g+p-1)$ and $H_1(M,\mathbb{Z}) \cong \mathbb{Z}^{p+2g}$. We shall write a typical element of $H_1(M,\mathbb{Z})$ as (α,β), where $\alpha \in \mathbb{Z}^p$ and $\beta \in \mathbb{Z}^{2g}$, and use $\pi_{(\alpha,\beta)}(T)$ to denote the number of (prime) closed geodesics in (α,β) of length at most T. Epstein [4] has shown that

$$\lim_{T\to\infty} \frac{T^{p+g+1}\pi_{(\alpha,\beta)}(T)}{e^T} = \frac{1}{2^{g+1}} \binom{2p}{p} (2g+p-1)^{p+g}. \tag{0.1}$$

In this paper, we shall be interested in refining Epstein's result to obtain a uniform estimate as the class (α,β) is allowed to vary. This is contained in the following theorem.

Theorem 1. *Let M be a finite area hyperbolic surface of genus g with $p+1$ cusps. Then there exists a strictly positive definite $2g \times 2g$ matrix A of inner products of cusp forms such that*

$$\lim_{T\to\infty} \sup_{(\alpha,\beta)\in\mathbb{Z}^{p+2g}} \left| \frac{T^{p+g+1}\pi_{(\alpha,\beta)}(T)}{e^T} - \frac{(2g+p-1)^{p+g}}{2^{g-p+1}} e^{-\langle\beta,A^{-1}\beta\rangle/4T}\mathcal{I}\left(\frac{2\mu(M)\alpha}{T}\right)\right| = 0,$$

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where

$$\mathcal{I}(x) = \int_{\mathbb{R}^p} e^{-i\langle x,\xi\rangle} e^{-(\sum_{j=1}^p |\xi_j| + |\xi_1+\cdots+\xi_p|)} d\xi.$$

(If $p = 0$ then we set $\mathcal{I}(x) \equiv 1$.)

Epstein has calculated that

$$\mathcal{I}(0) = \frac{1}{2^p}\binom{2p}{p}.$$

Thus, in particular, our result agrees with (0.1). The integral $\mathcal{I}(x)$ may be evaluated by means of a slightly more general version of the scheme considered in the appendix to [4]. The main point is that, for each subset $\mathcal{S} \subset \{1,2,\ldots,p\}$, one considers separately the integral over $\{(\xi_1,\ldots,\xi_p) \in \mathbb{R}^p : \xi_j \geq 0, j \in \mathcal{S}, \xi_j \leq 0, j \notin \mathcal{S}\}$. These calculations rapidly become complicated as p increases. Nevertheless, one can see that $\mathcal{I}(x)$ is a rational function and, for $p = 1,2$, one can calculate that $\mathcal{I}(x) = 4/(4+x^2)$ and

$$\mathcal{I}(x) = \frac{8(12 + x_1^2 + x_2^2 - x_1x_2)}{(4+x_1^2)(4+x_2^2)(4+(x_1-x_2)^2)},$$

respectively. We use this formula to give a more explicit estimate in the case of the thrice punctured sphere with hyperbolic metric. This surface is the quotient $\mathbb{H}^2/\Gamma(2)$, where $\Gamma(2)$ is the principal congruence subgroup $\Gamma(2) = \{\gamma \in PSL(2,\mathbb{Z}) : \gamma \equiv I \pmod 2\}$ of the modular group $PSL(2,\mathbb{Z})$. In this case we have $H_1(\mathbb{H}^2/\Gamma(2),\mathbb{Z}) \cong \mathbb{Z}^2$ and

$$\lim_{T\to\infty} \sup_{\alpha\in\mathbb{Z}^2} \left| \frac{T^3\pi_\alpha(T)}{e^T} - \frac{1}{2}\frac{3 + \frac{4\pi^2}{T^2}(\alpha_1^2+\alpha_2^2-\alpha_1\alpha_2)}{\left(1+\frac{4\pi^2\alpha_1^2}{T^2}\right)\left(1+\frac{4\pi^2\alpha_2^2}{T^2}\right)\left(1+\frac{4\pi^2(\alpha_1-\alpha_2)^2}{T^2}\right)} \right| = 0.$$

Theorem 1 may be used to describe the asymptotics the counting function for homology classes which are allowed to vary with T. Note that $e^{-\langle\beta,A^{-1}\beta\rangle/4T}\mathcal{I}(2\mu(M)\alpha/T)$ is a function of $(\alpha/T, \beta/\sqrt{T})$. Hence, if homology classes $(\alpha(T),\beta(T))$ are chosen so that $(\alpha(T)/T, \beta(T)/\sqrt{T}) \to (\theta,\varphi)$, as $T\to\infty$, then the leading asymptotic (0.1) is replaced by

$$\lim_{T\to\infty} \frac{T^{p+g+1}\pi_{(\alpha(T),\beta(T))}(T)}{e^T} = \frac{1}{2^{g+1}}\binom{2p}{p}(2g+p-1)^{p+g}e^{-\langle\theta,A^{-1}\theta\rangle/4}\mathcal{I}(2\mu(M)\varphi).$$

The analogue of (0.1) in the compact case was established by Katsuda and Sunada [11] and Phillips and Sarnak [15], where, for a surface of genus g, it takes the form

$$\lim_{T\to\infty} \frac{T^{g+1}\pi_\beta(T)}{e^T} = (g-1)^g.$$

In fact, [15] contains a more detailed asymptotic expansion and results valid for higher dimensional compact hyperbolic manifolds. (Related results for variable negative curvature surfaces and manifolds are contained in [1], [10], [12], [13], [16], [17].) Epstein's paper [4] also contains analogues of (0.1) for finite volume hyperbolic manifolds of dimension ≥ 3.

For such a manifold M, the most interesting new feature is that the polynomial term T^{p+g+1} has to be modified according to whether $\dim M = 3$ or $\dim M \geq 4$. More recently, Babillot and Peigné [2] have made a detailed study of the behaviour of $\pi_{(\alpha,\beta)}(T)$ for (infinite volume) quotients of hyperbolic space by Schottky groups with cusps. In particular, they have understood the dependence of the asymptotics on the ranks of the cusps. A version of Theorem 1 for compact variable negative curvature surfaces was obtained in [18]; however, in the constant curvature case the result may be more easily deduced directly from the analysis in [15].

It is interesting to compare Theorem 1 with the stable laws for the geodesic flow on surfaces with cusps obtained by Guivarc'h and Le Jan [7], [8], [14]. (More recent papers consider the stable laws relative to cusps associated to certain infinite volume surfaces and higher dimensional manifolds [3], [5].)

Notation. For given functions $A(T)$ and $B(T) > 0$, we shall write $A(T) = O(B(T))$ if $|A(T)| \leq CB(T)$, for some constant $C > 0$.

1. Preliminaries

The fundamental group $\pi_1 M$ has the simple presentation

$$\left\langle \gamma_1, \dots, \gamma_{2g}, \delta_0, \delta_1, \dots, \delta_p \;\middle|\; \prod_{i=1}^{g} \gamma_i \gamma_{i+g} \gamma_i^{-1} \gamma_{i+g}^{-1} \prod_{j=0}^{p} \delta_j = 1 \right\rangle.$$

The integer first homology group $H_1(M, \mathbb{Z})$ may be identified with the abelianization $\pi_1 M / [\pi_1 M, \pi_1 M]$ and this induces a map $[\cdot] : \pi_1 M \to H_1(M, \mathbb{Z})$, called the Hurewicz map [6, Chapter 12c]. Then $(\alpha, \beta) \in \mathbb{Z}^{p+2g}$ represents the homology class

$$(\alpha, \beta) = \sum_{j=1}^{p} \alpha_j [\delta_j] + \sum_{k=1}^{2g} \beta_k [\gamma_k].$$

The character group of $H_1(M, \mathbb{Z})$ is the torus $\mathbb{T}^{p+2g}$ and may be given co-ordinates (ξ, η) with $\xi = (\xi_1, \dots, \xi_p) \in [-\pi, \pi]^p$, $\eta = (\eta_1, \dots, \eta_{2g}) \in [-\pi, \pi]^{2g}$ by

$$\chi_{(\xi,\eta)}(\alpha, \beta) = e^{i(\sum_{j=1}^{p} \xi_j \alpha_j + \sum_{k=1}^{2g} \eta_k \beta_k)}.$$

For convenience, we shall write $\xi_0 = \xi_1 + \cdots + \xi_p$.

Choose simple closed curves $C_1, \dots, C_{2g}$ lying in $\gamma_1, \dots, \gamma_{2g}$, respectively. Let $\overline{M}$ denote the compactification of M and identify $H^1(\overline{M}, \mathbb{R})$ with the space of harmonic cusp forms on M (i.e. forms which vanish at the cusps of M). Introduce a basis $\omega_1, \dots, \omega_{2g}$ for $H^1(\overline{M}, \mathbb{R})$ by $\int_{C_i} \omega_j = \delta_{ij}$ and define a $2g \times 2g$ matrix $A = (a_{ij})$ by

$$a_{ij} = \frac{1}{\mu(M)} \int_M \omega_i \wedge * \omega_j.$$

Then $\det A = \mu(M)^{-2g}$. The matrix A is positive definite and defines the inner product

$$\langle \eta, A\eta \rangle = \frac{1}{\mu(M)} \int_M \eta \wedge *\eta$$

on $H^1(\overline{M}, \mathbb{R}) \cong \mathbb{R}^{2g}$.

We shall now summarize some results from [4]. Let Δ denote the Laplace-Beltrami operator on $\mathbb{H}^2$ and let $\mathcal{F}$ be a fundamental domain for the action of $\pi_1 M$ on $\mathbb{H}^2$. For $(\xi, \eta) \in \mathbb{T}^{p+2g}$ define the twisted Laplace operator $\Delta_{(\xi,\eta)}$ by $\Delta_{(\xi,\eta)} f = \Delta f$ for $f \in C^\infty(\mathbb{H}^2) \cap C_0^\infty(\overline{\mathcal{F}})$ with

$$f(\gamma x) = \chi_{(\xi,\eta)}([\gamma]) f(x), \qquad x \in \mathbb{H}^2, \ \gamma \in \pi_1 M.$$

(We have been deliberately vague about the domains of definition of these operators; full details may be found in [4].) Then, for (ξ, η) in a neighbourhood of $(0, 0)$, $\Delta_{(\xi,\eta)}$ has a unique eigenvalue $\lambda(\xi, \eta) \geq 0$ such that $(\xi, \eta) \mapsto \lambda(\xi, \eta)$ is continuous and $\lambda(0, 0) = 0$. Furthermore, λ is monotone on each ray $\{(t\xi, t\eta) : 0 \leq t \leq 1\}$.

We shall write $B(\epsilon_1, \epsilon_2) = \{(\xi, \eta) : ||\xi|| < \epsilon_1, \ ||\eta|| < \epsilon_2\}$ and, for $\delta > 0, \kappa > 0$,

$$D(\delta) = D(\delta, \kappa) = \bigcap_{j=0}^{p} \{(\xi, \eta) : |\xi_j| \geq e^{-\delta/(||\xi||^{1-\kappa} + ||\eta||^2)}\}.$$

The following estimates on $\lambda(\xi, \eta)$ are established in [4].

Proposition 1. *Given $\kappa > 0$, there exist $\epsilon_1, \epsilon_2 > 0$ such that the following hold:*
(i) there exist constants $C_1, C_2 > 0$ such that, for $(\xi, \eta) \in B(\epsilon_1, \epsilon_2)$,

$$C_1(||\xi|| + ||\eta||^2) \leq \lambda(\xi, \eta) \leq C_2(||\xi||^{1-\kappa} + ||\eta||^2);$$

(ii) given $\delta > 0$, there exists $\Delta(\epsilon_1, \epsilon_2, \delta) > 0$ such that, for $(\xi, \eta) \in B(\epsilon_1, \epsilon_2) \cap D(\delta)$,

$$\frac{1}{2\mu(M)} \sum_{j=0}^{p} |\xi_j|(1 - \Delta(\epsilon_1, \epsilon_2, \delta)) + \langle \eta, A\eta \rangle - \Delta(\epsilon_1, \epsilon_2, \delta)||\eta||^2 \leq \lambda(\xi, \eta)$$

$$\leq \frac{1}{2\mu(M)} \sum_{j=0}^{p} |\xi_j|(1 + \Delta(\epsilon_1, \epsilon_2, \delta)) + \langle \eta, A\eta \rangle + \Delta(\epsilon_1, \epsilon_2, \delta)||\eta||^2.$$

Furthermore, $\Delta(\epsilon_1, \epsilon_2, \delta) \to 0$ as $(\epsilon_1, \epsilon_2, \delta) \to 0$.

Write $\varrho(\xi, \eta) = i\sqrt{\lambda(\xi, \eta) - 1/4} + 1/2$, so that $\varrho(\xi, \eta) \geq 0$ and $\varrho(0, 0) = 0$. For future use, we have the estimates

$$\frac{e^{-\varrho(\xi,\eta)T}}{1/2 - \varrho(\xi, \eta)} = e^{-(\lambda(\xi,\eta) + O(\lambda^2(\xi,\eta)))T} (2 + O(\lambda(\xi, \eta))) = O(e^{-C\lambda(\xi,\eta)T}), \qquad (1.1)$$

for some $C > 0$.

2. An Auxiliary Function

In this section, we follow the lines of the analysis in Section 8 of [4] but taking into account the dependence of our quantities on (α, β). For a closed geodesic γ, let $l(\gamma)$ denote its length and $[\gamma]$ its homology class. Set

$$R_{(\alpha,\beta)}(T) = \sum_{\substack{l(\gamma)\leq T\\ [\gamma]=(\alpha,\beta)}} \frac{l(\gamma)}{2\sinh(l(\gamma)/2)},$$

where the sum is taken over prime closed geodesics γ of length $l(\gamma) \leq T$ and homology class $[\gamma] = (\alpha, \beta)$. Then, as in [4], the following estimate may be deduced from the Selberg Trace Formula for the twisted Laplacians $\Delta_{(\xi,\eta)}$ [9, p.302]. (The uniformity may be easily checked.)

Proposition 2 [4]**.**

$$\frac{R_{(\alpha,\beta)}(T)}{e^{T/2}} = \frac{1}{(2\pi)^{p+2g}} \int_N \frac{e^{-i\langle(\alpha,\beta),(\xi,\eta)\rangle} e^{-\varrho(\xi,\eta)T}}{1/2 - \varrho(\xi,\eta)} d\xi d\eta + O\left(e^{-aT}\right),$$

where N is a small neighbourhood of 0 in $\mathbb{T}^{p+2g}$ and $a > 0$. Furthermore, a and the implied constant in the big-O term are independent of (α, β).

Fix $0 < \kappa < 1$, $0 < \tau < 1/2$ and $\sigma \in (2\tau\kappa, 2\tau)$. Write $\epsilon_1 = 1/T^{2\tau}$, $\epsilon_2 = 1/T^{\tau}$, and $\delta = 1/T^{2\tau-\sigma}$. Then, for T sufficiently large, the estimates of Proposition 1 will hold. Write $\Delta(T) = \Delta(\epsilon_1, \epsilon_2, \delta)$. The next lemma allows us to replace N by the set $B(\epsilon_1, \epsilon_2) \cap D(\delta)$, where we have good estimates on $\lambda(\xi, \eta)$ and hence on $\varrho(\xi, \eta)$.

Lemma 1. *For any $k \geq 1$,*

$$\frac{R_{(\alpha,\beta)}(T)}{e^{T/2}} = \frac{1}{(2\pi)^{p+2g}} \int_{B(\epsilon_1,\epsilon_2)\cap D(\delta)} \frac{e^{-\varrho(\xi,\eta)T} e^{-i\langle(\alpha,\beta),(\xi,\eta)\rangle}}{1/2 - \varrho(\xi,\eta)} d\xi d\eta + O\left(\frac{1}{T^k}\right).$$

Furthermore, the implied constants in the big-O estimates are independent of (α, β).

Proof. Clearly,

$$\left| \frac{(2\pi)^{p+2g}}{e^{T/2}} R_{(\alpha,\beta)}(T) - \int_{B(\epsilon_1,\epsilon_2)\cap D(\delta)} \frac{e^{-i\langle(\alpha,\beta),(\xi,\eta)\rangle} e^{-\varrho(\xi,\eta)T}}{1/2 - \varrho(\xi,\eta)} d\xi d\eta \right|$$
$$\leq \int_{N\setminus B(\epsilon_1,\epsilon_2)} \frac{e^{-\varrho(\xi,\eta)T}}{1/2 - \varrho(\xi,\eta)} d\xi d\eta + \int_{B(\epsilon_1,\epsilon_2)\cap D(\delta)^c} \frac{e^{-\varrho(\xi,\eta)T}}{1/2 - \varrho(\xi,\eta)} d\xi d\eta + O(e^{-aT}).$$

To prove the lemma, we shall estimate the two integrals on the Right Hand Side.

In the first case we have, using Proposition 1(i) and the fact that λ is monotone on rays $\{(t\xi, t\eta) : 0 \leq t \leq 1\}$,

$$\int_{N\setminus B(\epsilon_1,\epsilon_2)} \frac{e^{-\varrho(\xi,\eta)T}}{1/2 - \varrho(\xi,\eta)} d\xi d\eta = O\left(\int_{N\setminus B(\epsilon_1,\epsilon_2)} e^{-C\lambda(\xi,\eta)T} d\xi d\eta \right)$$
$$= O(e^{-C(\epsilon_1+\epsilon_2^2)T}) = O(e^{-CT^{1-2\tau}}).$$

Since $\tau < 1/2$, this is $O(T^{-k})$, for any $k \geq 1$.

To estimate the second integral, notice first that

$$B(\epsilon_1, \epsilon_2) \cap D(\delta)^c \subset \bigcup_{j=0}^{p} \left\{(\xi, \eta) : |\xi_j| < e^{-\delta/(\epsilon_1^{1-\kappa}+\epsilon_2^2)}, ||\xi|| < \epsilon_1, ||\eta|| < \epsilon_2\right\}.$$

Thus

$$\begin{aligned}
&\int_{B(\epsilon_1,\epsilon_2)\cap D(\delta)^c} \frac{e^{-\varrho(\xi,\eta)T}}{1/2 - \varrho(\xi, \eta)} d\xi d\eta = O\left(\int_{B(\epsilon_1,\epsilon_2)\cap D(\delta)^c} e^{-C\lambda(\xi,\eta)T} d\xi d\eta\right) \\
&= O\left(\int_{||\eta||<\epsilon_2} \int_0^{e^{-\delta/(\epsilon_1^{1-\kappa}+\epsilon_2^2)}} \int_0^{\epsilon_1} \cdots \int_0^{\epsilon_1} e^{-C(\xi_1+\cdots+\xi_p+||\eta||^2)T} d\xi_1 \cdots d\xi_p d\eta\right) \\
&= O(\epsilon_1^{p-1}\epsilon_2^{2g} e^{-\delta/(\epsilon_1^{1-\kappa}+\epsilon_2^2)}) \\
&= O\left(\frac{e^{-T^{\sigma-2\tau\kappa}}}{T^{2\tau(g+p-1)}}\right).
\end{aligned}$$

Since $\sigma > 2\tau\kappa$, this last term is $O(T^{-k})$, for any $k \geq 1$.

Next we wish to replace the exponent $-\varrho(\xi, \eta)T$ in the integral over $B(\epsilon_1, \epsilon_2) \cap D(\delta)$ with the expression given in Proposition 1(ii). For simplicity, we shall write $\Xi(\xi) = (\sum_{j=0}^{p} |\xi_j|)/2\mu(M)$.

Lemma 2.

$$\lim_{T\to\infty} \sup_{(\alpha,\beta)\in\mathbb{Z}^{p+2g}} T^{p+g} \left| \int_{B(\epsilon_1,\epsilon_2)\cap D(\delta)} \frac{e^{-i\langle(\alpha,\beta),(\xi,\eta)\rangle} e^{-\varrho(\xi,\eta)T}}{1/2 - \varrho(\xi, \eta)} d\xi d\eta - \int_{B(\epsilon_1,\epsilon_2)} \frac{e^{-i\langle(\alpha,\beta),(\xi,\eta)\rangle} e^{-(\Xi(\xi)+\langle\eta,A\eta\rangle)T}}{1/2 - \varrho(\xi, \eta)} d\xi d\eta \right| = 0.$$

Proof. Applying equation (1.1) we have that, for any $k \geq 1$,

$$\begin{aligned}
&T^{p+g} \left| \int_{B(\epsilon_1,\epsilon_2)\cap D(\delta)} \left\{ \frac{e^{-i\langle(\alpha,\beta),(\xi,\eta)\rangle} e^{-\varrho(\xi,\eta)T}}{1/2 - \varrho(\xi, \eta)} - \frac{e^{-(\Xi(\xi)+\langle\eta,A\eta\rangle)T} e^{-i\langle(\alpha,\beta),(\xi,\eta)\rangle}}{1/2 - \varrho(\xi, \eta)} \right\} d\xi d\eta \right| \\
&\leq T^{p+g} \int_{B(\epsilon_1,\epsilon_2)\cap D(\delta)} \frac{e^{-(\Xi(\xi)+\langle\eta,A\eta\rangle)T}}{1/2 - \varrho(\xi, \eta)} \left(e^{(\Xi(\xi)+||\eta||^2)\Delta(T)} - 1\right) d\xi d\eta \\
&\leq T^{p+g} \int_{B(\epsilon_1,\epsilon_2)} e^{-(\Xi(\xi)+\langle\eta,A\eta\rangle)T} \left(e^{(\Xi(\xi)+||\eta||^2)\Delta(T)} - 1\right) d\xi d\eta + O\left(\frac{1}{T^k}\right) \\
&= \int_{B(\epsilon_1 T,\epsilon_2\sqrt{T})} e^{-(\Xi(\xi)+\langle\eta,A\eta\rangle)} \left(e^{(\Xi(\xi)+||\eta||^2)\Delta(T)/T} - 1\right) d\xi d\eta + O\left(\frac{1}{T^k}\right),
\end{aligned}$$

which converges to zero, as $T \to \infty$.

We can replace the integral over $B(\epsilon_1, \epsilon_2) \cap D(\delta)$ by one over $B(\epsilon_1, \epsilon_2)$ by observing that, as in the proof of Lemma 1, for any $k \geq 1$,

$$\int_{B(\epsilon_1,\epsilon_2)\cap D(\delta)} \frac{e^{-i\langle(\alpha,\beta),(\xi,\eta)\rangle} e^{-(\Xi(\xi)+\langle\eta,A\eta\rangle)T}}{1/2 - \varrho(\xi,\eta)} d\xi d\eta$$
$$= \int_{B(\epsilon_1,\epsilon_2)} \frac{e^{-i\langle(\alpha,\beta),(\xi,\eta)\rangle} e^{-(\Xi(\xi)+\langle\eta,A\eta\rangle)T}}{1/2 - \varrho(\xi,\eta)} d\xi d\eta + O\left(\frac{1}{T^k}\right).$$

The next result gives a uniform estimate on $e^{-T/2}T^{p+g}R_{(\alpha,\beta)}(T)$. Recall that

$$\mathcal{I}(x) = \int_{\mathbb{R}^p} e^{-i\langle x,\xi\rangle} e^{-\sum_{j=0}^p |\xi_j|} d\xi.$$

Proposition 3.

$$\lim_{T\to\infty} \sup_{(\alpha,\beta)\in\mathbb{Z}^{p+2g}} \left| \frac{T^{p+g}R_{(\alpha,\beta)}(T)}{e^{T/2}} - \frac{(2g+p-1)^{p+g}}{2^{g-p}} e^{-\langle\beta,A^{-1}\beta\rangle/4T} \mathcal{I}\left(\frac{2\mu(M)\alpha}{T}\right) \right| = 0.$$

Proof. Combining Lemma 1 and Lemma 2, we have that

$$\frac{T^{p+g}R_{(\alpha,\beta)}(T)}{e^{T/2}} - \frac{T^{p+g}}{(2\pi)^{p+2g}} \int_{B(\epsilon_1,\epsilon_2)} \frac{e^{-i\langle(\alpha,\beta),(\xi,\eta)\rangle} e^{-(\Xi(\xi)+\langle\eta,A\eta\rangle)T}}{1/2 - \varrho(\xi,\eta)} d\xi d\eta \to 0, \text{ as } T \to \infty,$$

uniformly in (α, β).

Our first step is to replace the above integral over $B(\epsilon_1, \epsilon_2)$ by one over $\mathbb{R}^{p+2g}$. First note that $|(1/2 - \varrho(\xi,\eta))^{-1} - 2| = O(\lambda(\xi,\eta)) = O(||\xi||^{1-\kappa} + ||\eta||^2))$ on $B(\epsilon_1, \epsilon_2)$, so that

$$T^{p+g} \left| \int_{B(\epsilon_1,\epsilon_2)} e^{-i\langle(\alpha,\beta),(\xi,\eta)\rangle} \left\{ \frac{e^{-(\Xi(\xi)+\langle\eta,A\eta\rangle)T}}{1/2 - \varrho(\xi,\eta)} - 2e^{-(\Xi(\xi)+\langle\eta,A\eta\rangle)T} \right\} d\xi d\eta \right|$$
$$= \left| \int_{B(\epsilon_1 T,\epsilon_2\sqrt{T})} e^{-i\langle(\alpha,\beta),(\xi/T,\eta/\sqrt{T})\rangle} \left\{ \frac{e^{-(\Xi(\xi)+\langle\eta,A\eta\rangle)}}{1/2 - \varrho(\xi/T,\eta/\sqrt{T})} - 2e^{-(\Xi(\xi)+\langle\eta,A\eta\rangle)} \right\} d\xi d\eta \right|$$
$$= O\left((\epsilon_1^{1-\kappa} + \epsilon_2^2) \int_{B(\epsilon_1 T,\epsilon_2\sqrt{T})} e^{-(\Xi(\xi)+\langle\eta,A\eta\rangle)} d\xi d\eta \right)$$
$$= O(\epsilon_1^{1-\kappa} + \epsilon_2^2) = O(T^{-2\tau(1-\kappa)})$$

(since $\int_{\mathbb{R}^p} e^{-(\Xi(\xi)+\langle\eta,A\eta\rangle)} d\xi d\eta < +\infty$). Next we observe that

$$\int_{\mathbb{R}^{p+2g}\setminus B(\epsilon_1,\epsilon_2)} e^{-i\langle(\alpha,\beta),(\xi,\eta)\rangle} e^{-(\Xi(\xi)+\langle\eta,A\eta\rangle)T} d\xi d\eta = O(e^{-cT}),$$

for some $c > 0$, uniformly in (α, β). Thus,

$$\frac{T^{p+g} R_{(\alpha,\beta)}(T)}{e^{T/2}} - \frac{2}{(2\pi)^{p+2g}} \int_{\mathbb{R}^{p+2g}} e^{-i\langle(\alpha,\beta),(\xi,\eta)\rangle} e^{-(\Xi(\xi)+\langle\eta,A\eta\rangle)T} d\xi d\eta \to 0, \text{ as } T \to \infty,$$

uniformly in (α, β).

It remains to evaluate the integral. Firstly,

$$\frac{T^g}{(2\pi)^{2g}} \int_{\mathbb{R}^{2g}} e^{-i\langle\beta,\eta\rangle} e^{-\langle\eta,A\eta\rangle T} d\eta = \frac{1}{(2\pi)^g} \frac{1}{2^g} \frac{1}{\sqrt{\det A}} e^{-\langle\beta,A^{-1}\beta\rangle/4T}.$$

Also

$$\frac{T^p}{(2\pi)^p} \int_{\mathbb{R}^p} e^{-i\langle\alpha,\xi\rangle} e^{-\Xi(\xi)T} d\xi = \frac{(2\mu(M))^p}{(2\pi)^p} \mathcal{I}\left(\frac{2\mu(M)\alpha}{T}\right).$$

Since $\det A = \mu(M)^{-2g}$, this completes the proof.

3. Proof of Theorem 1

In this section we shall transfer the uniform estimate on $R_{(\alpha,\beta)}(T)$ contained in Proposition 3 into the estimate on $\pi_{(\alpha,\beta)}(T)$ required by Theorem 1. All big-O estimates will be independent of (α, β). To simplify some expressions we shall write $n = p + g + 1$. First note that, using integration by parts,

$$\begin{aligned}
\pi_{(\alpha,\beta)}(T) &= \int_1^T \frac{2\sinh(t/2)}{t} dR_{(\alpha,\beta)}(t) + O(1) \\
&= \int_1^T \frac{e^{t/2}}{t} dR_{(\alpha,\beta)}(t) + O(e^{T/2}) \\
&= \frac{e^{T/2}}{T} R_{(\alpha,\beta)}(T) - \int_1^T \left(\frac{e^{t/2}}{2t} - \frac{e^{t/2}}{t^2}\right) R_{(\alpha,\beta)}(t) dt + O(e^{T/2}).
\end{aligned}$$

Thus, we have the estimate

$$\begin{aligned}
&\frac{T^n}{e^T} \pi_{(\alpha,\beta)}(T) - \frac{T^{n-1}}{2e^{T/2}} R_{(\alpha,\beta)}(T) \\
&= \frac{T^{n-1}}{e^{T/2}} R_{(\alpha,\beta)}(T) - \frac{T^n}{e^T} \int_1^T \left(\frac{e^{t/2}}{2t} - \frac{e^{t/2}}{t^2}\right) R_{(\alpha,\beta)}(t) dt + O(T^n e^{-T/2}).
\end{aligned}$$

Since $R_{(\alpha,\beta)}(T) = O(e^{T/2}/T^{p+g})$, we have that

$$\begin{aligned}
&\frac{T^n}{e^T} \int_1^T \frac{e^{t/2}}{t^2} R_{(\alpha,\beta)}(t) dt = \frac{T^n}{e^T} \left(\int_1^{T/2} + \int_{T/2}^T\right) \frac{e^{t/2}}{t^2} R_{(\alpha,\beta)}(t) dt \\
&= O\left(\frac{T^n}{e^T} \int_1^{T/2} \frac{e^t}{t^{n+1}} dt\right) + O\left(\frac{T^n}{e^T} \int_{T/2}^T \frac{e^t}{t^{n+1}} dt\right) = O(e^{-T/2}) + O(T^{-1}).
\end{aligned}$$

Thus, to prove Theorem 1, it suffices to show that

$$\lim_{T\to\infty} \sup_{(\alpha,\beta)\in\mathbb{Z}^{p+2g}} \left| \frac{T^{n-1}}{2e^{T/2}} R_{(\alpha,\beta)}(T) - \frac{T^n}{2e^T} \int_1^T \frac{e^{t/2}}{t} R_{(\alpha,\beta)}(t)dt \right| = 0.$$

By Proposition 3, we may write

$$\sup_{(\alpha,\beta)\in\mathbb{Z}^{p+2g}} \left| \frac{T^{n-1}}{e^{T/2}} R_{(\alpha,\beta)}(T) - C(p,g)e^{-\langle\beta,A^{-1}\beta\rangle/4T}\mathcal{I}(\alpha'/T) \right| \le \psi(T),$$

where $C(p,g) = 2^{-g+p+1}(2g+p-1)^{p+g}$, $\alpha' = 2\mu(M)\alpha$, and where $\psi(T)$ decreases to zero as $T \to \infty$. Hence

$$\begin{aligned}
&\sup_{(\alpha,\beta)\in\mathbb{Z}^{p+2g}} \left| \frac{T^{n-1}}{2e^{T/2}} R_{(\alpha,\beta)}(T) - \frac{T^n}{2e^T} \int_1^T \frac{e^{t/2}}{t} R_{(\alpha,\beta)}(t)dt \right| \\
&\le \frac{\psi(T)}{2} + \frac{T^n}{2e^T} \int_1^T \frac{e^t}{t^n}\psi(t)dt \\
&+ \frac{C(p,g)}{2} \sup_{(\alpha,\beta)\in\mathbb{Z}^{p+2g}} \left| e^{-\langle\beta,A^{-1}\beta\rangle/4T}\mathcal{I}(\alpha'/T) - \frac{T^n}{e^T}\int_1^T \frac{e^t}{t^n} e^{-\langle\beta,A^{-1}\beta\rangle/4t}\mathcal{I}(\alpha'/t)dt \right|.
\end{aligned}$$

We have

$$\begin{aligned}
\frac{T^n}{2e^T}\int_1^T \frac{e^t}{t^n}\psi(t)dt &= \frac{T^n}{2e^T}\left(\int_1^{T/2} + \int_{T/2}^T\right)\frac{e^t}{t^n}\psi(t)dt \\
&= O(T^n e^{-T/2}) + O(\psi(T/2)),
\end{aligned}$$

so that, to complete the proof, we need to show that

$$\lim_{T\to\infty} \sup_{(\alpha,\beta)\in\mathbb{Z}^{p+2g}} \left| e^{-\langle\beta,A^{-1}\beta\rangle/4T}\mathcal{I}(\alpha'/T) - \frac{T^n}{e^T}\int_1^T \frac{e^t}{t^n} e^{-\langle\beta,A^{-1}\beta\rangle/4t}\mathcal{I}(\alpha'/t)dt \right| = 0.$$

First note that

$$\frac{T^n}{e^T}\int_1^{T-T^{1/2}} \frac{e^t}{t^n} e^{-\langle\beta,A^{-1}\beta\rangle/4t}\mathcal{I}(\alpha'/t)dt = O(T^n e^{-T^{1/2}})$$

so we need only consider the integral between $T - T^{1/2}$ and T. However,

$$\begin{aligned}
&\left| e^{-\langle\beta,A^{-1}\beta\rangle/4T}\mathcal{I}(\alpha'/T) - \frac{T^n}{e^T}\int_{T-T^{1/2}}^T \frac{e^t}{t^n} e^{-\langle\beta,A^{-1}\beta\rangle/4t}\mathcal{I}(\alpha'/t)dt \right| \\
&\le \left| e^{-\langle\beta,A^{-1}\beta\rangle/4T}\mathcal{I}(\alpha'/T) - \frac{T^n}{e^T} e^{-\langle\beta,A^{-1}\beta\rangle/4T}\mathcal{I}(\alpha'/T)\int_{T-T^{1/2}}^T \frac{e^t}{t^n}dt \right| \\
&+ \frac{T^n}{e^T}\int_{T-T^{1/2}}^T \frac{e^t}{t^n}dt \sup_{t\in[T-T^{1/2},T]} \left| e^{-\langle\beta,A^{-1}\beta\rangle/4T}\mathcal{I}(\alpha'/T) - e^{-\langle\beta,A^{-1}\beta\rangle/4t}\mathcal{I}(\alpha'/t) \right|.
\end{aligned}$$

Clearly, the first term on the Right Hand Side above is of order $O(e^{-T^{1/2}})$.

Lemma 3.

$$\sup_{t\in[T-T^{1/2},T]} |e^{-\langle\beta,A^{-1}\beta\rangle/4T} - e^{-\langle\beta,A^{-1}\beta\rangle/4t}| = O(T^{-1/2}) \tag{3.1}$$

and

$$\sup_{t\in[T-T^{1/2},T]} |\mathcal{I}(\alpha'/T) - \mathcal{I}(\alpha'/t)| = O(T^{-1/2}). \tag{3.2}$$

Proof. By the Mean Value Theorem, the Left Hand Side in (3.1) is of order

$$O\left(\frac{\langle\beta, A^{-1}\beta\rangle}{T^{3/2}} e^{-\langle\beta,A^{-1}\beta\rangle/4T}\right)$$

and a simple calculation shows this is $O(T^{-1/2})$. Again by the Mean Value Theorem, the Left Hand Side in (3.2) is of order

$$O\left(T^{-1/2}\left|\sum_{j=1}^{p}\frac{\alpha_j}{\theta_T}\frac{\partial\mathcal{I}(\alpha/\theta_T)}{\partial x_j}\right|\right),$$

for some $\theta_T \in (T - T^{1/2}, T)$. To show that the required $O(T^{-1/2})$ estimate again holds, we shall show that

$$\sup_{x\in\mathbb{R}^p}\left|\sum_{j=1}^{p} x_j\frac{\partial\mathcal{I}(x)}{\partial x_j}\right| < +\infty. \tag{3.3}$$

We may write

$$\begin{aligned}\sum_{j=1}^{p} x_j\frac{\partial\mathcal{I}(x)}{\partial x_j} &= \frac{\partial}{\partial\tau}\int_{\mathbb{R}^p} e^{-i\tau\langle x,\xi\rangle}e^{-\sum_{j=0}^{p}|\xi_j|}d\xi\Big|_{\tau=1}\\ &= \frac{\partial}{\partial\tau}\left(\tau^{-p}\int_{\mathbb{R}^p} e^{-i\langle x,y\rangle}e^{-\sum_{j=0}^{p}|y_j|/\tau}dy\right)\Big|_{\tau=1},\end{aligned}$$

where we have made the substitution $y = \tau\xi$. The bound (3.3) now follows from the Riemann-Lebesgue Lemma.

Applying the lemma, we have that

$$\sup_{t\in[T-T^{1/2},T]}\left|e^{-\langle\beta,A^{-1}\beta\rangle/4T}\mathcal{I}(\alpha'/T) - e^{-\langle\beta,A^{-1}\beta\rangle/4t}\mathcal{I}(\alpha'/t)\right| = O(T^{-1/2})$$

and the proof of Theorem 1 is complete.

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