A LOCAL LIMIT THEOREM FOR CLOSED GEODESICS AND HOMOLOGY

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Abstract. In this paper, we study the distribution of closed geodesics on a compact negatively curved manifold. We concentrate on geodesics lying in a prescribed homology class and, under certain conditions, obtain a local limit theorem to describe the asymptotic behaviour of the associated counting function as the homology class varies.

0. Introduction

Let M be a compact smooth Riemannian manifold with first Betti number $k > 0$ and with negative sectional curvatures. Suppose also that either $\dim M = 2$ or that M is $1/4$-pinched, i.e., the sectional curvatures all lie in an interval $[-\kappa, -\kappa/4]$, for some $\kappa > 0$. Such a manifold contains a countable infinity of prime closed geodesics. (We say that a closed geodesic is prime if it is not a non-trivial multiple of another closed geodesic.) In this paper we shall be interested in how these closed geodesics are distributed with respect to homology.

The homology group $H_1(M, \mathbb{Z})$ is isomorphic to $\mathbb{Z}^k \oplus \mathrm{Tor}$, where Tor is the (finite) torsion subgroup. In this paper, it will be convenient to consider the torsion-free part of the homology, $H_1(M, \mathbb{Z})/\mathrm{Tor}$. We shall, in fact, assume that an isomorphism has been fixed and write $\mathbb{Z}^k$ instead of $H_1(M, \mathbb{Z})/\mathrm{Tor}$.

For a typical (prime) closed geodesic γ on M, let $l(\gamma)$ denote its length and $[\gamma] \in H_1(M, \mathbb{Z})/\mathrm{Tor} = \mathbb{Z}^k$ the torsion-free part of its homology class. For $\alpha \in \mathbb{Z}^k$, define a counting function

$$\pi(T, \alpha) = \#\{\gamma : l(\gamma) \leq T, [\gamma] = \alpha\}.$$

Recently, several papers have studied the asymptotics of this function as $T \to \infty$. In particular, Anantharaman [1] and Pollicott and Sharp [17] have shown that there exist constants $C_0 > 0$, independent of α, and $C_n(\alpha)$, $n \geq 1$, such that, for any $N \geq 1$, we have the asymptotic expansion

$$\pi(T, \alpha) = \frac{e^{hT}}{T^{k/2+1}} \left(C_0 + \frac{C_1(\alpha)}{T} + \frac{C_2(\alpha)}{T^2} + \cdots + \frac{C_N(\alpha)}{T^N} + O\left(\frac{1}{T^{N+1}}\right)\right), \tag{0.1}$$

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where $h > 0$ denotes the topological entropy of the geodesic flow over M. (In fact, the expansion in [17] contained some extra terms corresponding to powers of $T^{-1/2}$; a more careful analysis, as carried out in [11], shows that these terms vanish.) Furthermore, Kotani [11], has studied the dependence of the coefficients $C_n(\alpha)$ on $\alpha = (\alpha_1, \dots, \alpha_k)$, showing that they may be expressed as polynomials of degree $2n$ in $\alpha_1, \dots, \alpha_k$. In the special case of manifolds of constant negative curvature, the expansion (0.1) was obtained by Phillips and Sarnak [14] and, independently, Katsuda and Sunada [9] obtained the leading term. For manifolds of variable negative curvature (without the pinching condition) the leading term of (0.1) was obtained by Katsuda [8], Lalley [12] and Pollicott [15]. Analogous results for manifolds with cusps have been obtained by Epstein [6] and Babillot and Peigné [3].

In this note, we take a slightly different view and address the question of the behaviour of $\pi(T, \alpha)$ when α is allowed to vary independently of T. We obtain the following "local limit theorem".

Theorem 1. *Let M be a compact smooth Riemannian manifold with first Betti number $k > 0$ and with negative sectional curvatures. Suppose also that either $\dim M = 2$ or that M is 1/4-pinched. Then there exists a symmetric positive definite real matrix $\mathcal{D}$ such that,*

$$\lim_{T \to \infty} \left| \frac{h\sigma^k T^{k/2+1}}{e^{hT}} \pi(T, \alpha) - \frac{1}{(2\pi)^{k/2}} e^{-\langle \alpha, \mathcal{D}^{-1}\alpha \rangle / 2T} \right| = 0,$$

uniformly in $\alpha \in \mathbb{Z}^k$, where $\sigma > 0$ satisfies $\sigma^{2k} = \det \mathcal{D}$.

Here, $\langle \cdot, \cdot \rangle$ denotes the usual inner product $\langle x, y \rangle = x_1 y_1 + \cdots + x_k y_k$. As a particular consequence, we recover the leading term of the expansion (0.1), with $C_0 = h^{-1}\sigma^{-k}(2\pi)^{-k/2}$. Theorem 1 appears not to have been stated even for manifolds of constant negative curvature, although, in that case, the result can be easily deduced from the analysis contained in [14].

Remarks.
(*i*) If we take the torsion part of $H_1(M, \mathbb{Z})$ into account then we need to modify Theorem 1 to read

$$\lim_{T \to \infty} \left| \frac{h\sigma^k T^{k/2+1}}{e^{hT}} \pi(T, \alpha) - \frac{1}{(\#\mathrm{Tor})\ (2\pi)^{k/2}} e^{-\langle \alpha_F, \mathcal{D}^{-1}\alpha_F \rangle / 2T} \right| = 0,$$

uniformly in $\alpha \in H_1(M, \mathbb{Z})$, where $\alpha_F \in \mathbb{Z}^k$ denotes the torsion-free part of $\alpha \in H_1(M, \mathbb{Z})$.
(*ii*) In Kotani's formula for the term $C_n(\alpha)/T^n$ in (0.1), the highest power of α makes a contribution

$$\frac{1}{(2\pi)^{k/2} h\sigma^k} \frac{1}{n!} \left(-\frac{\langle \alpha, \mathcal{D}^{-1}\alpha \rangle}{2T} \right)^n.$$

As observed by Kotani in [11], formally summing these contributions gives the expression $e^{-\langle \alpha, \mathcal{D}^{-1}\alpha \rangle / 2T} / (2\pi)^{k/2} h\sigma^k$.

Theorem 1 should be compared with the results on homology classes varying linearly in T obtained by Lalley [12] and Babillot and Ledrappier [2]. Using these results, one can show that, for $\delta > 0$ sufficiently small,

$$\lim_{T \to \infty} \sup_{||\alpha|| \le \delta T} \left| \frac{T^{k/2+1}}{C(\alpha/T) e^{H(\alpha/T)T}} \pi(T, \alpha) - 1 \right| = 0, \qquad (0.2)$$

where $H(x)$ is an entropy function satisfying $H(0) = h$, $\nabla H(0) = 0$ and $\nabla^2 H(0) = -\mathcal{D}^{-1}$, and where $C(x)$ is continuous with $C(0) = C_0$. On the other hand, Theorem 1 is equivalent to

$$\lim_{T\to\infty} \sup_{||\alpha|| \le \delta T} \left| \frac{h\sigma^k T^{k/2+1}}{e^{hT}} \pi(T, \alpha) - \frac{1}{(2\pi)^{k/2}} e^{-\langle \alpha, \mathcal{D}^{-1}\alpha\rangle/2T} \right| = 0 \tag{0.3}$$

(as the supremum over $||\alpha|| > \delta T$ clearly tends to zero). However, even though $H(x) = H(0) - \langle x, \mathcal{D}^{-1}x\rangle/2 + O(||x||^3)$, which gives

$$\exp\{H(\alpha/T)T\} = \exp\{H(0)T - \langle \alpha, \mathcal{D}^{-1}\alpha\rangle/2T + O(||\alpha||^3/T^2)\},$$

the presence of the third order terms means that one cannot deduce (0.3) from (0.2).

The results of [2] and [12] do imply a central limit theorem: for $A \subset \mathbb{R}^k$,

$$\lim_{T\to\infty} \frac{\#\{\gamma : l(\gamma) \le T,\ [\gamma]/\sqrt{T} \in A\}}{\#\{\gamma : l(\gamma) \le T\}} = \frac{1}{(2\pi)^{k/2}\sigma^k} \int_A e^{-\langle x, \mathcal{D}^{-1}x\rangle/2} dx.$$

A key ingredient in the proof of Theorem 1 is an understanding of the analytic domain of a family of functions of a complex variable, called L-functions, indexed by the characters of $\mathbb{Z}^k$. In the next section, we shall define these functions and discuss their properties. In Section 2, we shall introduce a family of functions $\mathcal{S}_T(t)$, $t \in [-\pi, \pi]^k$, obtained by summing a suitably weighted character $e^{i\langle t, \cdot\rangle}$ over all (multiple) closed geodesics of length at most T, and show that they are related to contour integrals of the corresponding L-functions. The results in Section 1 are then used to estimate the sums $\mathcal{S}_T(t)$. In Section 3, we shall use an approach adapted from [18] to transfer information from the $\mathcal{S}_T(t)$ to an auxiliary function $\psi(T, \alpha)$, which is essentially a weighted version of $\pi(T, \alpha)$, and obtain an analogue of Theorem 1 valid for $\psi(T, \alpha)$. In Section 4, we shall complete the proof of Theorem 1 by elementary arguments. In the final section, we shall discuss the application of our method to homologically full Anosov flows, giving a new proof of the first order asymptotic formula for $\pi(T, \alpha)$ (but without uniformity) in that case.

Notation. For given functions $A(T)$ and $B(T)$, we shall write $A(T) \sim B(T)$, as $T \to \infty$, if $\lim_{T\to\infty} A(T)/B(T) = 1$, and $A(T) = O(B(T))$ if $|A(T)| \le CB(T)$, for some constant $C > 0$.

1. L-functions

In order to obtain our main result, we shall need to understand the analytic behaviour of a certain family of functions of a complex variable. We will identify the character group of $\mathbb{Z}^k$ with $[-\pi, \pi]^k$. For $t \in [-\pi, \pi]^k$, define

$$L(s, t) = \prod_\gamma \left(1 - e^{-sl(\gamma) + i\langle t, [\gamma]\rangle}\right)^{-1},$$

where the product is taken over all prime closed geodesics γ. This converges for $Re(s) > h$ and has a meromorphic extension to a strictly larger half-plane [13].

It will be convenient to consider multiple closed geodesics $\gamma' = \gamma^n$, $n \geq 1$. In this case we shall write $l(\gamma') = nl(\gamma)$, $[\gamma'] = n[\gamma]$, and $\Lambda(\gamma') = l(\gamma)$. (Note that Λ is analogous to the von Mangoldt function in number theory.)

We shall be interested in the logarithmic derivative $L'(s,t)/L(s,t)$ of $L(s,t)$. Whenever the summation converges, we have the identity

$$\frac{L'(s,t)}{L(s,t)} = -\sum_{\gamma'} \Lambda(\gamma') e^{-sl(\gamma')+i\langle t,[\gamma']\rangle}.$$

We shall make use of the properties of $L'(s,t)/L(s,t)$ described by the following two propositions. These results were obtained in [17] and rely heavily on the techniques of Dolgopyat [4]. We write $U(\delta) = \{t : ||t|| < \delta\}$.

Proposition 1 ([17]). *For all sufficiently small $\delta > 0$ the following statements are true.*

(i) *There exists $\epsilon > 0$ and an analytic function $s : U(\delta) \to (-\infty, h]$, satisfying $s(0) = h$ and $s(t) < h$ for $t \neq 0$, such that*

$$\frac{L'(s,t)}{L(s,t)} + \frac{1}{s - s(t)}$$

is analytic in $Re(s) > h - \epsilon$.

(ii) *There exists $\epsilon > 0$ such that, for $t \notin U(\delta)$, $L'(s,t)/L(s,t)$ is analytic in $Re(s) > h - \epsilon$.*

Proposition 2 ([17]). *There exists $\epsilon > 0$, $C > 0$, and $0 < \beta < 1$, such that, for all $t \in [-\pi,\pi]^k$,*

$$\left|\frac{L'(s,t)}{L(s,t)}\right| \leq C|Im(s)|^{\beta},$$

for $Re(s) > 1 - \epsilon$ and $|Im(s)| \geq 1$.

The function $s(t)$ enjoys the following properties.

Lemma 1. *$\nabla s(0) = 0$ and $\nabla^2 s(0)$ is strictly negative definite.*

We shall write $\mathcal{D} = -\nabla^2 s(0)$ and define $\sigma > 0$ by $\sigma^{2k} = \det \mathcal{D}$. The next result is crucial for our subsequent analysis.

Proposition 3. *There exists $\delta > 0$ such that, for $t \in U(\delta\sigma\sqrt{T})$,*

$$\lim_{T\to\infty} e^{(s(t/\sigma\sqrt{T})-h)T} = e^{-\langle t,\mathcal{D}t\rangle/2\sigma^2}.$$

Furthermore, $|e^{(s(t/\sigma\sqrt{T})-h)T}| \leq e^{-\langle t,\mathcal{D}t\rangle/4\sigma^2}$ and

$$\left|e^{(s(t/\sigma\sqrt{T})-h)T} - e^{-\langle t,\mathcal{D}t\rangle/2\sigma^2}\right| \leq 2e^{-\langle t,\mathcal{D}t\rangle/4\sigma^2}.$$

Proof. Let $f(t) = e^{s(t)-h}$. Then $f(0) = 1$, $\nabla f(0) = \nabla s(0) = 0$, and $\nabla^2 f(0) = \nabla^2 s(0) = -\mathcal{D}$. Applying Taylor's Theorem, we have that, for $||t/\sigma\sqrt{T}|| \leq \delta$,

$$f\left(\frac{t}{\sigma\sqrt{T}}\right) = 1 - \frac{\langle t, \mathcal{D}t\rangle}{2\sigma^2 T} + O\left(\frac{||t||^3}{T^{3/2}}\right)$$

(where the implied constant is independent of t). The first statement follows from the identity $\lim_{T\to\infty}(1 - x/T)^T = e^{-x}$.

Provided $\delta > 0$ is sufficiently small, for $||u|| \leq \delta$, we have

$$\langle u, \mathcal{D}u\rangle/2 + O(||u||^3) \geq \langle u, \mathcal{D}u\rangle/4.$$

Since $(1 - x/T)^T < e^{-x}$, this gives us $|f(t/\sigma\sqrt{T})| \leq e^{-\langle t,\mathcal{D}t\rangle/4\sigma^2}$. Applying the triangle inequality, we obtain

$$\begin{aligned}|f(t/\sigma\sqrt{T}) - e^{-\langle t,\mathcal{D}t\rangle/2\sigma^2}| &\leq e^{-\langle t,\mathcal{D}t\rangle/4\sigma^2} + e^{-\langle t,\mathcal{D}t\rangle/2\sigma^2} \\ &\leq 2e^{-\langle t,\mathcal{D}t\rangle/4\sigma^2}.\end{aligned}$$

Remark. The function $s(t)$ has an interpretation in terms of the thermodynamic formalism of the geodesic flow on SM. For a continuous function $G : SM \to \mathbb{R}$, define its pressure $P(G) = \sup_\mu \left\{h_\mu(\phi) + \int G d\mu\right\}$, where the supremum is taken over all probability measures invariant under the geodesic flow. We can define a (smooth) function $F : SM \to \mathbb{R}^k$ with the property that, for each closed geodesic γ, $\int_0^{l(\gamma)} F(\gamma(t), \dot{\gamma}(t))dt = [\gamma]$. Then $\mathbb{R}^k \ni z \mapsto P(\langle z, F\rangle)$ is real analytic and has an analytic extension to a neighbourhood of $\mathbb{R}^k$ in $\mathbb{C}^k$. We have that $s(t) = P(\langle it, F\rangle)$ and that $\mathcal{D} = \nabla^2 P(\langle z, F\rangle)|_{z=0}$ [10], [19].

2. Contour Integration

We shall now use the results on L-functions obtained in the preceding section to examine the behaviour of the summatory function

$$\mathcal{S}_T(t) = {\sum_{l(\gamma')\leq T}}' \Lambda(\gamma')e^{i\langle t,[\gamma']\rangle},$$

as $T \to \infty$. (Here, the $'$ on the summation sign denotes that the terms with $l(\gamma') = T$ are counted with weight 1/2.)

We begin by relating $\mathcal{S}_T(t)$ to $L'(s, t)/L(s, t)$. This is achieved through the following lemma.

Lemma 2 [20, p.132] **(Effective Perron Formula).** *Define a function* $\theta(y)$ *by*

$$\theta(y) = \begin{cases} 0 \ \textit{if}\ 0 < y < 1 \\ \frac{1}{2} \ \textit{if}\ y = 1 \\ 1 \ \textit{if}\ y > 1 \end{cases}.$$

Then, uniformly for $d > 0$, $R > 0$,

$$\left|\theta(y) - \frac{1}{2\pi i}\int_{d-iR}^{d+iR} \frac{y^s}{s} ds\right| = O\left(\frac{y^d}{1 + R|\log y|}\right).$$

Set $d = h + T^{-1}$ and $R = T^K$ (where $K > 0$ will be chosen later). Applying Lemma 2 term-by-term to $-L'(s,t)/L(s,t)$, we obtain

$$\mathcal{S}_T(t) = \frac{1}{2\pi i}\int_{d-iR}^{d+iR}\left(-\frac{L'(s,t)}{L(s,t)}\right)\frac{e^{sT}}{s}ds + O\left(\sum_{\gamma'}\frac{\Lambda(\gamma')e^{dT}e^{-dl(\gamma')}}{1 + R|T - l(\gamma')|}\right). \tag{2.1}$$

We will estimate the big-O term in this expression. First set $\epsilon = T^{-M}$ (where $M > 0$ will be chosen later) and consider the terms for which $|T - l(\gamma')| \le \epsilon$. We will use the following result contained in [16].

Proposition 4 [16]**.** *There exists $c < h$ such that*

$$\#\{\gamma' : l(\gamma') \le T\} = \int_2^{e^{hT}} \frac{1}{\log u} du + O(e^{cT}).$$

As a consequence, we may write

$$\#\{\gamma' : |T - l(\gamma')| \le \epsilon\} = \int_{e^{hT-h\epsilon}}^{e^{hT+h\epsilon}} \frac{1}{\log u} du + O(e^{cT}) = O\left(\frac{\epsilon e^{hT}}{T}\right).$$

Furthermore, if $|T - l(\gamma')| \le \epsilon$ then $e^{dT}e^{-dl(\gamma')} = O(1)$. Thus

$$\sum_{|T-l(\gamma')|\le\epsilon} \frac{\Lambda(\gamma')e^{dT}e^{-dl(\gamma')}}{1 + R|T - l(\gamma')|} = O\left(\frac{e^{hT}}{T^M}\right).$$

On the other hand,

$$\sum_{|T-l(\gamma')|>\epsilon} \frac{\Lambda(\gamma')e^{dT}e^{-dl(\gamma')}}{1 + R|T - l(\gamma')|} \le \frac{e^{dT}}{R\epsilon}\sum_{\gamma'}\Lambda(\gamma')e^{-dl(\gamma')} = O\left(\frac{e^{hT}}{T^{K-M-1}}\right),$$

where we have used the estimate

$$\left|\frac{L'(h + T^{-1}, 0)}{L(h + T^{-1}, 0)}\right| = O(T).$$

Combining the estimates above, equation (2.1) becomes

$$\mathcal{S}_T(t) = \frac{1}{2\pi i}\int_{d-iR}^{d+iR}\left(-\frac{L'(s,t)}{L(s,t)}\right)\frac{e^{sT}}{s}ds + O\left(\frac{e^{hT}}{T^{\min\{M,K-M-1\}}}\right). \tag{2.2}$$

Lemma 3. *For all $N \geq 1$ we have the following estimates. (The implied constants are independent of t.)*

(i) *For $t \in U(\delta)$,*

$$\mathcal{S}_T(t) = \frac{e^{s(t)T}}{s(t)} + O\left(\frac{e^{hT}}{T^N}\right);$$

(ii) *For $t \notin U(\delta)$,*

$$\mathcal{S}_T(t) = O\left(\frac{e^{hT}}{T^N}\right).$$

Proof. Choose $h - \epsilon < c < h$ and let Γ denote the contour formed by the rectangle with vertices at $d - iR$, $d + iR$, $c + iR$, and $c - iR$, oriented counter-clockwise.

(i) Suppose that $t \in U(\delta)$. By Proposition 1(i) we can choose $c < s(t)$ so that, using the Residue Theorem,

$$\frac{1}{2\pi i}\int_\Gamma \left(-\frac{L'(s,t)}{L(s,t)}\right)\frac{e^{sT}}{s}ds = \frac{e^{s(t)T}}{s(t)}.$$

Using Proposition 2, we also have the following bounds:

(a)

$$\left|\left(\int_{c+iR}^{d+iR} + \int_{c-iR}^{d-iR}\right)\left(-\frac{L'(s,t)}{L(s,t)}\right)\frac{e^{sT}}{s}ds\right| = O(R^{\beta-1}e^{hT}) = O\left(\frac{e^{hT}}{T^{K(1-\beta)}}\right);$$

(b)

$$\left|\int_{c-iR}^{c+iR}\left(-\frac{L'(s,t)}{L(s,t)}\right)\frac{e^{sT}}{s}ds\right| = O(R^\beta e^{cT}) = O(T^{\beta K}e^{cT}).$$

Combining this with (2.2) gives

$$\mathcal{S}_T(t) = \frac{e^{s(t)T}}{s(t)} + O\left(\frac{e^{hT}}{T^N}\right),$$

where

$$N = \min\{M, K - M - 1, K(1-\beta)\}.$$

Since K and M are arbitrary, we may take N as large as we please.

(ii) Suppose that $t \notin U(\delta)$. Then, by Proposition 1(ii),

$$\frac{1}{2\pi i}\int_\Gamma \left(-\frac{L'(s,t)}{L(s,t)}\right)\frac{e^{sT}}{s}ds = 0.$$

The result now follows as in the proof of (i).

3. An Auxiliary Function

In this section, we shall prove a result analogous to Theorem 1 but where $\pi(T,\alpha)$ is replaced by the auxiliary function

$$\psi(T,\alpha) = \sideset{}{'}\sum_{\substack{l(\gamma')\leq T\\ [\gamma']=\alpha}} \Lambda(\gamma'),$$

which can be related to the sums $\mathcal{S}_T(t)$ considered in the previous section. We shall adapt an approach used by Rousseau-Egele [18] to examine the quantity $\sigma^k T^{k/2} e^{-hT}\psi(T,\alpha)$. For $a>0$, write $I(a) = [-a,a]^k$. Using the orthogonality relationship

$$\frac{1}{(2\pi)^k}\int_{I(\pi)} e^{-i\langle t,\alpha\rangle} e^{i\langle t,y\rangle} dt = \begin{cases} 1 \text{ if } y=\alpha \\ 0 \text{ if } y\in\mathbb{Z}^k\backslash\alpha \end{cases},$$

we have that

$$\psi(T,\alpha) = \frac{1}{(2\pi)^k}\int_{I(\pi)} e^{-i\langle t,\alpha\rangle}\mathcal{S}_T(t)dt.$$

Making the substitution $t\mapsto t/\sigma\sqrt{T}$, we obtain

$$\sigma^k T^{k/2}\psi(T,\alpha) = \frac{1}{(2\pi)^k}\int_{I(\pi\sigma\sqrt{T})} e^{-i\langle t,\alpha\rangle/\sigma\sqrt{T}}\mathcal{S}_T(t/\sigma\sqrt{T})dt.$$

The next result is the key to the proof of Theorem 1.

Proposition 5.

$$\lim_{T\to\infty}\sup_{\alpha\in\mathbb{Z}^k}\left|\frac{h\sigma^k T^{k/2}}{e^{hT}}\psi(T,\alpha) - \frac{1}{(2\pi)^{k/2}}e^{-\langle\alpha,\mathcal{D}^{-1}\alpha\rangle/2T}\right| = 0.$$

Using the identity,

$$e^{-\langle\alpha,\mathcal{D}^{-1}\alpha\rangle/2T} = \frac{1}{(2\pi)^{k/2}}\int_{\mathbb{R}^k} e^{i\langle t,\alpha\rangle/\sigma\sqrt{T}} e^{-\langle t,\mathcal{D}t\rangle/2\sigma^2}dt,$$

we have established the bound

$$\begin{aligned}
&(2\pi)^k\left|\frac{h\sigma^k T^{k/2}}{e^{hT}}\psi(T,\alpha) - \frac{e^{-\langle\alpha,\mathcal{D}^{-1}\alpha\rangle/2T}}{(2\pi)^{k/2}}\right| \\
&\leq \left|\int_{U(\delta\sigma\sqrt{T})} e^{-i\langle t,\alpha\rangle/\sigma\sqrt{T}}\left\{he^{-hT}\mathcal{S}_T(t/\sigma\sqrt{T}) - e^{-\langle t,\mathcal{D}t\rangle/2\sigma^2}\right\}dt\right| \\
&+ \left|\int_{I(\pi\sigma\sqrt{T})\backslash U(\delta\sigma\sqrt{T})} e^{-i\langle t,\alpha\rangle/\sigma\sqrt{T}} he^{-hT}\mathcal{S}_T(t/\sigma\sqrt{T})dt\right| \\
&+ \left|\int_{\mathbb{R}^k\backslash U(\delta\sigma\sqrt{T})} e^{-i\langle t,\alpha\rangle/\sigma\sqrt{T}} e^{-\langle t,\mathcal{D}t\rangle/2\sigma^2}dt\right| \\
&= A_1(T,\alpha) + A_2(T,\alpha) + A_3(T,\alpha).
\end{aligned}$$

An easy calculation shows that

$$\lim_{T\to\infty}\sup_{\alpha\in\mathbb{Z}^k} A_3(T,\alpha)=0,$$

so, to complete the proof of Proposition 5, it remains to estimate $A_1(T,\alpha)$ and $A_2(T,\alpha)$. To do this we shall use the information on $s(t)$ and $\mathcal{S}_T(t)$ contained in Proposition 3 and Lemma 3.

Lemma 4. *There exists $C>0$ such that, for all sufficiently small $\delta>0$,*

$$\lim_{T\to\infty}\sup_{\alpha\in\mathbb{Z}^k} A_1(T,\alpha)\le C\left\{\int_{\mathbb{R}^k} e^{-\langle t,\mathcal{D}t\rangle/4\sigma^2}dt\right\}\delta^2.$$

Proof. By Lemma 3, we have that, for $t\in U(\delta\sigma\sqrt{T})$,

$$he^{-hT}\mathcal{S}_T(t/\sigma\sqrt{T})=\frac{he^{(s(t/\sigma\sqrt{T})-h)T}}{s(t/\sigma\sqrt{T})}+O(T^{-(k/2+1)}).$$

Using the analyticity of $s(t)$ and the fact that $\nabla s(0)=0$, we have

$$\left|e^{(s(t/\sigma\sqrt{T})-h)T}\left(\frac{h}{s(t/\sigma\sqrt{T})}-1\right)\right|\le C\delta^2 e^{-\langle t,\mathcal{D}t\rangle/4\sigma^2},$$

for some constant $C>0$. Thus,

$$A_1(T,\alpha)\le\int_{U(\delta\sigma\sqrt{T})}\left|e^{(s(t/\sigma\sqrt{T})-h)T}-e^{-\langle t,\mathcal{D}t\rangle/2\sigma^2}\right|dt$$
$$+C\delta^2\int_{U(\delta\sigma\sqrt{T})} e^{-\langle t,\mathcal{D}t\rangle/4\sigma^2}dt+O\left(\frac{1}{T}\right).$$

By Proposition 3, we know that $e^{(s(t/\sigma\sqrt{T})-h)T}$ converges to $e^{-\langle t,\mathcal{D}t\rangle/2\sigma^2}$, as $T\to\infty$. Furthermore, we have the estimate

$$\left|e^{(s(t/\sigma\sqrt{T})-h)T}-e^{-\langle t,\mathcal{D}t\rangle/2\sigma^2}\right|\le 2e^{-\langle t,\mathcal{D}t\rangle/4\sigma^2}.$$

Hence, applying the Dominated Convergence Theorem, we obtain the desired result.

Lemma 5.

$$\lim_{T\to\infty}\sup_{\alpha\in\mathbb{Z}^k} A_2(T,\alpha)=0.$$

Proof. By Lemma 3(ii), for $t\notin U(\delta/\sigma\sqrt{T})$,

$$e^{-hT}\mathcal{S}_T(t/\sigma\sqrt{T})=O(T^{-(k/2+1)}),$$

so that $\sup_{\alpha\in\mathbb{Z}^k} A_2(T,\alpha)=O(T^{-1})$.

Proof of Proposition 5. Combining the above results we have that

$$\lim_{T\to\infty}\sup_{\alpha\in\mathbb{Z}^k}\left|\frac{h\sigma^k T^{k/2}}{e^{hT}}\psi(T,\alpha)-\frac{1}{(2\pi)^{k/2}}e^{-\langle\alpha,\mathcal{D}^{-1}\alpha\rangle/2T}\right|\le C\left\{\int_{\mathbb{R}^k} e^{-\langle t,\mathcal{D}t\rangle/4\sigma^2}dt\right\}\delta^2.$$

Since this holds for all sufficiently small $\delta>0$, the proof is complete.

4. Proof of Theorem 1

In this section we will use elementary arguments to deduce Theorem 1 from Proposition 5. Whenever we make a big-O estimate, the implied constant will be independent of α.

Write

$$\psi^*(T,\alpha) = \sum_{\substack{l(\gamma)\le T\\ [\gamma]=\alpha}} l(\gamma).$$

An easy argument shows that

$$\psi(T,\alpha) = \psi^*(T,\alpha) + O(T^2 e^{hT/2}).$$

Thus Proposition 5 implies the following.

Proposition 6.

$$\lim_{T\to\infty} \sup_{\alpha\in\mathbb{Z}^k} \left| \frac{h\sigma^k T^{k/2}}{e^{hT}} \psi^*(T,\alpha) - \frac{1}{(2\pi)^{k/2}} e^{-\langle \alpha, \mathcal{D}^{-1}\alpha\rangle/2T} \right| = 0.$$

Finally we consider $\pi(T,\alpha)$. It is easy to see that

$$\psi^*(T,\alpha) \le T\pi(T,\alpha).$$

For the corresponding lower bound, choose $\tau > 0$ and set $\theta = (1+\tau)^{-1} < 1$. Then

$$\begin{aligned}
\frac{T\pi(T,\alpha)}{e^{hT}} &= \frac{T}{e^{hT}} \sum_{\substack{\theta T < l(\gamma)\le T\\ [\gamma]=\alpha}} 1 + \frac{T\pi(\theta T,\alpha)}{e^{hT}} \\
&\le \frac{1+\tau}{e^{hT}} \sum_{\substack{\theta T < l(\gamma)\le T\\ [\gamma]=\alpha}} l(\gamma') + \frac{T\pi(\theta T,\alpha)}{e^{hT}} \\
&\le \frac{(1+\tau)\psi^*(T,\alpha)}{e^{hT}} + \frac{T\#\{\gamma : l(\gamma)\le \theta T\}}{e^{hT}}.
\end{aligned}$$

Using the estimate $\#\{\gamma : l(\gamma) \le T\} = O(e^{hT}/T)$ [13], we have established

$$\begin{aligned}
0 &\le \frac{T^{k/2+1}}{e^{hT}} \pi(T,\alpha) - \frac{T^{k/2}}{e^{hT}} \psi^*(T,\alpha) \\
&\le \frac{\tau T^{k/2}}{e^{hT}} \psi^*(T,\alpha) + O(T^{k/2} e^{(\theta-1)hT}),
\end{aligned}$$

so that, by applying Proposition 6,

$$\limsup_{T\to\infty} \sup_{\alpha\in\mathbb{Z}^k} \left| \frac{T^{k/2+1}}{e^{hT}} \pi(T,\alpha) - \frac{T^{k/2}}{e^{hT}} \psi^*(T,\alpha) \right| \le \frac{\tau}{(2\pi)^{k/2} h\sigma^k}.$$

Since $\tau > 0$ is arbitrary, this proves Theorem 1.

5. Homologically Full Anosov Flows

The asymptotic identity identity (0.1) has been generalized to certain transitive Anosov flows $\phi_t : N \to N$, where N is a compact smooth Riemannian manifold. We now use γ to denote a (prime) periodic orbit of ϕ, with least period $l(\gamma)$. Once again, we write $[\gamma]$ for the torsion-free part of the homology class of γ in $H_1(N,\mathbb{Z}) \cong \mathbb{Z}^k \oplus \mathrm{Tor}$. We say that ϕ is *homologically full* if every homology class in $H_1(N,\mathbb{Z})$ is represented by a closed orbit. In this case, there exist $\xi \in H^1(N,\mathbb{R})$, $0 < h^* \leq h$ and $C_0 > 0$ such that

$$\pi(T,\alpha) \sim C_0 e^{-\langle \xi,\alpha\rangle} \frac{e^{h^*T}}{T^{k/2+1}}, \quad \text{as } T \to \infty. \tag{5.1}$$

This result was first proved in [19], drawing on ideas from [10]. An alternative proof was given in [2] and a more precise version is contained in [17]. In this section, we shall sketch a new proof of (5.1), using the techniques discussed above. However, we will not make any claims about uniformity.

Remark. We can define a function $\mathfrak{p} : H^1(N,\mathbb{R}) \to R$ by $\mathfrak{p}([\omega]) = P(\omega(\mathcal{X}))$, where ω is a closed 1-form representing the cohomology class $[\omega]$ and $\mathcal{X}$ is the vector field tangent to ϕ. Then ξ and h^* are characterized by the formulae

$$h^* = \mathfrak{p}(\xi) = \min\{\mathfrak{p}(\xi') : \xi' \in H^1(N,\mathbb{R})\}.$$

We begin by considering a modified family of L-functions. We define

$$L(s,t) = \prod_\gamma \left(1 - e^{-sl(\gamma)+\langle\xi,\alpha\rangle + i\langle t,[\gamma]\rangle}\right)^{-1}, \tag{5.2}$$

which converges for $Re(s) > h^*$. The extension of L'/L to a uniform strip, described in Propositions 1 and 2, is no longer valid; however, the next result provides a weaker substitute. As in the case of closed geodesics, an analysis due to Dolgopyat [5] is crucial here. For $\rho > 0$, write

$$\mathcal{R}(\rho) = \{s : Re(s) > h^* - |Im(s)|^{-\rho},\ |Im(s)| \geq 1\}.$$

Proposition 7 ([17])**.** *There exists a constant $\rho > 0$ such that, for all sufficiently small $\delta > 0$, the following statements are true.*

(i) *There exists an analytic function $s : U(\delta) \to \{z \in \mathbb{C} : Re(z) \leq z\}$, satisfying $s(0) = h^*$ and $Re(s(t)) < h^*$ for $t \neq 0$, such that*

$$\frac{L'(s,t)}{L(s,t)} + \frac{1}{s - s(t)}$$

is analytic in $\mathcal{R}(\rho)$.

(ii) *For $t \notin U(\delta)$, $L'(s,t)/L(s,t)$ is analytic in $\mathcal{R}(\rho)$.*

Proposition 8 ([17])**.** *There exist $C > 0$ and $\beta > 0$, such that, for all $t \in [-\pi, \pi]^k$,*

$$\left| \frac{L'(s,t)}{L(s,t)} \right| \le C|Im(s)|^\beta,$$

for $s \in \mathcal{R}(\rho)$.

Although the function $s(t)$ is now complex valued, it is still the case that $\nabla s(0) = 0$ and that $\nabla^2 s(0)$ is real and strictly negative definite. Moreover, again writing $\mathcal{D} = -\nabla^2 s(0)$ and $\sigma^{2k} = \det \mathcal{D}$, the function $e^{s(t)-h^*}$ still satisfies the conclusions of Proposition 3.

We shall now mimic the arguments of Section 2. However, the weaker bounds on $L'(s,t)/L(s,t)$ force us to use a more complicated auxiliary function. For $n \ge 0$, define

$$\psi_n(T,\alpha) = \frac{e^{\langle \xi, \alpha \rangle}}{n!} \sum_{\substack{l(\gamma') \le T \\ [\gamma'] = \alpha}}{}' \Lambda(\gamma') \left(e^{h^* T} - e^{h^* l(\gamma')} \right)^n.$$

Then we have the identity

$$\sigma^k T^{k/2} \psi_n(T,\alpha) = \frac{1}{(2\pi)^k} \int_{I(\pi\sigma\sqrt{T})} e^{-i\langle t,\alpha\rangle/\sigma\sqrt{T}} \mathcal{S}_T^*(t/\sigma\sqrt{T}) dt,$$

where

$$\mathcal{S}_T^*(t) = \sum_{l(\gamma') \le T}{}' \Lambda(\gamma') e^{\langle \xi, [\gamma'] \rangle + i\langle t, [\gamma'] \rangle} \left(e^{h^* T} - e^{h^* l(\gamma')} \right)^n.$$

In order to estimate the function $\mathcal{S}_T^*(t)$ we need the following identity, for $d > h^*$,

$$\mathcal{S}_T^*(t) = \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \left(-\frac{L'(s,t)}{L(s,t)} \right) \frac{e^{(s+n)T}}{s(s+1)\cdots(s+n)} ds, \tag{5.3}$$

where we have used the formula

$$\frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \frac{y^s}{s(s+1)\cdots(s+n)} ds = \begin{cases} 0 \text{ if } 0 < y < 1 \\ \frac{1}{n!} \left(1 - \frac{1}{y} \right)^n \text{ if } y \ge 1 \end{cases}.$$

Choose $0 < \epsilon < 1/\rho$ and set $R = T^\epsilon$ and $d = h^* + T^{-1}$. Then replacing the integral in (5.3) with the truncated integral $\int_{d-iR}^{d+iR}$ introduces an error of order $O(e^{(h^*+n)T}/T^{\epsilon n})$. Using the estimates

(a)

$$\left| \left(\int_{h^*-R^{-\rho}+iR}^{d+iR} + \int_{h^*-R^{-\rho}-iR}^{d-iR} \right) \left(-\frac{L'(s,t)}{L(s,t)} \right) \frac{e^{(s+n)T}}{s(s+1)\cdots(s+n)} ds \right|$$
$$= O(R^{\beta-\rho-n-1} e^{(h^*+n)T}) = O(e^{(h^*+n)T} T^{-\epsilon(\rho+n+1-\beta)});$$

(b)

$$\left| \int_{h^*-R^{-\rho}\pm i}^{h^*-R^{-\rho}\pm iR} \left(-\frac{L'(s,t)}{L(s,t)} \right) \frac{e^{(s+n)T}}{s(s+1)\cdots(s+n)} ds \right|$$
$$= O(R^\beta e^{(h^*+n-R^{-\rho})T}) = O(T^{\beta\epsilon} e^{(h^*+n)T} e^{-T^{1-\epsilon\rho}}),$$

we may repeat the proof of Lemma 3 to obtain the following lemma.

Lemma 6. *Setting $N = \min\{\epsilon n, \epsilon(\rho + n + 1 - \beta)\}$, we have the following. (The implied constants are independent of t.)*

(i) *For $t \in U(\delta)$,*

$$\mathcal{S}_T^*(t) = \frac{e^{(s(t)+n)T}}{s(t)(s(t)+1)\cdots(s(t)+n)} + O\left(\frac{e^{(h^*+n)T}}{T^N}\right);$$

(ii) *For $t \notin U(\delta)$, $\mathcal{S}_T^*(t) = O(e^{(h^*+n)T}/T^N)$.*

Provided n is sufficiently large that $N > k/2$, we may repeat the arguments used in the proof of Proposition 5 to obtain

$$\lim_{T\to\infty} \sup_{\alpha\in\mathbb{Z}^k} \left| \prod_{j=0}^{n}(h^*+j)\frac{\sigma^k T^{k/2}}{e^{(h^*+n)T}}\psi_n(T,\alpha) - \frac{1}{(2\pi)^{k/2}} e^{-\langle \alpha, \mathcal{D}^{-1}\alpha\rangle/2T} \right| = 0.$$

From this it immediately follows that

$$\psi_n(T,\alpha) \sim \frac{1}{(2\pi)^{k/2}\sigma^k} \prod_{j=0}^{n} \frac{1}{(h^*+j)} \frac{e^{(h^*+n)T}}{T^{k/2}}, \quad \text{as } T \to \infty.$$

The asymptotic formula

$$\psi_0(T,\alpha) \sim \frac{1}{(2\pi)^{k/2} h^* \sigma^k} \frac{e^{h^*T}}{T^{k/2}}, \quad \text{as } T \to \infty,$$

now follows by a standard inductive argument (cf. p.35 of [7]). Finally, (5.1) may be deduced as in section 3. (Note that one needs the *a priori* estimate $\limsup_{T\to\infty}(\pi(T,\alpha))^{1/T} \leq e^{h^*}$, which follows from the convergence of (5.2) for $Re(s) > h^*$.)

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