On the Hannay-Ozorio de Almeida sum formula

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Abstract

In this note we consider the well known Hannay-Ozorio de Almeida sum formula from a mathematically rigorous viewpoint. In particular, we discuss situations where we can obtain the Sinai-Ruelle-Bowen measure as a limit taken over periodic orbits with periods in an interval which shrinks as it moves to infinity.

1 Introduction

The Hannay-Ozorio de Almeida sum formula is a useful principle in the study of the distribution of closed orbits for Hamiltonian flows [7]. Roughly speaking, it asserts that an appropriately weighted sum of measures supported on periodic orbits converges to the physical measure as the periods become large. This formula was originally introduced and used in the study of Quantum Chaos. In particular, Berry used the so-called diagonal approximation and the Hannay-Ozorio de Almeida sum rule to determine the asymptotics of the spectral form factor, which is the Fourier transform of the two-point correlation function for the eigenvalues of the Laplacian [1], [8], [6]. The traditional setting is in the context of Hamiltonian flows, which include the canonical example of geodesic flows on negatively curved manifolds.

Let us now give a brief description of the formula in the context of a C^2 attracting hyperbolic flows $\phi_t : \Lambda \to \Lambda$, where the attractor Λ is contained in a Riemannian manifold M. Let τ denote a (prime) periodic orbit and let $\lambda(\tau)$ denote its least period. Let $f : \Lambda \to \mathbb{R}$ be a continuous function, then we can introduce a weighted period $\lambda_f(\tau) = \int_0^{\lambda(\tau)} f(\phi_t x_\tau)dt$, where $x_\tau \in \tau$. In particular, if we define the expansion coefficient $E : \Lambda \to \mathbb{R}$ by

$$E(x) := \lim_{t\to 0} \frac{1}{t} \log |\mathrm{Jac}(D\phi_t|E^u(x))|$$

then we shall write $\lambda^u(\tau) = \lambda_E(\tau)$. In this setting, one version of the Hannay-Ozorio de Almeida sum formula takes the following form:

$$\lim_{T\to+\infty} \frac{1}{\delta} \sum_{T-\frac{\delta}{2}\le\lambda(\tau)\le T+\frac{\delta}{2}} \lambda_f(\tau)e^{-\lambda^u(\tau)} = \int f \, d\mu, \tag{0.1}$$

where μ is the SRB (Sinai-Ruelle-Bowen) measure, i.e., the unique ϕ_t-invariant probability measure which is absolutely continuous with respect to the volume on M. William Parry was one of the first people to make a mathematically rigorous study of such results. In particular, he gave a completely rigorous proof of (0.1) in the very general setting of weak mixing Axiom A flows and a general class of Hölder weights [11], [12].

In this note we want to address the question of whether $\delta = \delta(T)$ can be allowed to shrink to zero as T increases and, if so, at what rate. This seems a natural question from both a mathematical and physical perspective, given that there is no natural choice of scale for δ.

Our main results are the following theorems which strengthen (0.1), in the appropriate settings. The first theorem is in the special case of geodesic flows.

Theorem 1.1. *Let $\phi_t : M \to M$ be the geodesic flow on the unit-tangent bundle over a compact negatively curved surface. If there exists $\epsilon > 0$ such that $\delta(T)^{-1} = O(e^{\epsilon T})$ then, for Hölder continuous functions $f : M \to \mathbb{R}$,*

$$\lim_{T\to+\infty} \frac{1}{\delta(T)} \sum_{T-\frac{\delta(T)}{2}\le\lambda(\tau)\le T+\frac{\delta(T)}{2}} \lambda_f(\tau)e^{-\lambda^u(\tau)} = \int f \, d\mu \tag{0.2}$$

The proof of Theorem 1.1 is based on estimates of Dolgopyat originally used in the proof of exponential mixing of geodesic flows [4]. In fact, the conclusion actually holds for any contact Anosov flows for which the stable and unstable foliations which are non-jointly integrable. In particular, it holds for the geodesic flow on the unit tangent bundle of a compact manifold with negative sectional curvatures, provided these curvatures are pinched between -1 and $-\frac{1}{4}$.

Definition 1.2. *We say that β is Diophantine if there exist $\alpha > 2$ and $C > 0$ for which there are no rationals p/q satisfying $|\beta - p/q| \le C/q^\alpha$.*

Our second theorem is the following.

Theorem 1.3. *Let $\phi_t : \Lambda \to \Lambda$ be a weak mixing C^2 hyperbolic attracting flow. Assume that we can chose two distinct closed orbits τ_1 and τ_2 such that $\beta = \lambda(\tau_1)/\lambda(\tau_2)$ is Diophantine. If there exists $\gamma > 0$ such that $\delta(T)^{-1} = O(T^\gamma)$ then, for Hölder continuous functions $f : \Lambda \to \mathbb{R}$,*

$$\lim_{T\to+\infty} \frac{1}{\delta(T)} \sum_{T-\frac{\delta(T)}{2}\le\lambda(\tau)\le T+\frac{\delta(T)}{2}} \lambda_f(\tau)e^{-\lambda^u(\tau)} = \int f \, d\mu \tag{0.3}$$

The proof of Theorem 1.3 is based on estimates of Dolgopyat used to establish polynomial rates of mixing in a wider setting [5]. In particular, the conclusion holds for any weak mixing C^2 Anosov flow.

Remark 1.4. Complementary results to Theorems 1.1 and 1.3 are obtained by fixing $\delta > 0$ and asking about the rate of convergence in (0.1). However, this follows easily using the ideas in [14], [15]. The results are the following.

1. Let ϕ_t be the geodesic flow on the unit-tangent bundle of a compact negatively curved surface and let $f : M \to \mathbb{R}$ be a Hölder continuous function. Then there exists $\epsilon > 0$ such that we have that

$$\frac{1}{\delta} \sum_{T-\frac{\delta}{2}\le\lambda(\tau)\le T+\frac{\delta}{2}} \lambda_f(\tau)e^{-\lambda^u(\tau)} = \int f \, d\mu + O(e^{-\epsilon T}), \text{ as } T \to +\infty.$$

2. Let ϕ_t be a weak mixing hyperbolic attracting flow and let $f : M \to \mathbb{R}$ be a Hölder continuous function. Assume that we can find two distinct closed orbits τ_1 and τ_2 such that $\beta = \lambda(\tau_1)/\lambda(\tau_2)$ is Diophantine. Then there exists $\eta > 0$ such that we have that

$$\frac{1}{\delta} \sum_{T-\frac{\delta(T)}{2} \le \lambda(\tau) \le T+\frac{\delta(T)}{2}} \lambda_f(\tau) e^{-\lambda^u(\tau)} = \int f \, d\mu + O(T^{-\eta}), \text{ as } T \to +\infty$$

Throughout the paper, we use the standard Landau big O and little o notation, i.e, we write $A(T) = O(B(T))$ if there exists $D > 0$ such that $|A(T)| \le DB(T)$ and $A(T) = o(B(T))$ if $|A(T)|/B(T) \to 0$, as $T \to +\infty$.

2 Hyperbolic flows and symbolic dynamics

Let $\phi_t : M \to M$ be a C^∞ flow on a compact manifold. Let Λ be a closed ϕ-invariant subset. We call the set Λ *hyperbolic* if:

1. there exists a $D\phi$-invariant splitting $T_\Lambda M = E^0 \oplus E^s \oplus E^u$ and constants $C > 0$ and $\lambda > 0$ such that
   (a) E^0 is tangent to the direction of the flow,
   (b) $\|D\phi_t | E^u\| \le Ce^{-\lambda t}$, for $t \ge 0$,
   (c) $\|D\phi_{-t} | E^s\| \le Ce^{-\lambda t}$, for $t \ge 0$;
2. the periodic orbits in Λ are dense;
3. the flow restricted to Λ has a dense orbit; and
4. there exists an open set $U \supset \Lambda$ such that $\Lambda = \cap_{t \in \mathbb{R}} \phi_t U$.

We call the restriction of the flow $\phi_t : \Lambda \to \Lambda$ a *hyperbolic* flow. If $\Lambda = \cap_{t>0} \phi_t U$ then we say that ϕ_t is an attracting hyperbolic flow or, more succinctly, a *hyperbolic attractor*. For any $x \in \Lambda$ we denote the associated unstable manifold by

$$W^u(x) = \{y \in M : \lim_{t\to\infty} d(\phi_t x, \phi_t y) = 0\},$$

and if Λ is an attractor then $W^u(x) \subset \Lambda$.

If a hyperbolic attractor is C^2 then it supports a unique probability measure which is both invariant and absolutely continuous with respect to the natural volume induced on each unstable manifold by the ambient Riemannian volume $\mathfrak{m}$. This measure, which we denote by μ, is called the Sinai-Ruelle-Bowen measure and describes the behaviour of $\mathfrak{m}$-almost every point in a neighbourhood of the attractor [3].

Example 2.1 (Geodesic flow). *Let M be the unit tangent bundle of a compact C^∞ surface V, i.e., the tangent vectors to V of unit length. The geodesic flow $\phi_t : M \to M$ is defined as follows. Given a unit tangent vector v we consider the unit speed geodesic $\gamma_v : M \to M$ such that $\dot\gamma_v(0) = v$. We then define $\phi_t(v) = \dot\gamma_v(t)$. If V has negative curvature then the associated geodesic flow is a hyperbolic attractor with $\Lambda = M$. Here μ is the Liouville measure.*

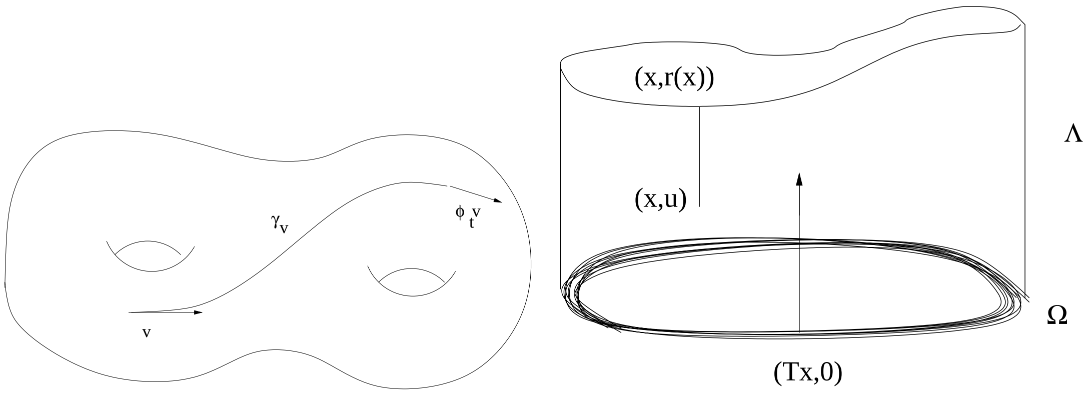

Figure 1: (i) The geodesic flow moves the unit tangent vector v along the geodesic γ_v to $\phi_t v$; (ii) The suspension flow is defined on the area under the graph of $r : \Omega \to \mathbb{R}$.

Example 2.2 (Suspension flow)**.** *Let $T : \Omega \to \Omega$ be a solenoid. Let $r : \Omega \to \mathbb{R}^+$ be a strictly positive Hölder continuous function. We define the flow space by*

$$\Lambda = \{(x, u) \in \Omega \times \mathbb{R} : 0 \leq u \leq r(x)\}$$

where we identify $(x, r(x))$ and $(T(x), 0)$. We define a flow by $\phi_t(x, u) = (x, u + t)$, subject to the identifications.

We shall prove our results via the symbolic description of a hyperbolic flow as a suspended flow over a subshift of finite type. We begin by recalling a few basic definitions and results. Let A be a $k \times k$ aperiodic matrix. We shall then let X be the space

$$X = \{x = (x_n)_{n=-\infty}^{\infty} \in \{1, \cdots, k\}^{\mathbb{Z}} : A(x_n, x_{n+1}) = 1 \text{ for all } n \in \mathbb{Z}\}$$

and define a metric on X by

$$d(x, y) = \sum_{n=-\infty}^{\infty} \frac{1 - \delta(x_n, y_n)}{2^{|n|}},$$

where $\delta(i, j) = 0$ if $i \neq j$ and $\delta(i, i) = 1$. The subshift of finite type $\sigma : X \to X$, defined by $(\sigma x)_n = x_{n+1}$, $n \in \mathbb{Z}$, is a homeomorphism. Given a strictly positive Hölder continuous function $r : X \to \mathbb{R}^+$, let us denote

$$X^r = \{(x, u) \in X \times \mathbb{R} : 0 \leq u \leq r(x)\},$$

where $(x, r(x))$ and $(\sigma x, 0)$ are identified. We can define the suspended flow $\sigma_t^r : X^r \to X^r$ by $\sigma_t^r(x, u) = (x, u + t)$, subject to the identifcations.

To proceed, we state the following, now classical, result.

Lemma 2.3. *Given a hyperbolic flow $\phi_t : \Lambda \to \Lambda$, there exists a subshift of finite type $\sigma : X \to X$, a strictly positive Hölder continuous function $r : X \to \mathbb{R}^+$ and a Hölder continuous semi-conjugacy $\pi : X \to \Lambda$ such that:*

1. *π is one-to-one on a residual set;*

2. *a closed σ-orbit $\{x, \sigma x, \cdots, \sigma^{n-1}x\}$ projects to a closed orbit τ of period $\lambda(\tau) = r^n(x) := r(x) + r(\sigma x) + ... + r(\sigma^{n-1}x)$. Moreover, if we define $g : X \to \mathbb{R}$ by*

$$g(x) = -\int_0^{r(x)} E(\pi(x, u))du$$

then g is Hölder continuous and $-\lambda^u(\tau) = g^n(x)$.

Proof. This follows from the work of Bowen [2] and Bowen-Ruelle [3]. □

3 Dirichet series

In order to understand the limits in Theorems 1.1 and 1.3, we shall need to study the analytic properties of certain complex functions. We start with the following definition.

Definition 3.1. *Given a non-negative Hölder continuous function $f : \Lambda \to \mathbb{R}$, we formally define an η-function for ϕ_t to be the Dirichlet series*

$$\eta(s) = \sum_{\tau} \sum_{m=1}^{\infty} \lambda_f(\tau) e^{m(-\lambda^u(\tau) - (s-1)\lambda(\tau))}, \ s \in \mathbb{C},$$

where the sum is taken over all prime periodic orbits of ϕ_t.

It is not difficult to show that $\eta(s)$ converges to an analytic function on the half-plane $\mathrm{Re}(s) > 1$, and thus the definition makes sense on this domain. (We refer the reader to [13] for the general theory.) The proofs of Theorems 1.1 and 1.3 require showing that $\eta(s)$ has an analytic extension to a larger domain. To achieve this we need to relate $\eta(s)$ to a complex function defined in terms of X and functions thereon.

Given $f : \Lambda \to \mathbb{R}$ we define $f_0 : X \to \mathbb{R}$ by $f_0(x) = \int_0^{r(x)} f(\pi(x, u))du$.

Definition 3.2. *We define a symbolic η-function by*

$$\eta_0(s) = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\sigma^n x = x} f_0^n(x) e^{g^n(x) - (s-1)r^n(x)}.$$

For a continuous function $w : X \to \mathbb{R}$, we define its pressure $P(w)$ by

$$P(w) = \sup\left\{h_\nu(\sigma) + \int w \, d\nu : \nu \text{ is a } \sigma\text{-invariant probability measure}\right\},$$

where $h_\nu(\sigma)$ denotes the entropy of σ with respect to ν. If w is Hölder continuous then there is a unique measure, called the equilibrium state for w, for which the supremum is attained.

It is a standard result that $\eta_0(s)$ converges to an analytic function for $P(g - \mathrm{Re}(s-1)r) < 0$ [13]. Since $P(g) = 0$, this holds for $\mathrm{Re}(s) > 1$.

The following lemma relates $\eta(s)$ and $\eta_0(s)$.

Lemma 3.3. *There exists $\epsilon > 0$ such that $\eta_0(s) - \eta(s)$ is analytic for* $\mathrm{Re}(s) > 1 - \epsilon$.

Proof. The functions $\eta(s)$ and $\eta_0(s)$ agree up to a small discrepancy (due to overcounting caused by orbits passing through the boundaries of the cross sections used to construct the symbolic dynamics in Lemma 2.3). This can be easily accounted for using the a construction of Bowen [2] (following [10]): the difference of the two functions can be written in terms of functions associated to a finite number of auxiliary subshifts of finite type. There are Hölder continuous maps from each of these to Λ but, crucially, they are not surjective. This forces a strict inequality of pressure functions which implies that the difference $\eta_0(s) - \eta(s)$ is analytic in a strictly larger half-plane than $\mathrm{Re}(s) > 1$. □

One of the interesting features of the present problem is the need to extend the region for which certain functions of two variable are bi-analytic. To address this problem, it is convenient to use some classical results in the theory of several complex variables [9]. We recall that a complex function of two variables is bi-analytic at a point $(z, s) \in \mathbb{C}^2$ if it has a uniformly convergent power series expansion (in two variables) in a neighourhood of the point. Let

$$D(r_1, r_2) = \{(z, w) \in \mathbb{C}^2 : |z| < r_1, |w| < r_2\}$$

denote a polydisc in $\mathbb{C}^2$, where $r_1, r_2 > 0$.

Lemma 3.4 (Hartog's Theorem). *Let $F : D(r_1, r_2) \to \mathbb{C}$ be a function such that*

(i) $F(z, w)$ is bi-analytic on the smaller polydisc $D(r, r_2)$ $(0 < r < r_1)$; and

(ii) for each $|w| < r_2$ the functions $f(\cdot, w) : \{z \in \mathbb{C} : |z| < r_1\} \to \mathbb{C}$ are analytic.

Then $F : D(r_1, r_2) \to \mathbb{C}$ is bi-analytic.

To prove Theorem 1.1 we shall require the following result on $\eta_0(s)$.

Lemma 3.5. *Let ϕ_t be a geodesic flow on a surface of negative curvature. We can write*

$$\eta_0(s) = \frac{\int f \, d\mu}{s - 1} + A(s),$$

where $A(s)$ is analytic for $Re(s) > 1 - \epsilon$, for some $\epsilon > 0$. Furthermore,

$$|\eta(s)| = O(\max\{|\mathrm{Im}(s)|^\rho, 1\}),$$

for some $0 < \rho < 1$.

Proof. Let us define

$$L(s, z) = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \sum_{\sigma^n x = x} e^{g^n(x) - (s-1)r^n(x) + z f_0^n(x)}\right)$$

with $s, z \in \mathbb{C}$ where $g : X \to \mathbb{R}$ and $f_0 : X \to \mathbb{R}$ are as defined above. It is easy to see that function $L(s, z)$ converges to a non-zero and bi-analytic function in (s, z) provided $\mathrm{Re}(s) > 0$ and $|z|$ sufficiently small [9]. Moreover, it follows from the approach in Dolgopyat's paper

[4] (explicitly in the case $z = 0$, or by a simple modification for any fixed z) that we have analyticity of $L(s, z)$ in s for $\mathrm{Re}(s) > 1 - \epsilon$, where $\epsilon > 0$ can be chosen independently of z. The key ingredients in this approach are estimates on the transfer operator $L_{g-zf_0-(s-1)r} : C^\alpha(X^+) \to C^\alpha(X^+)$ defined by

$$L_{g-zf_0-(s-1)r}w(x) = \sum_{\sigma y = x} e^{(g-zf_0-(s-1)r)(y)}w(y)$$

on a suitable family $C^\alpha(X^+)$ of Hölder continuous functions on the corresponding one-sided shift $\sigma : X^+ \to X^+$, where

$$X^+ = \left\{ x = (x_n)_{n=0}^\infty \in \{1, \cdots, k\}^{\mathbb{Z}^+} \; : \; A(x_n, x_{n+1}) = 1 \text{ for all } n \in \mathbb{Z}^+ \right\}.$$

(Here we assume that g, f_0 and r have been replaced by functions in $C^\alpha(X^+)$, chosen so that their sums around periodic orbits remain unchanged. We refer the reader to [13] for more details. To avoid complicating the exposition, we do not change the notation.)

The domain of analyticity corresponds to those (z, s) for which 1 is not in the spectrum of $L_{g-zf_0-(s-1)r}$. Moreover, one can also show that $L(s, z)$ is analytic in a neighbourhood of $s = 1$, provided $s \neq s(z)$, where $s(z)$ is an analytic function with $s(0) = 1$ satisfying $P(g - zf_0 - (s(z) - 1)r) = 0$, for $|z|$ sufficiently small, where $P(\cdot)$ is the analytic extension of the pressure function (i.e., the logarithm of the maximal eigenvalue of the associated transfer operator) [13]. We claim that $L(s, z)^{-1}$ can be differentiated in the second variable at $z = 0$. This is the point in the proof where it is convenient to use the Hartog's Theorem (Lemma 3.4). We have already observed that $L(s, z)^{-1}$ is bi-analytic in the pair of variables (s, z) for $\mathrm{Re}(s) > 0$ and $|z|$ then chosen sufficiently small. We can apply Hartog's Theorem to extend the domain of analyticity to $\mathrm{Re}(s) > 1 - \epsilon/2$, say, and $|z|$ sufficiently small (independent of s). It is now routine to show that $s = 0$ in a simple pole for $\eta_0(s)$ with the claimed residue. Briefly, for s in a suffciently small neighbourhood of 0 we can write

$$\begin{aligned}
\eta_0(s) &= \frac{\partial \log L(s, z)}{\partial z}|_{z=0} \\
&= -\frac{\partial \log(1 - e^{P(g-zf_0-(s-1)r)})}{\partial z}|_{z=0} + A_0(s) \\
&= \frac{1}{s-1} \int f d\mu + A_1(s),
\end{aligned}$$

where $A_0(s)$, $A_1(s)$ are analytic functions in a neighbourhood of $s = 1$ and $\int f d\mu = \int f_0 dm / \int r dm$, where m is the equilibrium state on X^+ for g [13].

To complete the proof, we need bounds on $\eta_0(s)$. There exists $\rho_0 > 0$ such that in the same region we have a bound $L(s, z) = O(|\mathrm{Im}(s)|^{\rho_0})$ for $|\mathrm{Im}(s)| \geq 1$. Moreover, the implied constants are uniform in z (in a small neighbourhood of 0). This is implicit in the details of the proof of the result cf. [4], [14]. Thus, we can use Cauchy's Theorem to obtain

$$\eta_0(s) = \frac{\partial}{\partial z} L(s, z)|_{z=0} = \frac{1}{2\pi i \delta} \int_{|\xi|=\delta} \frac{L(s, \xi)}{\xi^2} d\xi = O(|\mathrm{Im}(s)|^\rho),$$

where $\delta > 0$ is chosen suffciently small.

Finally, as in [14], we may use an argument based on the Phragmén-Lindelöf Theorem, to show that, decreasing ϵ if necessary, ρ may be chosen to be less than 1. □

To prove Theorem 1.3 we shall require the following, somewhat weaker, result on $\eta(s)$.

Lemma 3.6. *Let ϕ be a hyperbolic flow satisfying the hypotheses of Theorem 1.3. We can write*

$$\eta(s) = \frac{\int f d\mu}{s-1} + A(s),$$

where $A(s)$ is analytic for $\mathrm{Re}(s) > 1 - \epsilon \min\{|\mathrm{Im}(s)|^{-\alpha}, 1\}$, for some $\epsilon, \alpha > 0$. Furthermore,

$$|A(s)| = O(\max\{|\mathrm{Im}(s)|^{\rho}, 1\}),$$

for some $\rho > 0$.

Proof. The proof is similar to that of Lemma 3.5. Again the function $L(s,z)$ is bi-analytic in (s,z) for $\mathrm{Re}(s) > 1$ and $|z|$ suffciently small (cf. [13]). This time we apply the approach in Dolgopyat's paper [5] and for fixed z (with $|z|$ sufficiently small) we have analyticity in s for $\mathrm{Re}(s) > 1 - \epsilon \min\{|\mathrm{Im}(s)|^{-\alpha}, 1\}$, for some uniform (in z) choice of $\epsilon > 0$. The uniformity of the implied constants for small $|z|$ is implicit in the proofs. We can again apply Hartog's theorem for functions of two variables to deduce that $L(s,z)$ is bi-analytic in (s,z) for $\mathrm{Re}(s) > 1 - \frac{\epsilon}{2} \min\{|\mathrm{Im}(s)|^{-\alpha}, 1\}$, say, and $|z|$ sufficiently small. The pole free region for $\eta(s)$, the bounds on modulus $|\eta(s)|$ and the form of the pole and residue at $s = 1$ follow by arguments analogous to those in the previous case. □

4 Proof of Theorem 1.1

Given Lemma 3.5, the proof of Theorem 1.1 now follows fairly traditional lines. We recall the following standard identity [17].

Lemma 4.1. *Let $c > 0$ and $k \geq 1$. Then*

$$\frac{1}{2\pi i} \int_{c+i\infty}^{c+i\infty} \frac{T^{s+k}}{s(s+1)\cdots(s+k)} ds = \begin{cases} 0 & \text{if } 0 < T < 1 \\ \frac{1}{k!}(1 - \frac{1}{T})^k & \text{if } T \geq 1. \end{cases}$$

For $T > 0$, we shall write

$$\psi_0(T) = \sum_{e^{m\lambda(\tau)} \leq T} \lambda_f(\tau) e^{m(\lambda(\tau) - \lambda^u(\tau))},$$

where the summation is taken over all prime periodic orbits τ and all $m \geq 1$ satisfying $e^{m\lambda(\tau)} \leq T$.

Lemma 4.2. *Under the hypotheses of Theorem 1.1, there exists $\epsilon' > 0$ such that*

$$\psi_0(T) = \left(\int f \, d\mu \right) T + O(T^{1-\epsilon'}).$$

Proof. We introduce an auxiliary function $\psi_1(T) = \int_1^T \psi_0(u) du$. Using Lemma 3.1, with $k = 1$, we can write

$$\psi_1(T) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \eta(s) \frac{T^{s+1}}{s(s+1)} ds,$$

for any $c > 1$. We want to move the curve of integration to $d = 1 - \epsilon'$, say, where $0 < \epsilon' < \epsilon$, with ϵ as in Lemma 3.5. Since $\eta(s)$ has a simple pole at $s = 1$, we may use the Residue Theorem and the bound $|\eta(s)| = O(|\mathrm{Im}(s)|^\rho)$, for $\rho < 1$, to obtain

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \eta(s)\frac{T^{s+1}}{s(s+1)}ds = \left(\int f d\mu\right)\frac{T^2}{2} + \frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} \eta(s)\frac{T^{s+1}}{s(s+1)}ds. \tag{3.1}$$

Again using the bound on $|\eta(s)|$, the second term on the Right Hand Side of (3.1) can be estimated by

$$\left|\frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} \eta(s)\frac{T^{s+1}}{s(s+1)}ds\right| = O\left(T^{d+1}\int_1^\infty \frac{t^\rho}{t(t+1)}dt\right) = O(T^{2-\epsilon'}),$$

To finish the proof, we need to replace the estimate on $\psi_1(T)$ with one on $\psi_0(T)$. Since $\psi_0(T)$ and $\psi_1(T)$ are both monotone increasing, we may write

$$\begin{aligned}\psi_0(T) \leq \frac{\psi_1(T) - \psi_1(T-\Delta)}{\Delta} &= \left(\int f\ d\mu\right)\left(\frac{T^2 - (T-\Delta)^2}{2\Delta}\right) + O\left(\frac{T^{2-\epsilon'}}{\Delta}\right) \\ &= \left(\int f\ d\mu\right)T + O\left(\frac{T^{2-\epsilon'}}{\Delta}, \Delta\right).\end{aligned}$$

If we choose $\Delta = T^{1-\epsilon'/2}$ then we have that

$$\psi_0(T) \leq \left(\int f\ d\mu\right)T + O(T^{1-\epsilon'/2})$$

Similarly, we can show that

$$\psi_0(T) \geq \left(\int f\ d\mu\right)T + O(T^{1-\epsilon'/2})$$

and the result follows.

□

Lemma 4.3. *For some $\epsilon' > 0$, we have*

$$\sum_{\lambda(\tau)\leq T} \lambda_f(\tau)e^{\lambda(\tau)-\lambda^u(\tau)} = \left(\int f\ d\mu\right)e^T + O(e^{(1-\epsilon')T}).$$

Proof. By Lemma 3.2, we have

$$\sum_{m\lambda(\tau)\leq T} \lambda_f(\tau)e^{m(\lambda(\tau)-\lambda^u(\tau))} = \left(\int f\ d\mu\right)e^T + O(e^{(1-\epsilon')T}),$$

where m runs over all $m \geq 1$. We need to show that the terms with $m \geq 2$ make a contribution of smaller order. By simple estimates

$$\limsup_{T\to+\infty}\frac{1}{T}\log\left(\sum_{m\geq 2\ :\ m\lambda(\tau)\leq T} \lambda_f(\tau)e^{m(\lambda(\tau)-\lambda^u(\tau))}\right) \leq 1 + \sup_{m\geq 2}\frac{P(-mE)}{m},$$

where E is the function defined in the introduction and P denotes pressure. It follows from standard properties of pressure that

(a) $P(-mE) < 0$, for all $m \geq 2$;

(b)

$$\lim_{m\to+\infty} \frac{P(-mE)}{m} = e_- := \inf_\nu \int -E \, d\nu < 0,$$

where the infimum is taken over all ϕ_t-invariant probability measures.

In particular, there exists $N \geq 1$ such that

$$\frac{P(-mE)}{m} \leq \frac{e_-}{2}$$

for $m > N$ and so

$$1 + \sup_{m\geq 2} \frac{P(-mE)}{m} \leq 1 + \max\left\{\frac{P(-2E)}{2}, \ldots, \frac{P(-NE)}{N}, \frac{e_-}{2}\right\} < 1.$$

Decreasing ϵ' if necessary, this gives the required result. $\square$

Proof of Theorem 1.1. Lemma 4.3 shows that

$$\pi_f(T) := \sum_{\lambda(\tau)\leq T} \lambda_f(\tau) e^{\lambda(\tau)-\lambda^u(\tau)} = \left(\int f \, d\mu\right) e^T + O\left(e^{(1-\epsilon')T}\right), \text{ as } T \to +\infty.$$

Thus, for $\delta = \delta(T)$, we can write that

$$\begin{aligned}
\sum_{T-\delta/2\leq\lambda(\tau)\leq T+\delta/2} \lambda_f(\tau) e^{\lambda(\tau)-\lambda^u(\tau)} &= \pi_f\left(T + \frac{\delta}{2}\right) - \pi_f\left(T - \frac{\delta}{2}\right) \\
&= \left(\int f \, d\mu\right) \left(e^{(T+\delta/2)} - e^{(T-\delta/2)}\right) + O\left(e^{(1-\epsilon')T}\right) \\
&= \left(\int f \, d\mu\right) e^T \delta + O\left(e^{(1-\epsilon')T}, \delta^2 e^T\right).
\end{aligned}$$

We then have the asymptotic upper bound

$$\begin{aligned}
\sum_{T-\delta/2\leq\lambda(\tau)\leq T+\delta/2} \lambda_f(\tau) e^{-\lambda^u(\tau)} &\leq \exp\left(-T + \frac{\delta}{2}\right) \sum_{T-\delta/2\leq\lambda(\tau)\leq T+\delta/2} \lambda_f(\tau) e^{\lambda(\tau)-\lambda^u(\tau)} \\
&= \left(\int f \, d\mu\right) \delta \exp\left(\frac{\delta}{2}\right) + O\left(e^{-\epsilon' T}, \delta^2\right) \\
&= \left(\int f \, d\mu\right) \left(\delta + \frac{\delta^2}{2}\right) + O\left(e^{-\epsilon' T}, \delta^2\right) \\
&= \left(\int f \, d\mu\right) \delta + O\left(e^{-\epsilon' T}, \delta^2\right).
\end{aligned}$$

Similarly, we have an asymptotic lower bound

$$\begin{aligned}\sum_{T-\delta/2\le\lambda(\tau)\le T+\delta/2}\lambda_f(\tau)e^{-\lambda^u(\tau)} &\ge \exp\left(-T-\frac{\delta}{2}\right)\sum_{T-\delta/2\le\lambda(\tau)\le T+\delta/2}\lambda_f(\tau)e^{\lambda(\tau)-\lambda^u(\tau)}\\ &= \left(\int f\ d\mu\right)\delta\exp\left(-\frac{\delta}{2}\right)+O\left(e^{-\epsilon' T},\delta^2\right)\\ &= \left(\int f\ d\mu\right)\left(\delta-\frac{\delta^2}{2}\right)+O\left(e^{-\epsilon' T},\delta^2\right)\\ &= \left(\int f\ d\mu\right)\delta+O\left(e^{-\epsilon' T},\delta^2\right).\end{aligned}$$

Comparing these estimates, we see that

$$\frac{1}{\delta}\sum_{T-\delta/2\le\lambda(\tau)\le T+\delta/2}\lambda_f(\tau)e^{-\lambda^u(\tau)}=\int f\ d\mu+O\left(\frac{e^{-\epsilon' T}}{\delta},\delta\right).$$

In particular, providing $\delta(T)\to 0$ as $T\to+\infty$ with $\delta(T)^{-1}=o(e^{\epsilon' T})$ then the estimate (0.2) holds, provided f is non-negative. The result for general f follows from considering positive and negative parts. □

5 Proof of Theorem 1.3

We again write $\psi_0(T)=\sum_{e^{m\lambda(\tau)}\le T}\lambda_f(\tau)e^{m(\lambda(\tau)-\lambda^u(\tau))}$ and $\psi_1(T)=\int_1^T\psi_0(u)du$.

Lemma 5.1. *There exists $a>0$ such that*

$$\psi_0(T)=\left(\int f\ d\mu\right)T+O\left(\frac{T}{(\log T)^a}\right).$$

Proof. First, let us suppose the exponent $\rho>0$ in Lemma 3.6 satisfies $0<\rho<1$. For $c>1$ we can again write

$$\psi_1(T)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\eta(s)\frac{T^{s+1}}{s(s+1)}ds. \tag{3.1}$$

As before we want to move the line of integration to left, however, this time to a curve $\Gamma=\Gamma(T)$ depending on T . More precisely, Γ is the union of the arcs:

1. $\Gamma_0=[1+iR,1+i\infty]$;
2. $\Gamma_1=[d+iR,1+iR]$;
3. $\Gamma_2=[d-iR,d+iR]$;
4. $\Gamma_3=[1-iR,d-iR]$; and
5. $\Gamma_4=[1-i\infty,1-iR]$,

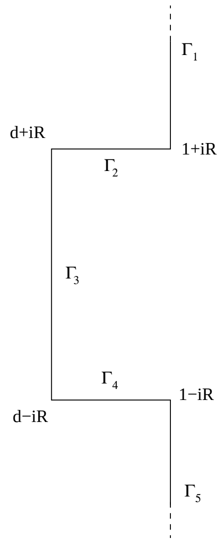

Figure 2: The curve of integration

where $R = R(T) = (\log T)^{\epsilon}$, with $0 < \epsilon < \min\{\frac{\alpha}{2}, \frac{1}{\rho}\}$ and $d = d(T) = 1 - (\log T)^{-1/2}$.

By the Residue Theorem, we can write

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \eta(s)\frac{T^{s+1}}{s(s+1)}ds = \left(\int f\ d\mu\right)\frac{T^2}{2} + \frac{1}{2\pi i}\int_{\Gamma} \eta(s)\frac{T^{s+1}}{s(s+1)}ds. \tag{3.2}$$

Moreover, we can bound

$$\left|\frac{1}{2\pi i}\int_{\Gamma_1\cup\Gamma_3} \frac{T^{s+1}}{s(s+1)}ds\right| = O(R^{\rho-2}T^2) = O\left(\frac{T^2}{(\log T)^{\epsilon(2-\rho)}}\right), \tag{3.3}$$

$$\left|\frac{1}{2\pi i}\int_{\Gamma_0\cup\Gamma_4} \frac{T^{s+1}}{s(s+1)}ds\right| = O\left(\frac{T^2}{R^{1-\rho}}\right) = O\left(\frac{T^2}{(\log T)^{\epsilon(1-\rho)}}\right), \tag{3.4}$$

$$\left|\frac{1}{2\pi i}\int_{\Gamma_2} \frac{T^{s+1}}{s(s+1)}ds\right| = O\left(\frac{T^{2-(\log T)^{-1/2}}}{R^{1-\rho}}\right) = O\left(\frac{T^2 e^{-(\log T)^{\frac{3}{2}}}}{(\log T)^{\epsilon(1-\rho)}}\right). \tag{3.5}$$

We can then estimate

$$\psi_1(T) = \left(\int f\ d\mu\right)\frac{T^2}{2} + O\left(\frac{T^2}{(\log T)^{-a}}\right),$$

for $a > 0$ chosen sufficiently small.

Using the same method as in the proof of Lemma 3.2 we can write

$$\begin{aligned}\psi_0(T-\Delta) \le \frac{\psi_1(T)-\psi_1(T-\Delta)}{\Delta} &= \int f d\mu\left(\frac{T^2-(T-\Delta)^2}{2\Delta}\right) + O\left(\frac{T^2}{\Delta(\log T)^a}\right) \\ &= T\int f d\mu + O\left(\frac{T^2}{\Delta(\log T)^a}, \Delta\right).\end{aligned}$$

If we choose $\Delta = T(\log T)^{-a/2}$ then we have that

$$\psi_0(T-\Delta) \le \left(\int f\ d\mu\right) T + O\left(\frac{T}{(\log T)^{a/2}}\right)$$

and thus

$$\psi_0(T) \le \left(\int f\ d\mu\right) T + O\left(\frac{T}{(\log T)^{a/2}}\right).$$

Modifying the proof of Lemma 3.2, we can also show that

$$\psi_0(T) \ge \left(\int f\ d\mu\right) T + O\left(\frac{T}{(\log T)^{a/2}}\right)$$

and the result follows.

More generally, if $k-1 \le \rho < k$ then we can inductively define a sequence of functions

$$\psi_2(T) = \int_1^T \psi_1(u)\ du, \ldots, \psi_k(T) = \int_1^T \psi_{k-1}(u)\ du.$$

By repeatedly using the above arguments we reach the same conclusion. □

Proof of Theorem 1.3. Lemma 4.1 and the arguments in Lemma 4.3 show that

$$\pi_f(x) := \sum_{\lambda(\tau)\le T} \lambda_f(\tau) e^{\lambda(\tau)-\lambda^u(\tau)} = \left(\int f\ d\mu\right) e^T + O\left(\frac{e^T}{T^a}\right), \text{ as } T \to +\infty,$$

for some choice of $a > 0$, and thus we can write that

$$\begin{aligned}\sum_{T-\delta/2\le\lambda(\tau)\le T+\delta/2} \lambda_f(\tau) e^{\lambda(\tau)-\lambda^u(\tau)} &= \pi_f\left(T+\frac{\delta}{2}\right) - \pi_f\left(T-\frac{\delta}{2}\right)\\ &= \left(\int f\ d\mu\right)\left(e^{(T+\delta/2)} - e^{(T-\delta/2)}\right) + O\left(\frac{e^T}{T^a}\right)\\ &= \left(\int f\ d\mu\right) e^T\delta + O\left(\frac{e^T}{T^a}, \delta^2 e^T\right).\end{aligned}$$

We can then write that

$$\begin{aligned}\sum_{T-\delta/2\le\lambda(\tau)\le T+\delta/2} \lambda_f(\tau) e^{-\lambda^u(\tau)} &\le \exp\left(-T+\frac{\delta}{2}\right) \sum_{T-\delta/2\le\lambda(\tau)\le T+\delta/2} \lambda_f(\tau) e^{\lambda(\tau)-\lambda^u(\tau)}\\ &= \left(\int f\ d\mu\right)\delta\exp\left(\frac{\delta}{2}\right) + O\left(\frac{1}{T^a}, \delta^2\right)\\ &= \left(\int f\ d\mu\right)\left(\delta+\frac{\delta^2}{2}\right) + O\left(\frac{1}{T^a}, \delta^2\right)\\ &= \left(\int f\ d\mu\right)\delta + O\left(\frac{1}{T^a}, \delta^2\right),\end{aligned}$$

with a similar lower bound.

Comparing these estimates, we see that

$$\frac{1}{\delta}\sum_{T-\delta/2\le\lambda(\tau)\le T+\delta/2}\lambda_f(\tau)e^{-\lambda^u(\tau)} = \int f\ d\mu + +O\left(\frac{1}{\delta T^a},\delta\right).$$

In particular, providing $\delta(T)\to 0$ as $T\to+\infty$ with $\delta(T)^{-1}=o(T^{-a})$, then the estimate (0.3) holds, provided f is non-negative. The result for general f follows from considering positive and negative parts. $\square$

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