ASYMPTOTIC EXPANSIONS FOR CLOSED ORBITS IN HOMOLOGY CLASSES

Mark Pollicott and Richard Sharp

University of Manchester

0. Introduction

Let V be a smooth compact manifold with first Betti number $b > 0$ equipped with a Riemannian metric of (possibly variable) negative curvature. Such a manifold has a countable infinity of closed geodesics and it is an important problem to understand their distribution. In this paper we concentrate on the distribution with respect to homology of V. A natural dynamical generalization of this problem is to study the distribution of closed orbits of Anosov flows, again with respect to homology.

The problem of the asymptotic estimates on the number of closed geodesics has been studied by a number of authors. Let γ denote a typical closed geodesic and $l(\gamma)$ its length. We denote by $\pi(T) = \{\gamma : l(\gamma) \le T\}$ the number of closed geodesics of length at most T. A classical result of Margulis states that $\pi(T) \sim e^{hT}/hT$, where $h > 0$ is the topological entropy of the geodesic flow on the unit-tangent bundle SV [18].

In this note we concentrate on the distribution with respect to homology. More precisely, for a fixed homology class $\alpha \in H_1(V, \mathbb{Z})$, we study the asymptotic behaviour of the counting function

$$\pi(T, \alpha) = \#\{\gamma : l(\gamma) \le T, [\gamma] = \alpha\},$$

where $[\gamma] \in H_1(V, \mathbb{Z})$ is the homology class of γ. It is well-known that there exists $C_0 > 0$ such that

$$\pi(T, \alpha) \sim C_0 \frac{e^{hT}}{T^{b/2+1}}, \text{ as } T \to \infty, \tag{0.1}$$

cf. [13],[16],[21]. The corresponding result was obtained earlier in the case of constant curvature in [14] and [20]. In fact, the result in [20] is somewhat more precise. An outstanding problem is to estimate other terms in the expansion of (0.1).

We say that the Riemannian manifold V is 1/4-pinched if the sectional curvatures lie in an interval $[-k, -k/4]$, for some $k > 0$. Our main result is the following.

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Theorem 1. *Let V be a compact manifold with first Betti number $b > 0$ and with negative sectional curvatures. Furthermore, if $\dim V \geq 3$ assume that V is $1/4$-pinched. Then there exist $C_0, C_1, C_2, \ldots$ (with $C_0 > 0$) such that*

$$\pi(T, \alpha) = \frac{e^{hT}}{T^{b/2}} \left(\sum_{n=0}^{N} \frac{C_n}{T^{n/2}} + o\left(\frac{1}{T^{N/2}} \right) \right) \tag{0.2}$$

for any $N > 0$.

The reason for the additional pinching condition in the higher dimensional case is to ensure the necessary technical condition that the stable and unstable foliations for the geodesic flow on SV are C^1.

Using an alternative approach based on renewal theory, N. Anantharaman has also obtained this result [1]. In fact, she has shown that the constants C_n vanish if n is odd.

Phillips and Sarnak were able to use non-commutative harmonic analysis (in the form of the Selberg trace formula) for manifolds of constant curvature, a method not available to us in the case of manifolds of variable curvature. Instead, we shall use a dynamical approach based on the geodesic flow on the unit-tangent bundle SV. In this setting we can identify $\pi(T, \alpha)$ with a counting function for closed orbits in a prescribed homology class.

We shall also prove a slightly less precise result in the more general context of (transitive) Anosov flows $\phi_t : M \to M$. We can associate to each closed orbit γ for $\phi_t : M \to M$ a homology class $[\gamma] \in H_1(M, \mathbb{Z})$. We say that the flow ϕ is *homologically full* if every element of $H_1(M, \mathbb{Z})$ contains a periodic orbit γ. Given $\alpha \in H_1(M, \mathbb{Z})$ we denote

$$\pi(T, \alpha) = \#\{\gamma : l(\gamma) \leq T, [\gamma] = \alpha\},$$

The next result gives a partial generalization of Theorem 1 to Anosov flows.

Theorem 2. *Let M be a compact manifold with first Betti number $b > 0$. Let $\phi_t : M \to M$ be a homologically full transitive Anosov flow. There exists $C_0 > 0$, $0 < h^* \leq h$ and $\delta > 0$ such that*

$$\pi(T, \alpha) = \frac{C_0 e^{h^* T}}{T^{b/2+1}} \left(1 + O\left(\frac{1}{T^{\delta}} \right) \right) \tag{0.3}$$

The proof of Theorem 2 actually yields a somewhat more precise result. Namely, if there exists $\rho > 0$ for which the associated L-functions $L(s, \chi)$ have an analytic extension to the region $s = \sigma + it$ with $|t| \geq 1$ and $\sigma > 1 - \frac{1}{|t|^{\rho}}$, then setting $\delta = [1/\rho] - 1$ and choosing $N < 2\delta$ there exist constants $C_1, C_2, \ldots C_N$ such that

$$\pi(T, \alpha) = \frac{e^{h^* T}}{T^{b/2+1}} \left(\sum_{n=0}^{N} \frac{C_n}{T^{n/2}} + O\left(\frac{1}{T^{\delta}} \right) \right)$$

The principal asymptotic was obtained by the second author [27]. There is no evidence that an asymptotic expansion such as the one in (0.2) is valid.

Remark. The quantity h^* can be characterized in terms of the winding cycles. These are elements $\Phi_\mu \in H_1(M, \mathbb{R})$ associated to ϕ-invariant probability measures μ and defined by $\Phi_\mu([\omega]) = \int \omega(\mathcal{X}) d\mu$, where ω is a closed 1-form, and $\mathcal{X}$ is the vector field generating the flow ϕ. We then have

$$h^* = \sup\{h_\mu(\phi) : \Phi_\mu = 0\}.$$

In the case of geodesic flows, the winding cycle associated to the measure of maximal entropy vanishes so that h^* is equal to the topological entropy.

The form of the asymptotic expansion is familiar in the context of a well-known problem in number theory. Consider the asymptotic behaviour of the number of integers less than x which can be written as the sum of two squares, i.e.,

$$B(x) = \#\{1 \leq n \leq x : n = u_1^2 + u_2^2, u_1, u_2 \in \mathbb{Z}\}.$$

Landau [17] showed that $B(x) \sim Kx/(\log x)^{1/2}$, for some $K > 0$, and the same result appears in Ramanujan's famous letter to Hardy in 1913 [2]. The fuller asymptotic expansion involves studying the complex function

$$f(s) = \frac{1}{1-2^{-s}} \prod_q \frac{1}{1-q^{-s}} \prod_r \frac{1}{1-r^{-2s}}$$

where q runs through all primes equal to 1 (mod 4) and r runs through all primes equal to 3 (mod 4). Observe that this differs from the Riemann zeta function only in the factor of 2 in the last exponent, the result being a singularity of the form $(s-1)^{-1/2}$. The asymptotic expansion for $B(x)$ has a similar form to that in (0.2) (after rescaling): for every $N \geq 1$

$$B(x) = \frac{Kx}{(\log x)^{1/2}} \left(1 + \sum_{n=1}^{N} \frac{\alpha_n}{(\log T)^n} + O\left(\frac{1}{(\log x)^N}\right)\right)$$

[11, pp.61-63], [26]. However, despite the analogy, the nature of the singularity in the complex function associated to our problem is somewhat more complicated and the detailed arguments are consequently more elaborate.

1. Geodesic flows and L-functions

Theorem 1 can be formulated in terms of the geodesic flow on of the unit-tangent bundle of the compact manifold V with negative sectional curvatures. Let $SV = \{(x, v) \in TV : ||v||_x = 1\}$ denote the unit-tangent bundle. Given $v \in SV$ we can choose a unique unit speed geodesic $\gamma : \mathbb{R} \to V$ with $\gamma'(0) = (x, v)$. We define the geodesic flow $\phi_t : SV \to SV$ by $\phi_t(x, v) = \gamma'(t)$ and let $h > 0$ denote its topological entropy.

There is a natural one-to-one correspondence between closed geodesics on V and closed orbits for ϕ. For such a closed orbit γ we denote its least period by $l(\gamma)$ and write $N(\gamma) = e^{hl(\gamma)}$. We shall also write $[\gamma] \in H_1(V, \mathbb{Z})$ for the homology class of the corresponding closed geodesic.

For convenience we shall write $H = H_1(V, \mathbb{Z})$. Then $H \cong \mathbb{Z}^b \oplus G$, where G is the finite torsion group. A key ingredient in our proof is an analysis of a family of L-functions defined with respect to characters in $\hat{H} \cong \mathbb{R}^b/\mathbb{Z}^b \oplus \hat{G}$.

For $\chi \in \hat{H}$, we define the *L-function*

$$L(s, \chi) = \prod_{\gamma} \left(1 - \chi([\gamma]) N(\gamma)^{-s}\right)^{-1},$$

where $s \in \mathbb{C}$. This infinite product converges to an analytic function for $\mathrm{Re} s > 1$. Note that if χ is the trivial character 1 then $L(s, 1)$ is the usual zeta function $\zeta(s) = \prod_{\gamma} \left(1 - N(\gamma)^{-s}\right)^{-1}$.

Proposition 1. *For a geodesic flow $\phi_t : SV \to SV$ the function $L(s, \chi)$ is meromorphic and non-zero in the domain $\{s = \sigma + it : \sigma > \sigma_0\}$. for some $\sigma_0 < 1$.*

1. *For $\chi \neq 1$, the function $L(s, \chi)$ is analytic and non-zero in the domain $\{s = \sigma + it : \sigma > \sigma_0, |t| > 1\}$.*
2. *The function $L(s, 1)$ is analytic and non-zero in the domain $\{s = \sigma + it : \sigma > \sigma_0\}$ apart from a simple pole at $s = 1$. Furthermore, for all $\chi \in \hat{H}$, $L(\sigma + it, \chi) = O(|t|^\beta)$ for some $0 < \beta < 1$, in the region $\sigma_0 < \sigma < 1$, $|t| > 1$.*

We defer a proof of this result until section 9.

The following standard lemma enables us to derive a bound on the logarithmic derivative of $L(s, \chi)$.

Lemma 1 ([9 , Theorem 4.2])**.** *Let $z \in \mathbb{C}$. Given $R > 0$ and $\epsilon > 0$ suppose that $F(s)$ is analytic on the disk $\Delta = \{s = \sigma + it : |s - z| \leq R(1 + \epsilon)^3\}$ and that there are no zeros for $F(s)$ on the open subset*

$$\{s = \sigma + it \in \mathbb{C} : |s - z| \leq R(1 + \epsilon)^2 \text{ and } \sigma > Re(z) - R(1 + \epsilon)\}.$$

Suppose in addition that $F(z) \neq 0$ and there exists a constant $U(z) \geq 0$ such that $\log |F(s)| \leq U(z) + \log |F(z)|$ on the set Δ.

Then we have the following bound for the logarithmic derivative on the disk $\{s = \sigma + it : |s - z| \leq R\}$:

$$\left|\frac{F'(s)}{F(s)}\right| \leq \frac{2 + \epsilon}{\epsilon} \left(\left|\frac{F'(z)}{F(z)}\right| + \left(2 + \frac{1}{(1 + \epsilon)^2 \log(1 + \epsilon)}\right) \frac{U(z)}{R\epsilon^2} \right). \tag{1.1}$$

We can apply the above lemma to $F(s) = L(s, \chi)$ to obtain a bound at $s = \sigma + it$ by using the choices $\epsilon = 1$, $R = (1 - \sigma_0)/12$ and $z = 1 + (1 - \sigma_0)/24 + it$ whenever $|t| > 1$. Notice that

$$\left|\frac{L'(z, \chi)}{L(z, \chi)}\right| \leq \left|\frac{L'(1 + (1 - \sigma_0)/24, 1)}{L(1 + (1 - \sigma_0)/24, 1)}\right|,$$

giving a bound on the Left Hand Side independent of t. Moreover, since ϵ and R are fixed the only term on the Right Hand Side of (1.1) depending on t is $U(z)$ for which we have the bound $U(z) = O(|t|^\beta)$. If we replace σ_0 by the larger value $1-(1-\sigma_0)/24$ then Lemma 1 gives the bound

$$\frac{L'(\sigma+it,\chi)}{L(\sigma+it,\chi)} \leq C|t|^\beta,$$

for some $0<\beta<1$ and $C>0$, in the region $\{s=\sigma+it : \sigma_0<\sigma<1, |t|>1\}$.

We define functions of a complex variable s by

$$\eta(s,\chi) = \frac{d^g}{ds^g}\frac{L'(s,\chi)}{L(s,\chi)} \text{ and } \eta(s) = \int_{\hat{H}} \chi^{-1}(\alpha)\eta(s,\chi)d\chi,$$

where we denote $g=[b/2]$. (The choice of the letter g is motivated by the fact that if V is a surface then $b/2$ is equal to the genus.)

Observe that by orthogonality, for $\mathrm{Re}s>1$, we have

$$\eta(s) = \sum_\gamma \sum_{\substack{n=1 \\ n[\gamma]=\alpha}}^\infty (-hl(\gamma))^{g+1} n^g e^{-shnl(\gamma)}.$$

The next result describes the behaviour of $\eta(s)$ when the imaginary part of s is large.

Proposition 2. *For the geodesic flow $\phi_t : SV \to SV$ we have the bound $\eta(s) = O(|t|^\beta)$, for some $0<\beta<1$, in the region $\sigma_1<\sigma<1$, $|t|>1$, for some $\sigma_1<1$*

Proof. It suffices to show the same bound holds for $\eta(s,\chi)$ in the region $\sigma_0<\sigma<1$, $|t|>1$, We do this by a simple application of Cauchy's theorem. In particular,

$$\begin{aligned} |\eta(s,\chi)| =& \frac{1}{2\pi}\left|\int_{|\xi-s|=\delta} \frac{L'(\xi,\chi)}{L(\xi,\chi)}\frac{1}{(s-\xi)^{g+1}}d\xi\right| \\ \leq& \frac{C|t|^\beta(g+1)!}{\delta^{g+1}} = O\left(|t|^\beta\right) \end{aligned}$$

Proposition 3. [13] *For a geodesic flow $\phi_t : SV \to SV$ there exists a neighbourhood U of $1\in\hat{H}$ and an analytic function $U \ni \chi \mapsto s(\chi)$ such that:*

1. *$s(\chi)$ is real valued and $s(1)=1$;*
2. *$s(\chi)$ is a simple pole for $L(s,\chi)$;*
3. *$\nabla s(1)=0$; and*
4. *$\nabla^2 s(1)$ is negative definite.*

2. Anosov flows and L-functions

Anosov flows provide a natural generalization of geodesics flows over negatively curved manifolds. Let M be a C^∞ compact manifold with first Betti number $b>0$. We call a C^1 flow $\phi_t : M\to M$ an *Anosov flow* if the tangent bundle TM has a continuous splitting $TM = E^0\oplus E^u\oplus E^s$ into $D\phi_t$-invariant subbundles such that:

1. E^0 is the one-dimensional bundle tangent to the flow;

(2) there exist $C, \lambda > 0$ such that

$$\begin{cases} ||D\phi_t|E^s|| \leq Ce^{-\lambda t} \text{ for } t \geq 0 \\ ||D\phi_{-t}|E^u|| \leq Ce^{-\lambda t} \text{ for } t \geq 0. \end{cases}$$

In addition, we say that ϕ is transitive is there is a dense orbit. We shall restrict our attention to transitive flows.

We shall say that an Anosov flow ϕ is homologically full if every homology class contains a closed orbit. If ϕ is homologically full then it is topologically weak mixing (i.e. there are no non-trivial continuous solutions $w \circ \phi_t = e^{iat}w$).

Given a continuous function $f : M \to \mathbb{R}$ we define the *pressure* by $P(f) = \sup\{h_\mu(\phi) + \int f d\mu : \mu \text{ is a } \phi\text{-invariant probability measure}\}$. Following [27], we shall use $\beta : H^1(M, \mathbb{R}) \to \mathbb{R}$ to denote the function $\beta(\omega) = P(\omega(\mathcal{X}))$, where $\mathcal{X}$ denotes the vector field generating ϕ.

For homologically full transitive Anosov flows there is unique $\xi \in H^1(M, \mathbb{R})$ such that

$$h^* = \beta(\xi) = \inf\{\beta(\xi') : \xi' \in H_1(M, \mathbb{R})\}.$$

As for geodesic flows, we shall use γ to denote a closed orbit and $l(\gamma)$ its least period. In this more general setting we shall write $N(\gamma) = e^{h^* l(\gamma)}$. We shall also use $[\gamma]$ to denote the its homology class in $H = H_1(M, \mathbb{Z})$ (Observe that we are using H to denote two slightly different homology groups, namely $H_1(V, \mathbb{Z})$ and $H_1(M, \mathbb{Z})$. We hope that this does not cause unnecessary confusion.)

In this setting we define the L-function by

$$L(s, \chi) = \prod_\gamma \left(1 - \chi([\gamma]) e^{\langle \xi, [\gamma] \rangle} N(\gamma)^{-s}\right)^{-1},$$

where $\chi \in \hat{H}$ and $\langle \xi, [\gamma] \rangle$ denotes the pairing of ξ with the torsion free part of $[\gamma]$. This converges to an analytic function for $\mathrm{Re} s > 1$.

The next result is the analogue of Proposition 1.

Proposition 4. *For a homologically full transitive Anosov flow the function $L(s, \chi)$ is meromorphic and non-zero in the domain $\{s = \sigma + it : \sigma > \sigma_0\}$, for some $\sigma_0 < 1$. Moreover, there exists $\sigma_0 : \mathbb{R}^+ \to (0, 1)$ with*

$$\sigma_0(|t|) \geq 1 - \frac{C}{|t|^\rho}, \text{ for } |t| \geq 1$$

for some $C > 0$ and $\rho > 0$ such that

(1) *for $\chi \neq 1$, the function $L(s, \chi)$ is analytic and non-zero in the domain $\{s = \sigma + it : \sigma > \sigma_0(|t|), |t| > 1\}$,*
(2) *the function $L(s, 1)$ is analytic and non-zero in the domain $\{s = \sigma + it : \sigma > \sigma_0(|t|)\}$,*
(3) *for all $\chi \in \hat{H}$, $L(\sigma + it, \chi) = O(|t|^\beta)$ for some $\beta > 01$, in the region $\sigma_0 < \sigma < 1$, $|t| > 1$.*

The statement concerning the meromorphic extension is well-known [19]. We defer the proof of the remainder of the proposition to section 10.

We can apply Lemma 1 to $F(s) = L(s,\chi)$ with the choices $\epsilon = 1$, $R = C/(8|t|^\rho)$ and $z = 1 + C/(16|t|^\rho)$, whenever $|t| > 1$. Observing that $|L'(z,\chi)/L(z,\chi)| = O(|t|^\rho)$ and that $U(z) = O(|t|^\beta)$, we have the bound

$$\left|\frac{L'(\sigma+it,\chi)}{L(\sigma+it,\chi)}\right| = O\left(|t|^{\rho+\beta}\right),$$

in the region $\{s = \sigma + it : \sigma_0(t) < \sigma < 1, |t| > 1\}$.

Without loss of generality, the exponent $\rho + \beta$ of $|t|$ in this bound can be improved to a value γ arbitrarily close to ρ. This is accomplished by applying the Hadamard Three Circle Theorem and decreasing the value of C.

Generalizing the case for geodesic flows, we define functions of a complex variable s by

$$\eta(s,\chi) = \frac{d^g}{ds^g}\frac{L'(s,\chi)}{L(s,\chi)} \text{ and } \eta(s) = \int_{\hat{H}} \chi^{-1}(\alpha)\eta(s,\chi)d\chi,$$

where we again denote $g = [b/2]$.

Observe that by orthogonality, for $\mathrm{Re} s > 1$, we have

$$\eta(s) = \sum_{\gamma} \sum_{\substack{n=1 \\ n[\gamma]=\alpha}}^{\infty} (-h^* l(\gamma))^{g+1} n^g e^{\langle \xi,\alpha\rangle} e^{-sh^* nl(\gamma)}.$$

Proposition 5. *For a weak-mixing Anosov flow $\phi_t : M \to M$ we have the bound $\eta(s) = O(|t|^\gamma)$ in the region*

$$\left\{\sigma + it : 1 - \frac{C}{|t|^\rho} < \sigma < 1, |t| > 1\right\}.$$

Proof. The proof is similar to that for Proposition 2.

The next result describes the dependence of the L-functions on χ in a neighbourhood of 1.

Proposition 6. *For an Anosov flow there exists a neighbourhood U of $1 \in \hat{H}$ and an analytic function $U \ni \chi \mapsto s(\chi)$ such that $s(1) = 1$ and*

(1) *$s(\chi)$ is a simple pole for $L(s,\chi)$;*
(2) *$\nabla\mathrm{Re}s(1) = 0$ and $\nabla\mathrm{Im}s(1) = 0$;*
(3) *$\nabla^2\mathrm{Re}s(1)$ is negative definite and $\nabla^2\mathrm{Im}s(1) = 0$*

Proof. This is shown in [15], for example.

We will need to understand how $\eta(s)$ behaves when s is close to 1. We shall carry out this analysis in the general setting of Anosov flows. The behaviour depends on whether the dimension b is even or odd. We shall consider the two cases separately in the next two sections.

3. The even dimensional case

In the case where b is even the behaviour of $\eta(s)$ is described by the following Proposition.

Proposition 7. *In a neighbourhood of $s = 1$, there exist analytic functions $f_1(s)$ and $f_2(s)$ such that*

$$\eta(s) = f_1(s) + \frac{c_0}{s-1} + \frac{c_1}{(s-1)^{1/2}} + f_2(s)\log(s-1) + \sum_{n=0}^{\infty} a_n (s-1)^{n+1/2},$$

where the branches of all functions considered are taken on the cut plane $\mathbb{C} - (-\infty, 1]$ and are real for real $s > 1$. Moreover, the coefficients a_n satisfy the bound $a_n = O(R^n)$, for some $R > 0$.

Proof. Applying Proposition 6 to the definition of $\eta(s)$ we obtain

$$\eta(s) = \int_U \chi^{-1}(\alpha) \frac{(-1)^g g!}{(s - s(\chi))^{g+1}} d\chi + A_1(s) \tag{3.1}$$

where $A_1(s)$ is analytic.

Using the Morse Lemma [12] there exists, without loss of generality, co-ordinates $\theta = (\theta_1, \dots, \theta_{2g})$ in U and $\delta > 0$ such that

$$s(\theta) = 1 - ||\theta||_2^2 + iQ(\theta)$$

where $Q(\theta)$ is an analytic function of third order in θ, for $||\theta||_2 \le \delta$, and where $||\theta||_2^2 = \theta_1^2 + \ldots + \theta_{2g}^2$. (In the special case of geodesic flows we have that $Q = 0$.) Applying this to (3.1) we may write

$$\begin{aligned}\eta(s) &= \int_{||\theta||_2 \le \delta} \frac{f(\theta)}{((s-1) + ||\theta||_2^2 + iQ(\theta))^{g+1}} d\theta + A_2(s), \\ &= \int_{||\theta||_2 \le \delta} \frac{f(\theta)}{((s-1) + ||\theta||_2^2)^{g+1}} \left(\sum_{n=0}^{\infty} \frac{(n+g+1)!}{n!(g+1)!} \left(\frac{iQ(\theta)}{(s-1) + ||\theta||_2^2} \right)^n \right) d\theta + A_2(s)\end{aligned}$$

where $f(\theta)$ and $A_2(s)$ are analytic.

Changing to spherical polar co-ordinates, we can obtain the formula

$$\eta(s) = \sum_{n=0}^{\infty} \int_0^{\delta} \frac{F_n(r)}{(s-1+r^2)^{g+n+1}} r^{2g-1} dr + A_2(s), \tag{3.2}$$

where $F_n(r) = i^n \frac{(n+g+1)!}{n!(g+1)!} \int_{S^{2g-1}} f(r\omega) Q(r\omega)^n d\omega$ and, in particular, $F_0(0) \neq 0$ (compare [15]). We can take the Taylor's series expansion $F_n(r) = \sum_{k=3n}^{\infty} a_k^{(n)} r^k$, where $a_k^{(n)} = O(R^k)$, for some $R > 0$. We may now rewrite (3.2) as

$$\eta(s) = \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} a_{3n+p}^{(n)} \int_0^{\delta} \frac{r^{2g-1+3n+p}}{(s-1+r^2)^{g+n+1}} dr + A_2(s),$$

where $a^{(n)}_{3n+p} = O(R^{3n+p})$. Thus we have to estimate the above integrals. Making the substitution $r = (s-1)^{1/2}x$ we obtain

$$\int_0^\delta \frac{r^{2g-1+k}}{(s-1+r^2)^{g+n+1}} dr = (s-1)^{\frac{k}{2}-n-1} I^{g+n+1}_{k+2g-1}(s), \quad k \geq 3n \geq 0 \tag{3.3}$$

where $I^l_m(s) = \int_0^{\xi(s)} \frac{x^m}{(1+x^2)^l} dx$, $l, m \geq 0$, and we denote $\xi(s) = \delta/(s-1)^{1/2}$. The analysis of the singularity of $\eta(s)$ reduces to understanding these integrals. This will occupy the remainder of this section.

Case I: $k = 0$**.** In this case we also know that $n = 0$ and the following lemma allows us to estimate $I^{(g+1)}_{2g-1}(s)$.

Lemma 2. *We can write*

$$\frac{1}{(s-1)} I^{(g+1)}_{2g-1}(s) = -\frac{(2g-2)\cdots 2}{(2g)\cdots 4} \frac{1}{s-1} \frac{1}{2} \left(1 - \frac{s-1}{s-1+\delta^2}\right),$$

up to an analytic function in a neighbourhood of $s = 1$.

Proof. By integration by parts we have that

$$\frac{1}{(s-1)} I^{(g+1)}_{2g-1}(s) = C_k(s) + \frac{(2g-2)\cdots 2}{(2g)\cdots 4} I^2_1(s) \frac{1}{(s-1)} \tag{3.4}$$

where $A^l_m(s) = \frac{\delta^{m-1}}{(l-1)(s-1+\delta)^{l-1}}$ and

$$C_k(s) = \left\{ A^{g+1}_{2g-1}(s) + \left(\frac{2g-2}{2g}\right) A^g_{k+2g-3}(s) + \ldots \ldots + \left(\frac{(2g-1)\ldots 2}{(2g)\cdots 4}\right) A^2_1(s) \right\}.$$

is an analytic function in a neighbourhood of $s = 1$. Notice that

$$I^2_1(s) = \int_0^{\xi(s)} \frac{x}{(1+x^2)^2} dx = \frac{1}{2}\left(1 - \frac{1}{1+\xi(s)^2}\right) = \frac{1}{2}\left(1 - \frac{s-1}{s-1+\delta^2}\right). \tag{3.5}$$

Comparing (3.4) and (3.5) completes the proof of the lemma.

An inductive step. To deal with the case $k \geq 1$ it is convenient to reduce the expressions involving $I^l_m(s)$, $l, m \geq 0$, to ones involving $I^1_m(s)$, $m \geq 0$. Observe that for $m \geq 2$ and $l \geq 1$,

$$I^l_m(s) = -A^l_m(s) \frac{1}{(s-1)^{m/2-l+1/2}} + \frac{(m-1)}{2(l-1)} I^{l-1}_{m-2}(s)$$

where $|A^l_m(s)| = O(\delta^{m-1})$. In particular,

$$I^{g+n+1}_{k+2g-1}(s) = -A^{g+n+1}_{k+2g-1}(s) \frac{1}{(s-1)^{(k/2-1-n)}} + \frac{(2g+k-2)}{2(g+n)} I^{g+n}_{k+2g-3}(s) \tag{3.6}$$

where $|A^{g+n+1}_{k+2g-1}(s)| = \frac{\delta^{k+2g-1}}{2g(s-1+\delta)^2} = O(\delta^{k+2g-1})$.

A repeated application of (3.6) gives the required reduction. In particular, we have that

$$\begin{aligned}(s-1)^{k/2-n-1} & I^{g+n+1}_{k+2g-1}(s) \\ &= B_{k,n}(s) + \frac{(k+2g-2)\cdots(k-2n)(k-2n-2)}{(2g+2n)\cdots 4.2} I^1_{k-2n-1}(s)(s-1)^{k/2-n-1}\end{aligned} \tag{3.7}$$

where

$$\begin{aligned}B_{k,n}(s) = \Bigg\{ & A^{g+n+1}_{k+2g-1}(s) + \left(\frac{k+2g-2}{2g+2n}\right) A^{g+n}_{k+2g-3}(s) + \dots \\ & \dots + \left(\frac{(k+2g-2)\dots(k-2n+2)\cdot(k-2n)}{(2g+2n)\cdots 6\cdot 4}\right) A^2_{k-2n+1}(s) \Bigg\}\end{aligned}$$

is an analytic function with $|B_{k,n}(s)| = O\left(\delta_0^{k+1}\right)$ for any $\delta < \delta_0 < 1$.

We are led to estimate expressions of the form $I^1_m(s)$, where $m = k-2n-1$.

Case II: $m = k-2n-1 = 0$**.**

In this case we can directly evaluate

$$I^1_0(s) = \int_0^{\xi(s)} \frac{1}{1+x^2} = \tan^{-1}\xi(x)$$

Case III: $m = k-2n-1 \geq 1$**.** In this final case, the following lemma gives the required estimates.

Lemma 3.

(i) *If m is odd then*

$$\begin{aligned}(s-1)^{m/2-1/2} I^1_m(s) = & \frac{\delta^{m-1}}{m-1} - \frac{\delta^{m-3}}{m-3}(s-1) + \dots \pm \frac{\delta^2}{2}(s-1)^{m/2-3/2} \\ & \pm (s-1)^{m/2-1/2}\log(s-1)/2 \mp (s-1)^{m/2-1/2}\log(s-1+\delta^2)/2,\end{aligned}$$

up to an analytic function in a neighbourhood of $s=1$ (where, in particular, in the first part of this expression consists of integer powers of $(s-1)$).

(ii) *If m is even then*

$$\begin{aligned}(s-1)^{m/2-1/2} I^1_m(s) = & \frac{\xi(s)^{m-1}}{m-1}(s-1) - \frac{\xi(s)^{m-3}}{m-3} + \dots \pm \xi(s)(s-1)^{m/2-1} \\ & \mp \tan^{-1}\left(\frac{\delta}{(s-1)^{1/2}}\right)(s-1)^{m/2-1/2},\end{aligned}$$

up to an analytic function in a neighbourhood of $s=1$.

Proof. By a simple calculation we have that

(a)

$$I^1_m(s) + I^1_{m-2}(s) = \frac{\xi(s)^{m-1}}{m-1} = \frac{\delta^{m-1}}{m-1}\frac{1}{(s-1)^{(m-1)/2}},$$

for $m \geq 2$

(b) $I^1_1(s) = \log(1+\xi(s)^2)/2 = -\log(s-1)/2 + \log(s-1+\delta^2)/2$

(c) $I^1_0(s) = \tan^{-1}(\xi(s))$

Observe that $\tan^{-1}\left(\frac{\delta}{(s-1)^{1/2}}\right) = \frac{\pi}{2} - \tan^{-1}\left(\frac{(s-1)^{1/2}}{\delta}\right)$. For part (i), a repeated application of part (a) (and a final application of (b)) gives

$$I^1_m(s) = \frac{\xi(s)^{m-1}}{m-1} - \frac{\xi(s)^{m-3}}{m-3} + \cdots \pm \frac{\xi(s)^2}{2} \mp \frac{1}{2}\log(1+\xi(s)^2),$$

up to an analytic function. Similarly, we can estimate $I^1_m(s)$ in part (ii). This completes the proof of the lemma.

Lemma 2 and Lemma 3 correspond to the statements in Proposition 7.

4. The odd dimensional case

A slight modification is required in the case where $b = 2g+1$ is odd. The leading singularity of $\eta(s)$ at $s=1$ takes the form $c_1(s-1)^{-1/2}$ rather than a simple pole. More precisely, we have the following result.

Proposition 8. *In a neighbourhood of $s=1$, there exist analytic functions $f_1(s)$ and $f_2(s)$ such that*

$$\eta(s) = f_1(s) + \frac{c_1}{(s-1)^{1/2}} + f_2(s)\log(s-1) + \sum_{n=0}^{\infty} a_n (s-1)^{n+1/2},$$

where the branches of all functions considered are taken on the cut plane $\mathbb{C} - (-\infty, 1]$ and are real for real $s > 1$.

In this case the analogue of (3.2) is

$$\eta(s) = \sum_{n=0}^{\infty}\int_0^\delta \frac{F_n(r)}{(s-1+r^2)^{g+n+1}} r^{2g}\,dr + A_2(s). \tag{4.1}$$

Observe that (4.1) differs from (3.2) in that the exponent of r changes from $2g-1$ to $2g$. As before, we can expand $F_n(r) = \sum_{k=3n}^\infty a^{(n)}_k r^k$ and rewrite a typical term in (4.1) as

$$\sum_{k=3n}^{\infty} a^{(n)}_k \int_0^\delta \frac{r^{2g+k}}{(s-1+r^2)^{g+n+1}}\,dr.$$

The result now follows by a similar calculation to that in the previous section, with k replaced by $k+1$.

5. Auxiliary functions

In order to study the counting function $\pi(T,\alpha)$ in Theorem 1 we shall introduce an auxiliary function $\psi(T,\alpha)$ defined by

$$\psi(T,\alpha)=\sum_{n=1}^{\infty}\sum_{\substack{N(\gamma)^n\le T\\ n[\gamma]=\alpha}}{}' (h^* l(\gamma))^{g+1}\, n^g$$

where the prime means that for terms where $N(\gamma)^n = T$ the summand is multiplied by 1/2.

We shall relate $\psi(T,\alpha)$ to $\eta(s)$ by applying the formula

$$\frac{1}{2\pi}\int_{d-i\infty}^{d-i\infty}\frac{y^s}{s}ds=\begin{cases}0 & \text{if } 0<y<1\\ \frac{1}{2} & \text{if } y=1\\ 1 & y>1\end{cases}$$

term by term to the series defining $\eta(s)$. The functions $\psi(T,\alpha)$ and $\eta(s)$ are related by the following lemma.

Lemma 4. *For $d>1$ we may write*

$$\psi(T,\alpha)=\frac{1}{2\pi i}\int_{d+i\infty}^{d-i\infty}\frac{\eta(s)}{s}T^s ds. \tag{5.1}$$

To evaluate this integral, we shall change the contour of integration. The first step is to replace this integral by a finite integral. Write $R=R(T)=(\log T)^K$, for some fixed $K>0$, and truncate the integral in (5.1) to obtain the estimate

$$\left|\psi(T)-\frac{1}{2\pi i}\int_{d-iR}^{d+iR}\frac{\eta(s)}{s}T^s ds\right|\le|\eta(d)|\frac{T^d}{R^{1/2}} \tag{5.2}$$

(cf. [6, pp. 105-106]). If we choose $d=1+\frac{1}{\log T}$, then since $\eta(d)=O((d-1)^{-1})$ we can bound the Right Hand Side of (5.2) by a term which is $O\left(\frac{T\log T}{R^{1/2}}\right)=O\left(\frac{T}{(\log T)^{K/2-1}}\right)$.

For the remainder of this section, we shall concentrate on the case of geodesic flows.

If we fix $c<1$ then we can replace

$$\frac{1}{2\pi i}\int_{d-iR}^{d+iR}\frac{\eta(s)}{s}T^s ds$$

by the integral along the curves

(i) $[c+iR,d+iR]$ and $[d-iR,c-iR]$, for $\sigma_0<c<1$
(ii) $[c-iR,c+i\delta]$ and $[c+i\delta,c+iR]$
(iii) $\left[c+i\delta,1-r/\sqrt{1+(\delta/r)^2}+i\delta\right]$ and $\left[1-r/\sqrt{1+(\delta/r)^2}-i\delta],c-i\delta\right]$, for $r>0$ arbitrarily small
(iv) $\mathcal{C}_r=\{1+re^{2\pi i\theta}:-\pi+\tan^{-1}(\delta/r)\le\theta\le\pi-\tan^{-1}(\delta/r)\}$,

where $\delta = \delta(T)$ is chosen to decrease exponentially fast as $T \to +\infty$. In the subsequent discussion we shall let $r \to 0$.

We can bound

$$\left| \frac{1}{2\pi i} \int_{[d+iR, c+iR] \cup [c-iR, d-iR]} \frac{\eta(s)}{s} T^s ds \right| = O\left(\frac{T^d}{R^{1-\beta}} \right)$$
$$= O\left(\frac{T}{(\log T)^{(1-\beta)K}} \right),$$

using the estimate $|\eta(s)| = O(|t|^\beta)$, for $s = \sigma + it$.

We can also bound

$$\left| \frac{1}{2\pi i} \int_{[c-iR, c-i\delta] \cup [c+i\delta, c+iR]} \frac{\eta(s)}{s} T^s ds \right| = O\left(T^c \int_1^R t^{\beta-1} dt \right) = O\left(T^c R^\beta \right)$$

The contribution from the integral around $\mathcal{C}_r$ can be evaluated as follows. On this curve for $s = 1 + re^{2\pi i\theta}$ we can bound

$$\left| \left(\eta(s) - \frac{C_0}{s-1} \right) \frac{T^s}{s} \right| = O\left(\frac{1}{r^{1/2}} \right)$$

and thus $\frac{1}{2\pi i} \int_{\mathcal{C}_r} \frac{\eta(s)}{s} T^s ds = C_0 T + O(r^{1/2})$, by Cauchy's theorem.

The contribution from (iii) requires a more detailed analysis which will appear in section 6.

Our aim is to prove the following result.

Proposition 9. *Let $\phi_t : SV \to SV$ be a geodesic flow and let $N \geq 0$.*

(1) *If b is even then we can write*

$$\psi(T, \alpha) = \sum_{n=0}^{N} C_n \frac{T}{(\log T)^{n/2}} + o\left(\frac{T}{(\log T)^{N/2}} \right)$$

(2) *If b is odd then we can write*

$$\psi(T, \alpha) = \sum_{n=1}^{N} C_n \frac{T}{(\log T)^{n/2}} + o\left(\frac{T}{(\log T)^{N/2}} \right)$$

The rather involved proof constitutes most of section 6.

6. Contour integrals and the proof of Proposition 9

We shall now evaluate the contribution to the contour integral from the curves in (iii) by considering the expansion of $\eta(s)$ for s close to 1. This then leads directly to a proof of Proposition 9.

6.1 The contribution from $(s-1)^{-1}$. This is the easiest singularity to deal with. Observe that

$$\begin{aligned}&\frac{1}{2\pi i}\int_{c+i\delta}^{1-r+i\delta}\frac{T^s}{s(s-1)}ds+\frac{1}{2\pi i}\int_{c-i\delta}^{1-r-i\delta}\frac{T^s}{s(s-1)}ds\\&=\frac{1}{2\pi i}\int_{c}^{1-r}\frac{T^s}{s(s-1)}ds+\frac{1}{2\pi i}\int_{1-r}^{c}\frac{T^s}{s(s-1)}ds+O\left(T\delta(T)\right)\end{aligned}$$

It is immediately apparent that the integral $\frac{1}{2\pi i}\int_c^{1-r}\frac{T^s}{s(s-1)}ds$ cancels with the integral $\frac{1}{2\pi i}\int_{1-r}^{c}\frac{T^s}{s(s-1)}ds$, to give a contribution which tends to zero exponentially fast as $T\to+\infty$.

6.2 The contribution from $(s-1)^{-1/2}$. The complication in this case is that the function $(s-1)^{-1/2}$ is multiple valued. The term $(s-1)^{1/2}$ gives rise to the integral

$$\begin{aligned}&\frac{1}{2\pi i}\int_{c+i\delta}^{1-r/\sqrt{1+(\delta/r)^2}+i\delta}\left(\frac{1}{(s-1)^{1/2}}\right)\frac{T^s}{s}ds+\frac{1}{2\pi i}\int_{1-r/\sqrt{1+(\delta/r)^2}-i\delta}^{c-i\delta}\left(\frac{1}{(s-1)^{1/2}}\right)\frac{T^s}{s}ds\\&=-\frac{1}{\pi}\int_c^1\frac{1}{\sigma(1-\sigma)^{1/2}}e^{\sigma(\log T)}d\sigma+O\left(T\delta(T)\right)\end{aligned}\tag{6.1}$$

Lemma 5. *Given $K\geq 1$ there exist constants w_{2n}, $n=0,\ldots,K$ such that*

$$\int_c^1\frac{1}{\sigma(1-\sigma)^{1/2}}e^{\sigma(\log T)}d\sigma=\sum_{n=0}^{K}2w_{2n}\frac{T}{(\log T)^{n+1/2}}+O\left(\frac{1}{(\log T)^K}\right)$$

Proof. We can introduce a new variable y and use the substitution $1-\sigma=y^2/\log T$ to rewrite the Left Hand Side of the expression in Lemma 5 as

$$\frac{2T}{(\log T)^{1/2}}\int_0^{\epsilon(\log T)^{1/2}}\left(\frac{e^{-y^2}}{1-\frac{y^2}{\log T}}\right)dy\tag{6.2}$$

where $\epsilon=(1-c)^{1/2}$. For any arbitrary $0<\alpha<1/2$ we can write

$$\begin{aligned}&\int_0^{\epsilon(\log T)^{1/2}}\left(\frac{e^{-y^2}}{1-\frac{y^2}{\log T}}\right)dy\\&=\int_0^{\epsilon(\log T)^{\alpha}}\left(\frac{e^{-y^2}}{1-\frac{y^2}{\log x}}\right)dy+\int_{\epsilon(\log T)^{\alpha}}^{\epsilon(\log T)^{1/2}}\left(\frac{e^{-y^2}}{1-\frac{y^2}{\log T}}\right)dy.\end{aligned}\tag{6.3}$$

Let K be as in the definition of $R(T)$ and set $M=K/(1-2\alpha)$. We can rewrite the first term on the Right Hand Side of (6.3) as

$$
\begin{aligned}
&\int_0^{\epsilon(\log T)^\alpha} \left(\frac{e^{-y^2}}{1-\frac{y^2}{\log T}} \right) dy \\
&= \int_0^{\epsilon(\log T)^\alpha} \left(\sum_{n=0}^{M} \left(\frac{y^2}{\log T} \right)^n \right) e^{-y^2} dy + O\left(\left(\frac{(\log T)^{2\alpha}}{\log T} \right)^M \right) \\
&= \int_0^{\epsilon(\log T)^\alpha} \left(\sum_{n=0}^{M} \left(\frac{y^2}{\log T} \right)^n \right) e^{-y^2} dy + O\left(\frac{1}{(\log T)^K} \right)
\end{aligned} \tag{6.4}
$$

using

$$
\frac{1}{1-\left(\frac{y^2}{\log T}\right)} = \sum_{n=0}^{M} \left(\frac{y^2}{\log T} \right)^n + O\left(\left(\frac{y^2}{\log T} \right)^M \right)
$$

which converges since $0 < y < \epsilon(\log T)^\alpha$. Moreover, observe that for each $0 \le n \le M$ we can write

$$
\int_0^{\epsilon(\log T)^\alpha} y^{2n} e^{-y^2} dy = w_{2n} + O\left(\frac{1}{T^{\epsilon^2}} \right) \tag{6.5}
$$

where $w_{2n} = \int_0^\infty y^{2n} e^{-y^2} dy$. Thus applying (6.5) to (6.4) we see that

$$
\int_0^{\epsilon(\log T)^\alpha} \left(\frac{e^{-y^2}}{1-\frac{y^2}{\log T}} \right) dy = \sum_{n=0}^{M} \frac{w_{2n}}{(\log T)^n} + O\left(\frac{1}{(\log T)^K} \right). \tag{6.6}
$$

(Observe that the terms involving $K < n \le N$ are already dominated by the error term.)

The second term in (6.3) can be bounded by

$$
\int_{\epsilon(\log T)^\alpha}^{\epsilon(\log T)^{1/2}} \left(\frac{e^{-y^2}}{1-\frac{y^2}{\log T}} \right) dy = O\left(\int_{\epsilon(\log T)^\alpha}^{\epsilon(\log T)^{1/2}} e^{-y^2} dy \right) = O\left(\frac{1}{T^{\epsilon^2}} \right). \tag{6.7}
$$

Comparing (6.6) and (6.7) with (6.3) we see that we obtain

$$
\int_c^1 \frac{1}{\sigma(1-\sigma)^{1/2}} e^{\sigma(\log T)} d\sigma = \sum_{n=0}^{K} 2w_{2n} \frac{T}{(\log T)^{n+1/2}} + O\left(\frac{1}{(\log T)^K} \right) \tag{6.8}
$$

This completes the proof of lemma.

6.3 The contribution from $(s-1)^n \log(s-1)$**.** The function $\log(s-1)$ is again multiple valued and so the integrals contribute

$$
\begin{aligned}
&\frac{1}{2\pi i} \int_{c+i\delta}^{1-r/\sqrt{1+(\delta/r)^2}+i\delta} (s-1)^n \log(s-1) \frac{T^s}{s} ds + \frac{1}{2\pi i} \int_1^c (s-1)^n \log(s-1) \frac{T^s}{s} ds \\
&= \int_{1-r/\sqrt{1+(\delta/r)^2}-i\delta}^{c-i\delta} \frac{(\sigma-1)^n e^{\sigma \log T}}{\sigma} d\sigma + O\left(T\delta(T)\right).
\end{aligned} \tag{6.9}
$$

Lemma 6. *Given $K \geq n+1$, there exist constants c_m, $m = n+1, \ldots, K$ such that*

$$\int_c^1 \frac{(\sigma-1)^n e^{\sigma \log T}}{\sigma} d\sigma = \sum_{m=n+1}^{K} c_m \frac{T}{(\log T)^m} + O\left(\frac{T}{(\log T)^K}\right).$$

Proof. Consider first the case $n = 0$. We can easily evaluate the integral on the Right Hand Side of (6.9) to have the estimate

$$\int_c^1 \frac{e^{\sigma \log T}}{\sigma} d\sigma = \frac{T}{\log T} + \frac{T}{(\log T)^2} + \cdots + (K-1)! \frac{T}{(\log T)^K} + O\left(\frac{T}{(\log T)^{K+1}}\right) \tag{6.10}$$

by a simple inductive argument.

More generally, if $n \geq 1$ then we can rewrite the Right Hand Side of (6.9) as

$$\begin{aligned}
&\int_c^1 \frac{(\sigma-1)^n e^{\sigma \log T}}{\sigma} d\sigma \\
&= \int_c^1 (\sigma-1)^{n-1} e^{\sigma \log T} d\sigma - \int_c^1 \frac{(\sigma-1)^{n-1} e^{\sigma \log T}}{\sigma} d\sigma \\
&= \left(\sum_{k=0}^{n-1} (-1)^{n-1-k} \binom{n-1}{k} \int_c^1 \sigma^k e^{\sigma(\log T)} d\sigma \right. \\
&\left. - \int_c^1 \frac{(\sigma-1)^{n-1} e^{\sigma \log T}}{\sigma} d\sigma \right)
\end{aligned} \tag{6.11}$$

The integrals which appear in the first term on the Right Hand Side of (6.11) can easily be shown to satisfy the estimate

$$\int_c^1 \sigma^k e^{\sigma(\log T)} d\sigma = \frac{T}{\log T} - k \frac{T}{(\log T)^2} + k(k-1) \frac{T}{(\log T)^3} + \cdots + O\left(\frac{T}{(\log T)^K}\right).$$

Furthermore, letting $\epsilon(T) = \min\{1-c, 1/\log T\}$ we can bound

$$\begin{aligned}
\int_c^1 \frac{(1-\sigma)^n e^{\sigma \log T}}{\sigma} d\sigma &= \int_{1-\epsilon(T)}^1 \frac{(1-\sigma)^n e^{\sigma \log T}}{\sigma} d\sigma + \int_c^{1-\epsilon(T)} \frac{(1-\sigma)^n e^{\sigma \log T}}{\sigma} d\sigma \\
&= O\left(\frac{T}{(\log T)^{n+1}}\right) + O(T^c) \\
&= O\left(\frac{T}{(\log T)^{n+1}}\right).
\end{aligned}$$

This completes the proof of the lemma.

6.4 The contribution of terms $(s-1)^{n+1/2}$, $n \geq 1$. Finally we have to consider the contribution of terms $(s-1)^{n+1/2}$, $n \geq 0$ to the integrals in (iii). The integral in question has the following estimate

$$\begin{aligned}&\frac{1}{2\pi i}\int_{c+i\delta}^{1-r/\sqrt{1+(\delta/r)^2}+i\delta}(s-1)^{n+1/2}\frac{T^s}{s}ds+\frac{1}{2\pi i}\int_{1-r/\sqrt{1+(\delta/r)^2}-i\delta}^{c-i\delta}(s-1)^{n+1/2}\frac{T^s}{s}ds\\&=\frac{1}{\pi}\int_c^1\frac{(1-\sigma)^{n+1/2}}{\sigma}e^{\sigma(\log T)}d\sigma+O\left(T\delta(T)\right)\end{aligned} \tag{6.12}$$

Lemma 7. *Given $K \geq n+1$, there exists constants d_m, $m = n+1, \dots, K$ such that*

$$\int_c^1\frac{(1-\sigma)^{n+1/2}}{\sigma}e^{\sigma(\log T)}d\sigma=\sum_{m=n+1}^{K}d_m\frac{T}{(\log T)^{m+1/2}}+O\left(\frac{T}{(\log T)^K}\right).$$

Proof. We consider first the case $n = 0$. Using the simple identity

$$(1-\sigma)^{1/2}=\frac{-\sigma}{(1-\sigma)^{1/2}}+\frac{1}{(1-\sigma)^{1/2}}$$

we can write

$$\begin{aligned}&\int_c^1\frac{(1-\sigma)^{1/2}}{\sigma}e^{\sigma(\log T)}d\sigma\\&=-\int_c^1\frac{1}{(1-\sigma)^{1/2}}e^{\sigma(\log T)}d\sigma+\int_c^1\frac{1}{\sigma(1-\sigma)^{1/2}}e^{\sigma(\log T)}d\sigma.\end{aligned} \tag{6.13}$$

We recall that by (6.8) we can estimate the second term on the Right Hand Side of (6.13) by

$$\int_c^1\frac{1}{\sigma(1-\sigma)^{1/2}}e^{\sigma(\log T)}d\sigma=\sum_{m=0}^{K}2w_{2m}\frac{T}{(\log T)^{m+1/2}}+O\left(\frac{T}{(\log T)^K}\right) \tag{6.14}$$

To estimate the the first term on the Right Hand Side of (6.13) we can substitute $1-\sigma=y^2$ to get

$$\begin{aligned}\int_c^1\frac{1}{(1-\sigma)^{1/2}}e^{\sigma(\log T)}d\sigma&=2T\int_0^{(1-c)^{1/2}}e^{-y^2(\log T)}dy\\&=\frac{2T}{(\log T)^{1/2}}\left(w_0+O\left(\frac{1}{T^{\epsilon^2}}\right)\right)\end{aligned} \tag{6.15}$$

where $w_0=\int_0^\infty e^{-y^2}dy$.

Thus we see that the terms with $m = 0$ cancel and we have

$$\int_c^1\frac{(1-\sigma)^{1/2}}{\sigma}e^{\sigma(\log T)}d\sigma=\sum_{m=1}^{K}2w_{2m}\frac{T}{(\log T)^{m+1/2}}+O\left(\frac{T}{(\log T)^K}\right)$$

We shall now consider the terms $(s-1)^{n+1/2}$ for $n \geq 1$. Let us denote

$$B_{k+1} = \int_c^1 \frac{(1-\sigma)^{k+1/2}}{\sigma} e^{\sigma(\log T)} d\sigma$$

for $0 \leq k \leq n$ then using integration by parts we see that

$$\begin{aligned} B_{k+1} &= B_k + \frac{2T}{(\log T)^{k+3/2}} \int_0^{\epsilon(\log T)^{1/2}} y^{2k+2} e^{-y^2} dy \\ &= B_k + \frac{2T}{(\log T)^{k+3/2}} w_{2k+2} + O\left(\frac{1}{T^{\epsilon^2}}\right) \end{aligned}$$

(using (6.5)). Hence

$$\begin{aligned} B_{n+1} &= \sum_{k=1}^{n} (B_{k+1} - B_k) + B_1 \\ &= \sum_{k=1}^{K} \frac{2T}{(\log T)^{k+3/2}} w_{2k+2} + O\left(\frac{T}{(\log T)^K}\right) \end{aligned}$$

Furthermore, again setting $\epsilon(T) = \min\{1-c, 1/\log T\}$, we have the bound

$$\begin{aligned} &\int_c^1 \frac{(1-\sigma)^{n+1/2} e^{\sigma \log T}}{\sigma} d\sigma \\ &= \int_{1-\epsilon(T)}^1 \frac{(1-\sigma)^{n+1/2} e^{\sigma \log T}}{\sigma} d\sigma + \int_c^{(1-\epsilon(T))} \frac{(1-\sigma)^{n+1/2} e^{\sigma \log T}}{\sigma} d\sigma \\ &= O\left(\frac{T}{(\log T)^{n+3/2}}\right) \end{aligned}$$

This completes the proof of Lemma 7.

Comparing Lemmas 5, 6 and 7 completes the proof of Proposition 9.

7. Proof of Theorem 1

We are now in a position to prove Theorem 1, modulo the proof of Proposition 1 (which will appear in section 9). It is convenient to work with the auxiliary function

$$\Pi(T, \alpha) = \#\{\gamma : N(\gamma) \leq T, [\gamma] = \alpha\}.$$

This may be expressed as a Stieltjes integral

$$\Pi(T, \alpha) = \int_2^T \frac{1}{(\log x)^{g+1}} d\tilde{\psi}(x, \alpha) + O(1),$$

where

$$\tilde{\psi}(T, \alpha) = \sum_{\substack{N(\gamma) \leq T \\ [\gamma] = \alpha}} (hl(\gamma))^{g+1}.$$

The following lemma tells us that the asymptotic expression that we have obtained for $\psi(T, \alpha)$ also holds for $\tilde{\psi}(T, \alpha)$.

Lemma 8. *$\tilde{\psi}(T,\alpha) = \psi(T,\alpha) + O\left(T^{\theta}\right)$, for any $\theta > 1/2$.*

Proof. We can write

$$\begin{aligned} |\psi(T,\alpha) - \tilde{\psi}(T,\alpha)| &= \sum_{n=2}^{\infty} \sum_{\substack{N(\gamma)^n \le T \\ n[\gamma]=\alpha}} (hl(\gamma))^{g+1} n^g \\ &= O\left(\Pi(T^{1/2})(\log T)^{2g+2}\right) \\ &= O\left(T^{1/2}(\log T)^{2g+1}\right) \end{aligned}$$

by Margulis' theorem [18].

In particular, by Proposition 9 we have the following.

Proposition 10. *Let $N \ge 0$.*

(1) *If b is even then we can write*

$$\tilde{\psi}(T,\alpha) = \sum_{n=0}^{N} C_n \frac{T}{(\log T)^{n/2}} + o\left(\frac{T}{(\log T)^{N/2}}\right)$$

(2) *If b is odd then we can write*

$$\tilde{\psi}(T,\alpha) = \sum_{n=1}^{N} C_n \frac{T}{(\log T)^{n/2}} + o\left(\frac{T}{(\log T)^{N/2}}\right)$$

We may calculate

$$\begin{aligned} \Pi(T,\alpha) &= \int_2^T \frac{1}{(\log x)^{g+1}} d\tilde{\psi}(x,\alpha) + O(1) \\ &= \frac{\tilde{\psi}(T,\alpha)}{(\log T)^{g+1}} + (g+1)\int_2^T \frac{\tilde{\psi}(x,\alpha)}{x(\log x)^{g+2}} dx + O(1) \\ &= \frac{\tilde{\psi}(T,\alpha)}{(\log T)^{g+1}} + (g+1)\left(\sum_{n=0}^{N} C_n \int_2^T \frac{1}{(\log x)^{n/2+g+2}} dx\right) \\ &\quad + O\left(\int_2^T \frac{1}{(\log x)^{N/2+g+2}}\right) \end{aligned} \tag{7.1}$$

using Proposition 10.

We can estimate the error term in (7.1) by splitting

$$\int_2^T \frac{1}{(\log x)^{N/2+g+2}} dx = \int_2^{T^{1/2}} \frac{1}{(\log x)^{N/2+g+2}} dx + \int_{T^{1/2}}^T \frac{1}{(\log x)^{N/2+g+2}} dx.$$

Observe that:

(1) $\int_2^{T^{1/2}} \frac{1}{(\log x)^{N/2+g+2}} dx = O\left((\log T) T^{1/2}\right)$; and
(2) $\int_{T^{1/2}}^T \frac{1}{(\log x)^{N/2+g+2}} dx = O\left(\frac{T}{(\log T)^{N/2+g+2}}\right)$

and so we see that the error term is of order $O\left(\frac{T}{(\log T)^{N/2+g+2}}\right)$.

To deal with the terms $= \int_2^T \frac{1}{(\log x)^{n/2+g+2}} dx$ we can integrate by parts to write

$$
\begin{aligned}
&\int_2^T \frac{1}{(\log x)^{n/2+g+2}} dx \\
&= \frac{\mathrm{li}(T)}{(\log T)^{n/2+g+1}} - \left(\frac{n}{2}+g+1\right) \int_2^T \frac{\mathrm{li}(x)}{x(\log x)^{n/2+g+2}} dx \\
&= \sum_{k=0}^{N} \frac{(k-1)!T}{(\log T)^{n/2+g+1+k}} - \left(\frac{n}{2}+g+1\right) \sum_{k=0}^{N} \int_2^T \frac{1}{(\log x)^{n/2+g+3+k}} dx \\
&\quad + O\left(\frac{T}{(\log T)^N}\right)
\end{aligned}
\tag{7.2}
$$

where we have used the well-know identities

$$
\mathrm{li}(T) = \int_2^T \frac{1}{\log x} dx = \frac{T}{\log T} + \frac{T}{(\log T)^2} + \ldots + \frac{(k-1)!T}{(\log T)^k} + \ldots
$$

(cf. [9].)

By induction on (7.2) we see that we can write

$$
\int_2^T \frac{1}{(\log x)^{n/2+g+2}} dx = \sum_{k=0}^{N} c_k \frac{T}{(\log T)^{n/2+g+1+k}} + O\left(\frac{T}{(\log T)^N}\right), \tag{7.3}
$$

for some c_k, $k \geq 0$.

Comparing (7.1), (7.2) and (7.3) completes the proof of Theorem 1.

8. Proof of Theorem 2

The general method of the proof of Theorem 1 applies for Anosov flows in the case that $0 < \rho < 1$. However, for Anosov flows we have available to us only the information on $\eta(s)$ contained in Proposition 4, rather than the stronger results in Proposition 1. In particular, we can replace the integral

$$
\psi(T, \alpha) = \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \frac{\eta(s)}{s} T^s ds. \tag{8.1}
$$

by the integral along the curves

(i) $[c(R) + iR, d + iR]$ and $[d - iR, c(R) - iR]$,
(ii) $[c(R) - iR, c(R) + iR]$
(iii) $[c(R), 1 - r]$ and $[1 - r, c(R)]$, for $r > 0$ arbitrarily small
(iv) $\mathcal{C}_r = \{1 + re^{2\pi i\theta} : -\pi \leq \theta \leq \pi\}$,

where $c(R) = 1 - \frac{c}{R^\rho}$.

Consider first the integral along the line segment $[c(R) - iR, c(R) + iR]$. We have the estimate

$$\left| \frac{1}{2\pi i} \int_{[c(R)-iR,c(R)+iR]} \frac{\eta(s)}{s} T^s ds \right| = O\left(T^{c(R)} \int_1^R t^{\gamma-1} dt \right)$$
$$= O\left(T^{c(R)} R^\gamma \right).$$

Since $R(T) = (\log T)^K$, for some $K > 0$, we can bound

$$T^{c(R)} R^\gamma = T^{1 - \frac{1}{(\log T)^{K\rho}}} (\log T)^{K\gamma} = T \left(e^{-\frac{c \log T}{(\log T)^{K\rho}}} (\log T)^{K\gamma} \right).$$

Thus, providing $K < \rho^{-1}$ we see that the contribution to $\psi(T, \alpha)$ is $O(T/(\log T)^N)$, for any $N > 0$. Unfortunately, the restriction on K leads to a less satisfactory estimate on the next contribution.

More precisely, if we consider the contours $[c(R) + iR, d + iR]$ and $[d - iR, c(R) - iR]$ then we have the bound

$$\left| \int_{[c(R)+iR,d+iR] \cup [d-iR,c(R)-iR]} \frac{\eta(s)}{s} T^s ds \right| = O\left(R^{(\gamma-1)} T^d \right) = O\left(\frac{T}{(\log T)^{(1-\gamma)K}} \right).$$

Since we must take $K < 1/\rho$ this gives us an error term which is not better than

$$O\left(\frac{T}{(\log T)^{[(1-\gamma)1/\rho]}} \right).$$

The other contributions can be estimated as in section 6 to give us

$$\psi(T, \alpha) = \sum_{n=0}^{[1/\rho]-1} C_n \frac{T}{(\log T)^{n/2}} + o\left(\frac{T}{(\log T)^{[1/\rho - \gamma/\rho]}} \right).$$

Recall that we can choose γ/ρ arbitrarily close to 1. Theorem 2 can now be deduced using a similar argument to that in section 7.

If, on the other hand, $\rho \geq 1$ then we need to slightly modify the above analysis. Let $k = [\gamma] + 1$ and define ψ_k inductively by $\psi_k(T, \alpha) = \int_0^T \psi_{k-1}(u, \alpha) du$, where $\psi_1(T, \alpha) = \psi(T, \alpha)$. We then have the identity

$$\psi_k(T, \alpha) = \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \frac{\eta(s)}{s(s+1)\cdots(s+k)} T^s ds.$$

By an analysis similar to the above, we obtain the following estimates

(1) If b is even then we can choose $0 < \delta < 1$ such that

$$\psi_k(T, \alpha) = C_0 T \left(1 + +O\left(\frac{1}{(\log T)^\delta} \right) \right)$$

(2) If b is odd then we can choose $0 < \delta < 1$ such that

$$\psi_k(T, \alpha) = C_0 \frac{T}{(\log T)^{1/2}} \left(+O\left(\frac{1}{(\log T)^\delta} \right) \right)$$

The required estimates for $\Pi(T)$, and hence $\pi(t)$, now follow by iterating a standard argument, to be found, for example, in [9, p. 52]. Note that the value of δ above will be reduced by a factor of 2 at each of the k steps.

9. Proof of Proposition 1

In this section we outline the proof of Proposition 1. In the interests of clarity, we shall assume that $H_1(V, \mathbb{Z})$ is torsion free, however this is no great loss in generality.

Poincaré sections to the flow give rise to an C^1 expanding map $f : \coprod_{i=1}^k U_i \to \coprod_{i=1}^k U_i$, where $\{U_i\}$ are the projections onto the unstable directions of the Markov sections [3], [23]. The pinching condition on the curvatures ensures that f is C^1 [7].

We fix $n_0 > 0$ such that for $n \geq n_0$ we can find distinct points $y_1, y_2 \in \coprod_{i=1}^k U_i$ such that there exists $1 \leq i \leq k$ with $f^n(y_1), f^n(y_2) \in U_i$ and $f^n(y_1) = f^n(y_2)$. We can fix a reference point $x^0 \in \coprod_{i=1}^k U_i$ and choose y_1^0, y_2^0 (corresponding to y_1, y_2, respectively) such that $x^0 = f^n(y_1^0) = f^n(y_2^0)$.

Let $r : \coprod_{i=1}^k U_i \to \mathbb{R}$ be given by the return time between sections and define a function

$$\psi(x, x^0) = (r^n(y_1) - r^n(y_2)) - \left(r^n(y_1^0) - r^n(y_2^0)\right),$$

where $r^n(z) = r(z) + r(fz) + \cdots + r(f^{n-1}z)$.

For each x^0 (and all sufficiently close x), and choices of y_1, y_2, y_1^0, y_2^0 as above, the maps $U_i \ni x \mapsto \psi(x, x^0)$ are C^1. An essential feature of the stable and unstable foliations for the geodesic flow is their uniform non-integrability [5], [10]. In particular, there exists constants $B_1, B_2 > 0$ such that if we look at the one dimensional gradient lines for ψ restricted to U_i then along these curves

$$B_1|x - x^0| \leq |\psi(x, x^0)| \leq B_2|x - x^0|. \tag{9.1}$$

Periodic points $f^n(x) = x$ correspond to closed orbits γ in M, with associated homology classes $[\gamma] \in \mathbb{Z}^b$. We can choose a function $g : \coprod_{i=1}^\infty U_i \to \mathbb{R}^b$ such that g is constant on each set U_i and $g^n(x) = g(x) + g(fx) + \ldots + g(f^{n-1}x) = [\gamma]$. In particular, given $\omega \in \mathbb{R}^b/\mathbb{Z}^b$ we can associate a character $\chi_\omega : H_1(M, \mathbb{Z}) \to \mathbb{R}$ such that $\chi_\omega([\gamma]) = e^{2\pi i\langle\omega, g^n(x)\rangle}$.

We can consider the *transfer operators* $L_{-sr+2\pi i\langle\omega,g\rangle} : C^1(\coprod_{i=1}^k U_i) \to C^1(\coprod_{i=1}^k U_i)$ defined by

$$L_{-sr+2\pi i\langle\omega,g\rangle}w(x) = \sum_{fy=x} e^{-sr(y)+2\pi i\langle\omega,g(y)\rangle}w(y)$$

for $\omega \in \mathbb{R}^b/\mathbb{Z}^b$. Our proof of Proposition 1 involves a bound on the norm of iterates of $L_{-sr+2\pi i\langle\omega,g\rangle}$.

In the proof the function r must be replaced by a function r_σ, which depends on σ. In fact, r_σ differs from r by a coboundary and for $\sigma_0 \leq \sigma \leq h$, we have $L_{-\sigma r_\sigma}1 = e^{P(-\sigma r_\sigma)}1$, where $P(-\sigma r_\sigma)$ denotes the pressure of $-\sigma r_\sigma$ [4]. In particular, $r_h = r$ and $L_{-hr_h}1 = 1$ [19].

Lemma 9. *There exists $C > 1$ and $0 < \theta < 1$ (both independent of $|t| \geq 1$ and $\sigma_0 \leq \sigma \leq h$) such that*

$$||D(L^n_{-(\sigma+it)r_\sigma+2\pi i\langle\omega,g\rangle}w)||_\infty \leq e^{nP(-\sigma r_\sigma)}\left(C|t|||w||_\infty + \theta^n||Dw||_\infty\right), \quad \textit{for } n \geq 0.$$

Remark. For simplicity, we will ignore the contributions of $e^{P(-\sigma r_\sigma)}$ (since this can be made to grow at arbitrarily slow rates by choosing σ sufficiently close to 1).

We shall show that the operator $L_{-(\sigma+2\pi it)r_\sigma}$ is a contraction in norm $||w|| = ||w||_\infty + \frac{1}{|t|}||Dw||_\infty$ on C^1-functions. There are two separate cases depending on whether $2C|t|||w||_\infty$ is bigger or smaller than $||Dw||_\infty$.

Case I: $2C|t|||w||_\infty \leq ||Dw||_\infty$. We can fix $\frac{1}{2} < \eta < 1$ and choose $k > 0$ and $\sigma_0 < h$ sufficiently large that $\left(\frac{1}{2} + \theta^k\right) < \eta$. Lemma 9 gives that

$$\frac{1}{|t|}||D(L^k_{-(\sigma+2\pi it)r_\sigma+i\langle\omega,g\rangle>}w)||_\infty \leq C||w||_\infty + \theta^k\frac{1}{|t|}||Dw||_\infty \leq \left(\frac{1}{2} + \theta^k\right)\frac{1}{|t|}||Dw||_\infty.$$

In addition, $||L^k_{-(\sigma+2\pi it)r_\sigma+i\langle\omega,g\rangle>}w||_\infty \leq ||w||_\infty \leq \frac{1}{2C}\frac{1}{|t|}||Dw||_\infty$.

Case II : $2C|t|||w||_\infty \geq ||Dw||_\infty$. To prove $||\cdot||_\infty$-contraction (and subsequently $||\cdot||$-contraction) we first establish $L^1(\mu_\sigma)$-contraction with respect to an appropriate measure μ_σ.

We want to associate a sequence of functions $u_N > 0$, $N \geq 0$, such that:

1. $0 \leq |L^{nN}_{-(\sigma+2\pi it)r_\sigma+i\langle\omega,g\rangle}w(x)| \leq u_N(x)$;
2. there exists $0 < \beta < 1$ such that $\int u_N(x)d\mu(x) \leq \beta^N$ (and β is independent of w, t and σ).

In addition, the functions u_N are constructed so that they are C^1 on each set U_i and

3. $||Du_N(x)|| \leq 2C|t||u_N(x)|$.

The functions u_N are defined inductively as follows:

(i) Fix $u_0 = 1$;

(ii) Given the C^1 function $u_N(x)$ we want to choose a pair of pre-images y_1, y_2, say, for each point x. Consider the corresponding terms in $(L^n_{-(\sigma+it)r_\sigma+2\pi i\langle\omega,g\rangle}u_N)(x)$. We can choose $0 < \eta_0 < 1$, $\pi/2 \leq \theta_0 \leq 3\pi/2$ and $\frac{\pi}{2} > \delta > 0$ such that whenever

$$|t[r^n(y_1) - r^n(y_2)] + 2\pi[\langle\omega, g^n(y_1)\rangle - \langle\omega, g^n(y_2)\rangle]| \in [\theta_0 - \delta, \theta_0 + \delta] \tag{9.2}$$

(mod 2π) then we have

$$|e^{2\pi\langle\omega,g^n(y_1)\rangle-(\sigma+it)r^n(y_1)}u_N(y_1) + e^{2\pi\langle\omega,g^n(y_1)\rangle-(\sigma+it)r^n(y_2)}u_N(y_2)|$$
$$\leq \eta_0\left(e^{2\pi\langle\omega,g^n(y_1)\rangle-\sigma r^n(y_1)}u_N(y_1) + e^{2\pi\langle\omega,g^n(y_2)\rangle-\sigma r^n(y_2)}u_N(y_2)\right).$$

Let $\mathcal{A}_{t,\theta_0,\delta}$ denote the set of x with pre-images y_1 and y_2 satisfying (9.2) then we can choose a smooth function $\eta_0 \le \eta(y) \le 1$ such that

$$\eta(y) = \begin{cases} \eta_0 & \text{if } x \in \mathcal{A}_{t,\theta_0,\frac{\delta}{2}} \\ 1 & \text{if } x \in \left(\cup_{i=1}^k U_i\right) - \mathcal{A}_{t,\theta_0,\delta} \end{cases}.$$

We then set $u_{N+1}(x) = L^n_{-\sigma r_\sigma}\left(\eta u_N\right)(x)$.

The following lemma gives important estimates on the properties of the probability measures satisfying $L^*_{-\sigma r_\sigma}\mu_\sigma = \mu_\sigma$ cf. [7].

Lemma 10. *There exist $R_1, R_2 > 0$ such that for any $x \in \coprod_{i=1}^k U_i$ there exists $x' \in \coprod_{i=1}^k U_i$ with $d(x,x') \le R_1/|t|$ and $B(x', R_2/|t|) \subset \mathcal{A}_{t,\theta_0,\frac{\delta}{2}}$ such that for all $\sigma_0 < \sigma < h$:*

(a) *There exists $0 < C_1 < C_2 < 1$ such that*

$$C_1 \le \frac{\mu_\sigma(B(x', R_2/|t|))}{\mu_\sigma(B(x, R_1/|t|))} \le C_2;$$

(b) *There exists $C_3, C_4 > 0$ such that*

$$C_3 \le \left|\frac{u_n(z')}{u_n(z)}\right| \le C_4,$$

for $d(z,z') \le R_1/|t|$ and for all $n \ge 0$; and

(c) *There exists $0 < \alpha < 1$ such that*

$$\int_{\mathcal{A}_{t,\theta_0,\frac{\delta}{2}}} u_{n+1}(x)d\mu_\sigma \le \alpha \int_{\mathcal{A}_{t,\theta_0,\frac{\delta}{2}}} u_n(x)d\mu_\sigma, \qquad \text{and}$$

$$\int_{\mathcal{A}^c_{t,\theta_0,\frac{\delta}{2}}} u_{n+1}(x)d\mu_\sigma \le \int_{\mathcal{A}^c_{t,\theta_0,\frac{\delta}{2}}} u_n(x)d\mu_\sigma.$$

In the proof of this lemma, the point x' is chosen to lie on the same gradient line for ψ as x. This enables us to employ (9.1).

We now establish contraction in $L^1(\mu_\sigma)$. From estimates (a) and (b) above we have

$$C_1C_3 \le \frac{\int_{\mathcal{A}_{t,\theta_0,\frac{\delta}{2}}} u_N d\mu_\sigma}{\int_{\mathcal{A}^c_{t,\theta_0,\frac{\delta}{2}}} u_N d\mu_\sigma} \le C_2C_4.$$

Estimate (c) then shows that for some $\alpha < \beta < 1$ we have $\int u_{N+1}(x)d\mu_\sigma \le \beta \int u_N(x)d\mu_\sigma$ and so

$$\int |L^{nN}_{-(\sigma+it)r_\sigma + 2\pi i\langle\omega,g\rangle} w|d\mu \le \beta^N,$$

where we can assume for simplicity the normalization $||w||_\infty = 1$. We refer the reader to [7] for details.

We now establish uniform contraction of $L_{-(\sigma+it)r_\sigma+2\pi i\langle\omega,g\rangle}$. We can use the quasi-compactness of $L_{-\sigma r_\sigma}$ to choose $c > 0$ and $0 < \rho < 1$ (independent of $\sigma_0 \le \sigma \le h$) such that

$$\left|\left|L^n_{-\sigma r_\sigma} w - \int w d\mu_\sigma\right|\right|_\infty \le c||w||\rho^n, \quad \forall n \ge 0$$

We can use this to write

$$\begin{aligned}
&||L^{2nN}_{-(\sigma+it)r_\sigma+2\pi i\langle\omega,g\rangle} w||_\infty \\
&\le ||L^{nN}_{-\sigma r_\sigma}\left(L^{nN}_{-(\sigma+it)r_\sigma+2\pi i\langle\omega,g\rangle} w\right)||_\infty \\
&\le \int |L^{nN}_{-(\sigma+it)r_\sigma+2\pi i\langle\omega,g\rangle} w| d\mu_\sigma + c||L^{nN}_{-(\sigma+it)r_\sigma+2\pi i\langle\omega,g\rangle} w||\rho^{nN} \\
&\le \beta^N + c\left(C|t|||w||_\infty + ||Dw||_\infty\right)\rho^{nN} \\
&\le \left(\beta^N + 3cC|t|\right)\rho^{nN} \\
&\le E\gamma^N = E\gamma^N||w||_\infty
\end{aligned} \tag{9.3}$$

with $E > 0$ and $\max(\beta, \rho^n) < \gamma < 1$ (and where $E > 0$ can be assumed to be independent of $|t|$ provided we allow that N does depend on $|t|$ i.e. $N = O(\log|t|)$).

We now establish norm contraction of $L_{-(\sigma+it)r_\sigma++2\pi i\langle\omega,g\rangle}$ in Case II. We can use Lemma 9 to write

$$\begin{aligned}
&\frac{1}{|t|}||D(L^{2nN}_{-(\sigma+it)r_\sigma+2\pi i\langle\omega,g\rangle} w)||_\infty \\
&\le \left(C||L^{Nn}_{-(\sigma+it)r_\sigma+2\pi i\langle\omega,g\rangle} w||_\infty + \theta^{Nn}\frac{1}{|t|}||D(L^{Nn}_{-(\sigma+it)r_\sigma} w)||_\infty\right) \\
&\le CE\gamma^{N/2}||w||_\infty + \theta^{Nn}\left(C||w||_\infty + \frac{1}{|t|}||Dw||_\infty\right) \\
&\le F\tau^N||w||
\end{aligned}$$

using (9.3), with $F > 0$ and $\max(\gamma^{1/2}, \theta^n) < \tau < 1$. Finally, all that remains is to convert this spectral estimate into a bound on the L-function, which can be expressed as

$$L(s, \chi_\omega) = \exp\left(\sum_{n=1}^\infty \frac{1}{n}\sum_{f^n x = x} e^{-sr^n(x)+2\pi i\langle\omega,g^n(x)\rangle}\right).$$

This uses the following result, which is essentially due to Ruelle.

Lemma 11. [22], [25] *There exists $0 < \alpha < 1$ such that for any points $x_i \in U_i$ ($i = 1, \dots, k$) we can estimate*

$$\sum_{T^n x = x} e^{-sr^n(x)+2\pi i\langle\omega,g^n(x)\rangle} = \sum_{i=1}^k \left(L^n_{-sr+2\pi i\langle\omega,g\rangle}\chi_{U_i}\right)(x_i) + O\left(|t|\gamma^n\right)$$

for $n \ge 1$.

10. Proof of Proposition 4

In this section we give the proof of Proposition 4, which gives bounds on the L-functions in the greater generality of Anosov flows. This method is due to Dolgopyat [8]. In this situation, the bounds we obtain are somewhat weaker. For convenience we shall assume that $H_1(M, \mathbb{Z})$ is torsion free.

Our proof depends on the reduction to a symbolic model. Given a $k \times k$ aperiodic matrix A we can define a space

$$X_A = \{x \in \prod_{n=-\infty}^{\infty} \{1, \ldots, k\} : A(x_n, x_{n+1}) = 1, n \in \mathbb{Z}\}$$

with a metric $d(x,y) = \sum_{n=-\infty}^{\infty} \frac{1-\delta(x_n,y_n)}{2^{|n|}}$, where $\delta(i,j)$ is the Kronecker symbol. The shift map $\sigma : X_A \to X_A$ given by $(\sigma x)_n = x_{n+1}$ is a homeomorphism. Assume that $r : X_A \to \mathbb{R}$ a strictly positive Hölder continuous function then we define $X_A^r = \{(x,t) : 0 \leq t \leq r(x)\}$, subject to the identification $(x, r(x)) \sim (\sigma x, 0)$, and a flow $\sigma_t^r : X_A^r \to X_A^r$ defined locally by $\sigma_t^r(x,u) = (x, u+t)$, up to the identifications. It follows by results of Ratner and Bowen that any Anosov flow can be modeled by such a symbolic flow (for complete details see [3], [23]). In particular, periodic points $\sigma^n x = x$ correspond to periodic orbits γ for the Anosov flow $\phi : M \to M$ with $l(\gamma) = r^n(x)$. (In fact, this correspondence is not one-to-one but there is a standard technique for dealing with this discrepancy and we will ignore it [3].)

We can also define a corresponding one-sided shift space

$$X_A^+ = \{x \in \prod_{n=0}^{\infty} \{1, \ldots, k\} : A(x_n, x_{n+1}) = 1, n \in \mathbb{Z}\}$$

with a metric $d(x,y) = \sum_{n=0}^{\infty} \frac{1-\delta(x_n,y_n)}{2^n}$. The shift map $\sigma : X_A \to X_A$ given by $(\sigma x)_n = x_{n+1}$ is a local homeomorphism. Given the Hölder continuous function $r : X_A \to \mathbb{R}$ above, we may assume without loss of generality that it depends only on future co-ordinates (i.e. $r(x) = r(y)$, if $x_n = y_n$ for $n \geq 0$). We may therefore consider it defined as a function $r : X_A^+ \to \mathbb{R}^+$.

There is a locally constant function $g : X_A \to \mathbb{Z}^b$ such that if the periodic orbit $\sigma^n x = x$ corresponds to a closed orbit γ for $\phi_t : M \to M$ then $g^n(x) = [\gamma] \in H_1(M, \mathbb{Z}) \cong \mathbb{Z}^b$. In particular, we have

$$L(s, \chi_\omega) = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \sum_{\sigma^n x = x} e^{-sr^n(x) + 2\pi i \langle \omega, g^n(x)\rangle}\right), \qquad w \in \mathbb{R}^d/\mathbb{Z}^d$$

Given $\alpha > 0$, let $C^\alpha(X_A^+)$ denote the space of Hölder continuous functions $w : X_A^+ \to \mathbb{C}$ with exponent α. This is a Banach space with respect to the norm $||w|| = ||w||_\infty + ||w||_\alpha$, where

$$||w||_\infty = \sup\{|w(x)| : x \in X_A^+\} \text{ and } ||w||_\alpha = \sup\left\{\frac{|w(x) - w(y)|}{d(x,y)^\alpha} : x \neq y\right\}.$$

We can consider the transfer operator $L_{s,\omega}: C^\alpha(X_A^+) \to C^\alpha(X_A^+)$ defined as before by the formula

$$L_{s,\omega}w(x) = \sum_{\sigma y = x} e^{-sh^* r(y) + 2\pi i \langle \omega, g(y)\rangle + \langle \xi, g\rangle} w(y).$$

The following lemma links the transfer operator and the L-function.

Lemma 12. *For each one cylinder $[i] = \{x \in X_A^+ : x_0 = i\}$ fix a point $x^i \in [i]$ (i.e. the first term of the sequence is $(x^i)_0 = i$). There exists a constant $K > 0$ such that*

$$\left| \sum_{\sigma^n x = x} e^{-sh^* r^n(x) + 2\pi i \langle \omega, g^n(x)\rangle + \langle \xi, g^n(x)\rangle} - \sum_{i=1}^{k} L_{s,\omega}^n \chi_{[i]}(x_i) \right| \leq K|t|^2 e^{nP(-\sigma r)} \left(1 - \frac{1}{|t|^\tau}\right)^{\frac{n}{2N}}$$

where $\chi_{[i]}$ is the characteristic function for $[i]$, and where $N = [C \log |t|]$ for some $C > 0$.

Lemma 12 is a consequence of an estimate Ruelle in [25], and Proposition 11 below. From Lemma 12, the bounds in Proposition 4 readily follow.

The next three lemmas give estimates which will be useful to us later.

Lemma 13. *(cf. [24]) Let μ be the unique equilibrium measure for $-h^* r + \langle \xi, g\rangle$ Assume $w \in C^\alpha(X_A^+)$ satisfies $||w||_\infty \leq 1$ and $||w||_\alpha \leq |t|$ then for $0 \leq n \leq 2N$:*

$$||L_{1+it,\omega}^{2N} w||_\infty \leq \int |L_{1+it,\omega}^n w| d\mu + O\left(||L_{1+it,\omega}^n|| \delta^{2N-n}\right) \tag{10.1}$$

for any δ chosen larger than the modulus of the second eigenvalue of L_{-hr}.

Proof. For $x \in X_A^+$, we can bound

$$\begin{aligned} |L_{1+it,\omega}^{2N} w(x)| &\leq L_{1,0}^{2N-n}(|L_{1+it,\omega}^n w|)(x) \\ &\leq \int |L_{1+it,\omega}^n w| d\mu + O\left(||L_{1+it,\omega}^n|| \delta^{2N-n}\right). \end{aligned}$$

We can assume for simplicity that $L_{\sigma,0}1 = 1$, by replacing $-\sigma h^* r + \langle \xi, g\rangle$ by $-\sigma h^* r + \langle \xi, g\rangle + u \circ \sigma - u - P(-\sigma r)$, for appropriate u [19]. With this simplification the following result is well-known.

Lemma 14. *There exists $C_0 > 0$ such that*

$$||L_{1+it,\omega}^n w|| \leq C_0 |t| ||w||_\infty + \left(\frac{1}{2}\right)^{n\alpha} ||w||_\alpha, \quad \forall n \geq 0 \tag{10.2}$$

(The constant $C_0 > 0$ will be the constant in the statement of Proposition 11. It is independent of w and n [19].)

Lemma 15. *Given $\tau > 0$ there exists $\tau_0 > 0$ such that provided*

(1) $||w||_\infty = 1$ *and* $||w||_\alpha \le |t|$,

(2) *there exists* $x \in X_A^+$ *and* $0 \le n \le N$ *with* $|L^{2N}_{1+it,\omega} w(x)| \le 1 - \frac{1}{|t|^\tau}$,

then

$$||L^{2N}_{1+it,\omega} w||_\infty \le 1 - \frac{1}{|t|^{\tau_0}} \tag{10.3}$$

for $|t|$ sufficiently large.

Proof. As is easily observed from (10.2), $||L^n_{1+it,\omega}|| \le (C_0+1)|t|$, for all $n \ge 0$. In particular, we see that whenever $y \in B(x, \epsilon) = \{y : d(x, y) \le \epsilon\}$, where $\epsilon > 0$ is chosen such that $\epsilon^\alpha = \frac{1}{2(C_0+1)|t|^{\tau+1}}$, we have by hypothesis

$$\begin{aligned} |L^{2N}_{1+it,\omega} w(y)| &\le |L^{2N}_{1+it,\omega} w(x)| + (C_0 + 1)|t|\epsilon^\alpha \\ &\le \left(1 - \frac{1}{|t|^\tau}\right) + (C_0 + 1)|t|\epsilon^\alpha, \end{aligned} \tag{10.4}$$

for $n \ge 0$ and w with $||w||_\infty = 1$ and $||w||_\alpha \le |t|$. Furthermore, from the definition of Gibbs measures, we see that there exists $D > 0$ (independent of $\epsilon > 0$)

$$\mu(B(x, \epsilon)) \ge \epsilon^{D\alpha} = \left(\frac{1}{2(C_0 + 1)|t|^{\tau+1}}\right)^D \tag{10.5}$$

Thus we have from (10.4) and (10.5) that

$$\begin{aligned} \int |L^{2N}_{1+it,\omega} w| d\mu &\le \int_{B(x,\epsilon)^c} |L^{2N}_{1+it,\omega} w| d\mu + \int_{B(x,\epsilon)} |L^{2N}_{1+it,\omega} w| d\mu \\ &\le (1 - \mu(B(x, \epsilon))) + \mu(B(x, \epsilon)) \left(1 - \frac{1}{2|t|^\tau}\right) \\ &\le 1 - \frac{1}{2^{1+D}(C_0 + 1)^D |t|^{\tau(1+D)+D}} \end{aligned} \tag{10.7}$$

for $|t|$ sufficiently large. Thus comparing (10.1) and (10.7) we see that

$$\begin{aligned} ||L^{2N}_{1+it,\omega} w||_\infty &\le \left(1 - \frac{1}{2^{1+D}(C_0 + 1)^D |t|^{\tau(1+D)+D}}\right) + O\left(|t|\delta^N\right) \\ &\le \left(1 - \frac{1}{2^{1+D}(C_0 + 1)^D |t|^{\tau(1+D)+D}}\right) + O\left(|t||t|^{C \log \delta}\right) \\ &\le 1 - \frac{1}{|t|^{\tau_0}} \end{aligned} \tag{10.8}$$

for $|t|$ sufficiently large, where we choose $\tau_0 > \min\{\tau(1 + D) + D, C|\log \delta| - 1\}$, provided only that we previously choose $C > 0$ sufficiently large that $C|\log \delta| > 1$. This completes the proof of Lemma 15.

The next result gives important information on the spectrum of the operator.

Proposition 11. *$\exists \tau > 0$, $\exists C_0 > 0$, and $\exists C > 0$, such that $\forall |t| \geq 1$, $\forall n \geq 1$,*

$$||L^{2Nn}_{\sigma+it,\omega}|| \leq C_0 |t| e^{2NnP(-\sigma r)} \left(1 - \frac{1}{|t|^\tau}\right)^{n-1}$$

where $\sigma \geq 1 - \frac{1}{|t|^\rho}$ and $N = [C \log |t|]$.

Proof. With the earlier reductions it suffices to show that there exist $\tau > 0$, and $C > 0$ such that for $m \geq 1$,

$$||L^{2mN}_{1+it_k,\omega}|| \leq C_0 |t| \left(1 - \frac{1}{|t|^\tau}\right)^{m-1}, \tag{10.9}$$

where and $|t|$ is sufficiently large. The proof is by contradiction. The first step is to show that if we assume that (10.9) does not hold then there exist

(1) $\tau > ||r||_\infty / |\log \delta|$,
(2) $1/|\log \delta| < C < \tau/||r||_\infty$,
(3) $|t_k| \to +\infty$,
(4) $w_k \in C^\alpha(X_A^+)$, with $||w_k||_\infty = 1$ and $||w_k||_\alpha \leq t_k$

such that for all $0 \leq n \leq N$,

$$\inf_{x \in X_A^+} |L^n_{1+it_k,\omega} w_k(x)| \geq 1 - \frac{1}{|t_k|^\tau}. \tag{10.10}$$

From the assumption that (10.9) is false we can deduce that for all $\tau > ||r||_\infty / |\log \delta|$, and $C > 0$ there exist sequences t_k with $|t_k| \to +\infty$ and $m_k \to +\infty$ such that

$$||L^{m_k N}_{1+it_k,\omega}|| > 2C_0 |t| \left(1 - \frac{1}{|t_k^\tau|}\right)^{m_k - 1}.$$

In order to establish the existence of objects in (1)-(4) satisfying (10.10) let us assume, for a contradiction, that it is false. In particular, we then have for all $\tau > ||r||_\infty / |\log \delta|$ and $1/|\log \delta| < C < \tau/||r||_\infty$, for all sufficiently large $|t|$ and functions w with $||w||_\infty = 1$ and $||w||_\alpha \leq |t|$ there exists $0 \leq n \leq N$ and $x \in X_A^+$ such that

$$|L^n_{1+it_k,\omega} w(x)| \leq 1 - \frac{1}{|t|^\tau}.$$

We can apply Lemma 15 to get the supremum bound in (10.3). Moreover, we can improve this to a norm estimate as follows. By (10.2) we have for any $m \geq 1$,

$$\begin{aligned}
&||L^{2Nm}_{1+it,\omega} w|| \\
&\leq C_0 |t| ||L^{2N(m-1)}_{1+it_k,\omega} w||_\infty + \left(\frac{1}{2}\right)^{2N(m-1)} ||L^{2N}_{1+it_k,\omega} w||_\alpha \\
&\leq C_0 |t| \left(\left(1 - \frac{1}{|t|^{\tau_0}}\right)^{(m-1)} + \left(\frac{1}{2}\right)^{2N(m-1)} \right) \\
&\leq C_0 |t| \left(1 - \frac{1}{|t|^{\tau_1}}\right)^{(m-1)},
\end{aligned}$$

for any $\tau_1 > \min\{\tau_0, C\log 4\}$, and we assume that $|t|$ is sufficiently large. However, this contradicts our assumption that (10.9) does not hold, and thus shows the validity of (10.10).

For the second step in the proof of Proposition 11 let $w_k \in C^\alpha(X_A^+)$ and t_k be the sequences we have just established for (10.10). Then we can write:

$$\begin{aligned}
&w_k(x) = R_0(x)e^{i\theta_0(x)},\\
&L^N_{1+it_k,\omega}w_k(x) = R_1(x)e^{i\theta_1(x)}, \text{ and}\\
&L^{2N}_{1+it_k,\omega}w_k(x) = R_2(x)e^{i\theta_2(x)},
\end{aligned}$$

where R_0, R_1, R_2 are the moduli of these functions, and $\theta_0, \theta_1, \theta_2$ are the arguments. We now show the existence of such w_k leads to an estimate (denoted (10.11) below) which we shall subsequently show cannot hold in the case of weak-mixing transitive Anosov flows. This contradiction will complete the proof of Proposition 11.

Set $\tau' = \tau - C||r||_\infty > 0$ and $n_k = [\log|t_k|]$. We claim that whenever $\sigma^{n_k}y = x$ then

$$\exp\left(i\Theta_1(y,x)\right) = 1 + O\left(\frac{1}{|t_k|^{\tau'}}\right) \qquad \text{and} \qquad \exp\left(i\Theta_2(y,x)\right) = 1 + O\left(\frac{1}{|t_k|^{\tau'}}\right). \tag{10.11}$$

where we denote

$$\Theta_1(y,x) = t_k r^{n_k}(y) + 2\pi\langle w, g^{n_k}(y)\rangle - \theta_1(x) + \theta_0(y),$$

$$\Theta_2(y,x) = t_k r^{n_k}(y) + 2\pi\langle w, g^{n_k}(y)\rangle - \theta_2(x) + \theta_1(y).$$

Since we are assuming $L^{n_k}_{1,0}1 = 1$ we can write

$$\begin{aligned}
&\sum_{\sigma^{n_k}y_1 = x} e^{-r^{n_k}(y_1)}\left(1 - \exp\left(-i\Theta_1(y_1,x)\right)r_0(y_1)\right)\\
&= 1 - e^{-i\theta_1(x)}L^{n_k}_{1+it_k,\omega}w(x)\\
&= 1 - r_1(x).
\end{aligned} \tag{10.12}$$

Since by estimate (10.10) we can bound $1 - r_1(x) = O\left(1/|t_k|^\tau\right)$, we can estimate from (10.12) that for each $\sigma^{n_k}y_1 = x$:

$$\begin{aligned}
1 - \exp\left(-i\Theta_1(y_1,x)\right)r(y_1) &= O\left(\frac{e^{n_k||r||_\infty}}{|t_k|^\tau}\right)\\
&= O\left(|t_k|^{C||r||_\infty}\frac{1}{|t_k|^\tau}\right) = O\left(\frac{1}{|t_k|^{\tau'}}\right).
\end{aligned}$$

This proves the first part of (10.11). The second part follows similarly.

Finally, we shall show that (10.11) is inconsistent with $\phi_t : M \to M$ being a transitive weak-mixing Anosov flow. We begin by making choices such that $\sigma^{n_k}y_0 = \sigma^{n_k}y_1 = x$ and

$\sigma^{n_k} y_2 = \sigma^{n_k} y_3 = z$ where $d(y_0, y_2) = \left(\frac{1}{2}\right)^{n_k}$ and $d(y_1, y_3) = \left(\frac{1}{2}\right)^{n_k}$. We shall use Ξ to denote the set of values

$$\Delta(y_0, y_1, y_2, y_3) = r^{n_k}(y_0) + r^{n_k}(y_3) - r^{n_k}(y_1) - r^{n_k}(y_2),$$

where y_0, y_1, y_2, y_3 range over the above choices. For transitive Anosov flows it is obvious that Ξ contains an interval. However, we shall show that if (10.11) holds then we obtain a contradiction.

From (10.11) we see that:

$$\exp\left(i\Theta_1(y_0, x)\right) = 1 + O\left(\frac{1}{|t_k|^{\tau'}}\right) \text{ and } \exp\left(i\Theta_2(y_2, z))\right) = 1 + O\left(\frac{1}{|t_k|^{\tau'}}\right);$$

$$\exp\left(i\Theta_1(y_1, x)\right) = 1 + O\left(\frac{1}{|t|^{\tau'}}\right) \text{ and } \exp\left(i\Theta_2(y_3, z)\right) = 1 + O\left(\frac{1}{|t_k|^{\tau'}}\right).$$

Taking ratios of the first pair of expressions we see that

$$\exp\left(i(\Theta_1(y_0, x) - \Theta_2(y_2, z))\right) = 1 + O\left(\frac{1}{|t_k|^{\tau'}}\right)$$

and taking ratios of the second pair of expressions we see that

$$\exp\left(i(\Theta_1(y_1, x) - \Theta_2(y_3, z))\right) = 1 + O\left(\frac{1}{|t_k|^{\tau'}}\right).$$

Taking further ratios we get

$$\begin{aligned} &\exp\left(i(\Theta_1(y_0, x) - \Theta_2(y_2, z)) - i(\Theta_1(y_1, x) - \Theta_2(y_3, z))\right) \\ &= \exp\left(it_k\Delta(y_0, y_1, y_2, y_3)\right)\exp\left(2\pi i\langle \omega, g^{n_k}(y_0) + g^{n_k}(y_3) - g^{n_k}(y_1) - g^{n_k}(y_2)\rangle)\right) \\ &= 1 + O\left(\frac{1}{|t_k|^{\tau'}}\right) \end{aligned} \tag{10.13}$$

Since the range of g is a finite set, we can take it to be equal to $\Lambda = \{\lambda_1, \dots, \lambda_N\}$, say, then

$$\langle \omega, g^{n_k}(y_0) + g^{n_k}(y_3) - g^{n_k}(y_1) - g^{n_k}(y_2)\rangle \in \Lambda^{2n_k} - \Lambda^{2n_k}$$

where $\Lambda^{2n_k} = \{\lambda_{i_0} + \ldots + \lambda_{i_{2n_k}} : \lambda_{i_0}, \dots, \lambda_{i_{2n_k}} \in \Lambda\}$.

In particular, we can assume that for $|t_k|$ sufficiently large,

$$\Xi \subset \bigcup_{n\in\mathbb{Z}} \bigcup_{\lambda\in\Lambda^{2n_k}-\Lambda^{2n_k}} \left[\frac{2\pi n + \lambda}{|t_k|} - \frac{1}{|t_k|^{\tau'}}, \frac{2\pi n + \lambda}{|t_k|} + \frac{1}{|t_k|^{\tau'}}\right]$$

However, the cardinality of $\Lambda^{2n_k} - \Lambda^{2n_k}$ is at most N^{4n_k}. Since $n_k \asymp \log|t_k|$, we see that provided τ' is sufficiently large this contradicts Ξ containing an interval.

This contradiction completes the proof of Proposition 11.

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Department of Mathematics, University of Manchester, Oxford Road, Manchester M13 9PL, UK