# INVARIANCE PRINCIPLES FOR INTERVAL MAPS WITH AN INDIFFERENT FIXED POINT

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Abstract. In this note we establish an almost sure invariance principle for a large class of interval maps with indifferent fixed points, including the Manneville-Pomeau map. This implies a number of well-known corollaries, including the Weak Invariance Principle and the Law of the Interated Logarithm.

## 0. Introduction

It is a classical problem in ergodic theory to understand the statistical properties of typical orbits. For example, the Birkhoff ergodic theorem describes the average behaviour of such orbits and the Central Limit Theorem describes the deviation from this average. These results are subsumed by more general invariance principles.

The situation for uniformly hyperbolic systems is reasonably well understood. In this note, we shall study a particular class of non-uniformly hyperbolic systems. Let $T : X \to X$ be a continuous transformation of the interval $X = [0, 1]$ preserving an absolutely continuous probability measure $\mu$. Assume that $T$ is expanding, except at an indifferent fixed point. More precisely, for $0 < \alpha < 1$, we consider the class $\mathfrak{I}_\alpha$ of $C^2$ interval maps $T : X \to X$, with a fixed point $T(0) = 0$, such that:

(i) $T'(0) = 1$;
(ii) $T'(x) > 1$ for $0 < x \leq 1$;
(iii) there exists $c \neq 0$ such that $\lim_{x\searrow 0} T''(x)x^{1-\alpha} = c$.

Any transformation $T \in \mathfrak{I}_\alpha$ has an absolutely continuous invariant probability measure $\mu$. A simple example is provided by the Manneville-Pomeau map $T_\alpha : [0, 1] \to [0, 1]$ defined by $T_\alpha(x) = x + x^{1+\alpha} \pmod 1$, for $0 < \alpha < 1$.

Let $\phi : X \to \mathbb{R}$ be a Hölder continuous function with $\int \phi d\mu = 0$. We say that $\phi$ is a coboundary if there exists $u \in C^0(X, \mathbb{R})$ such that $\phi = u \circ T - u$. We introduce the sequence

$$\phi^n(x) = \phi(x) + \phi(Tx) + \ldots + \phi(T^{n-1}x), \text{ for each } n \geq 1.$$

Under the hypothesis that $0 < \alpha < \frac{1}{2}$, Young [19] and Liverani, Saussol and Vaienti [14] established the Central Limit Theorem, i.e., provided $\phi$ is not a coboundary then

$$\lim_{n\to\infty} \mu \left\{x : \phi^n(x) < \sqrt{n}\sigma t\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2} du,$$

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where $\sigma^2 = \int \phi(x)^2 d\mu + 2\sum_{n=1}^{\infty} \int \phi(T^n x)\phi(x) d\mu > 0$.

We shall now consider a stronger result than the Central Limit Theorem (cf. [4, p35]). Define a sequence of functions $\zeta_n : X \to C^0([0,1],\mathbb{R})$ by

$$\zeta_n(x) : t \mapsto \frac{1}{\sigma\sqrt{n}}\left(\phi^{[tn]}(x) + (nt - [nt])\phi(T^{[nt]+1}x)\right), n \geq 1,$$

for $0 \leq t \leq 1$, i.e., $\zeta_n(x)$ is the piecewise linear function on $[0,1]$ defined by interpolating between the values $\zeta_n(x)(k/n) = (\sigma\sqrt{n})^{-1}\phi^k(x)$, for $k = 0, \dots, n$. The Weak Invariance Principle asserts that the measures $(\zeta_n)_*\mu$ converges weakly on $C^0([0,1],\mathbb{R})$ to the standard Wiener measure [9].

Our first main result is the following.

**Theorem 1.** *Suppose that $0 < \alpha < \frac{1}{3}$. Let $T \in \mathfrak{I}_\alpha$ and let $\phi : X \to \mathbb{R}$ be a Hölder continuous function with $\int \phi d\mu = 0$. Then the Weak Invariance Principle holds provided $\phi$ is not a coboundary.*

Our second theorem relates to the Law of the Iterated Logarithm. This describes the growth of the sums $\phi^n$ and asserts that

$$\limsup_{n\to+\infty} \frac{1}{\sigma\sqrt{2n\log\log n}}\phi^n(x) = 1, \text{ a.e.}(\mu).$$

**Theorem 2.** *Suppose that $0 < \alpha < \frac{1}{3}$. Let $T \in \mathfrak{I}_\alpha$ and let $\phi : X \to \mathbb{R}$ be a Hölder continuous function with $\int \phi d\mu = 0$. Then the Law of the Iterated Logarithm holds provided $\phi$ is not a coboundary.*

A functional form of the Law of the Iterated Logarithm can also be deduced [4, p.36].

Our interest in these problems was motivated by the earlier work of Liverani, Saussol and Vaienti [14], Young [19] and Isola [11] on central limit theorems for indifferent maps, and the papers of Denker and Philipp [5] and Field, Melbourne and Török [7] on almost sure invariance principles for flows and skew products. Invariance principles for maps admitting an infinite invariant measure were studied in [3].

## 1. Transfer operators, martingales and invariance principles

In this section we shall first describe the induced transformation and associated transfer operator. Let us choose $a_0$ with $T(a_0) = 0$ such that, for the subinterval $Y = [a_0, 1] \subset X$, the map $T : Y \to X$ is a homeomorphism. Given any point $x \in Y$ we can define the first return time to $Y$ by

$$R(x) = \inf\{n \geq 1 : T^n x \in Y\}.$$

We can define an *induced transformation* $S : Y \to Y$ by $S(x) = T^{R(x)}(x)$. The transformation $S : Y \to Y$ preserves an absolutely continuous invariant measure $m$, which is the (normalized) restriction to $Y$ of the measure $\mu$ on $X$.

Let us assume, for simplicity of notation, that $T$ has two inverse branches $T_0 : X \to [0, a_0]$ and $T_1 : X \to [a_0, 1]$ and let us denote $a_n = T_0^n(a_0)$. In particular, this sequence converges monotonically to 0 at a rate $a_n = O(n^{-\frac{1}{\alpha}})$. We can write the inverse branches to the induced map $S$ in the form $T_1T_0^n(x)$, for $x \in [T_1(a_{n+1}), T_1(a_n)]$. In particular, $S : [T_1(a_{n+1}), T_1(a_n)] \to Y$ is continuous and surjective.

Theorems 1 and 2 are standard consequences of the Technical Theorem below, which relates the summations $\phi^n(x)$ to a sequence $\chi^n(x)$ which has the stronger property of being approximated by a Brownian motion.

**Technical Theorem.** *Suppose that $0 < \alpha < \frac{1}{3}$. Let $T \in \mathfrak{I}_\alpha$ and let $\phi : X \to \mathbb{R}$ be a Hölder continuous function with $\int \phi d\mu = 0$. Then there exists a Hölder continuous function $\chi : Y \to \mathbb{R}$, a one-dimensional Brownian motion $W : \Omega \to C^0(\mathbb{R}^+, \mathbb{R})$ on some probability space $(\Omega, \nu)$, such that $W(\cdot)(t)$ has variance $t\sigma^2$, and a sequence of random variables $\Phi_n : \Omega \to \mathbb{R}$ such that:*

(a) *for some $\delta > 0$ and a.e.(m) $x \in Y$ there exists a sequence $k = k(n)$ such that $\phi^n(x) = \chi^k(x) + O(n^{\frac{1}{2}-\delta})$ and $n/k = \int Rdm + o(1)$;*
(b) *the families $\{\Phi_k : k \geq 1\}$ and $\{\chi^k : k \geq 1\}$ have the same distribution (i.e., for every Borel set $A \subset \mathbb{R}$ we have $m\{x \in Y : \widehat{\chi}^k(x) \in A\} = \nu\{\omega \in \Omega : \Phi_k(\omega) \in A\}$, for all $k \geq 1$); and*
(c) *$\Phi_k(\cdot) = W(\cdot)(n) + o(n^{\frac{1}{2}})$, for $n \geq 0$, a.e. $(\nu)$*

*provided $\phi$ is not a coboundary.*

The derivation of Theorems 1 and 2 from almost sure invariance principles of this type is explained in Chapter 1 of [16]. Many other consequences are also discussed there.

Given $0 < \beta \leq 1$, let $C^\beta(Y)$ be the Banach space of Hölder continuous functions of exponent $\beta$ on $Y$ with respect to the usual norm $||f||_\beta = ||f||_\infty + |f|_\beta$ where $|f|_\beta = \sup_{x \neq y} \frac{|f(x)-f(y)|}{|x-y|^\beta}$. We can associated to $S$ a *transfer operator* $\mathcal{L} : C^\beta(Y) \to C^\beta(Y)$ defined by

$$\mathcal{L}f(x) = \sum_{Sx=y} \frac{1}{|S'(y)|} f(y).$$

This operator is well defined (cf. [11] and [17]).

The advantage of studying $S$ instead of $T$ is that we are helped by the additional feature of hyperbolicity and, in particular, by the resulting estimates on the iterates of $\mathcal{L}$. The following lemma collects together some useful estimates.

**Lemma 1.**

(1) *There exists $\lambda > 1$ such that $\inf_{x \in Y} |S'(x)| \geq \lambda$.*
(2) *There exists a constant $C > 0$ such that $\log |S'(x)/S'(y)| \leq C|x - y|$.*
(3) *If $\phi : X \to \mathbb{R}$ is a Hölder continuous function then the function $\psi : Y \to \mathbb{R}$ defined by $\psi(x) = \phi^{R(x)}(x)$ has the property that $(\mathcal{L}\psi) : Y \to \mathbb{R}$ is Hölder continuous.*
(4) *There exists a positive Hölder continuous function $h > 0$ such that $\mathcal{L}h = h$.*
(5) *If we define $Pf(x) = \frac{1}{h}\mathcal{L}(hf)$ then $P1 = 1$ and there exists $0 < \theta < 1$ such that if $\int \phi d\mu = 0$ then $||P^n\psi||_\infty = O(\theta^n)$.*

*Proof.* Since the restriction of $S$ to the interval $[T_1(a_{n+1}), T_1(a_n)] \subset Y$ is a composition of the expanding map $T : Y \to X$ and the non-contracting map $T^n : [a_{n+1}, a_n] \to Y$, part (1) is immediate.

For part (2), we observe that for $x \in [T_1(a_{n+1}), T_1(a_n)]$ we can use the chain rule to write $\log |S'(x)| = \log |(T^n)'(x)| = \log |(T^{n-1})'(Tx)| + \log |T'(x)|$. Since

$T : Y \to X$ is uniformly expanding and $C^2$, the term $\log|T'(x)|$ is Lipschitz. Moreover, the function $\log|(T^{n-1})'|$ is Lipschitz on $[a_{n+1}, a_n]$ by direct calculation, cf. [19, Lemma 5].

For part (3), observe that if $x, y \in [T_1(a_{n+1}), T_1(a_n)]$ then

$$\begin{aligned}|\psi(x) - \psi(y)| &= \sum_{i=0}^{n-1} |\phi(T^i x) - \phi(T^i y)| \\ &\leq ||\phi||_\beta \sum_{i=0}^{n-1} |T^i x - T^i y|^\beta \\ &\leq ||\phi||_\beta \left( \sum_{i=0}^{n-1} \sup_{x \in [T_1(a_{n+1}), T_1(a_n)]} |(T^i)'(x)|^\beta \right) |x-y|^\beta.\end{aligned}$$

The function $\psi$ may have discontinuities at the points $T_1(a_n)$. However, since the map $S : [T_1(a_{n+1}), T_1(a_n)] \to Y$ is surjective we see that $\mathcal{L}\psi : Y \to \mathbb{R}$ is Hölder continuous.

For part (4), the existence of a positive Hölder continuous eigenfunction $\mathcal{L}h = h$ is proved in [17].

Finally, for part (5) it follows using the bounds in parts (1) and (2) that there exists $K > 0$, such that $||P^n\psi|| \leq K||\psi||_\infty + \lambda^{-\beta n}||\psi||_\beta$, for all $n \geq 1$, cf. [15]. The result immediately follows by a standard argument [18, Proposition 5.24].

If we write $U_S f(x) = f(Sx)$ then the condition $P1 = 1$ implies that $P$ is a left inverse to $U_S$, i.e., $PU_S = I$.

A consequence of Lemma 1 is the following result.

**Lemma 2.** *There exists $w \in C^\beta(Y)$ such that if we set $\chi := \psi + (w - U_{\widetilde{T}} w)$ then $P\chi = 0$ and $\psi^n(x) = \chi^n(x) + O(1)$.*

*Proof.* We can define $w \in C^\beta(Y)$ by the series $w := \sum_{n=1}^\infty P^n\psi$ which converges since, by part (5) of Lemma 1, $||P^n\psi||_\beta = O\left(\lambda^{-\beta n}\right)$. We observe that

$$Pw - w = \left(\sum_{n=1}^\infty P^{n+1}\psi\right) - \left(\sum_{n=1}^\infty P^n\psi\right) = -P\psi.$$

Since $PU_S = I$ we see that

$$P\chi = P\psi + P(w - U_S w) = P\psi + (Pw - w) = 0.$$

Finally, we notice that $\psi^n(x) - \chi^n(x) = U_S^n w(x) - w(x)$, so that $|\psi^n(x) - \chi^n(x)| \leq 2||w||_\infty$.

*Remark.* Rather than working with the induced map $S : Y \to Y$, an alternative approach would be to consider the original map $T : X \to X$ and an associated transfer operator $\mathcal{L} : L^p(X, \mu) \to L^p(X, \mu)$, where $p < (2\alpha)^{-1}$. However, using this approach we are only able to show $||\mathcal{L}^n\phi||_p = O(n^{-\gamma})$, for some $\gamma > 1$, and $\psi \in L^p(X, \mu)$. In particular, in this case we can only prove the conclusion of the Technical Theorem in the smaller range $0 < \alpha < \frac{1}{4}$.

To proceed with the analysis, we need to replace the sequence of functions $\chi^n(x)$ with another sequence (on a related space) which forms a martingale. We consider the natural extension $\overline{S} : \overline{Y} \to \overline{Y}$ of $S : Y \to Y$, which is the space consisting of all sequences $\underline{x} = (x_n)_{n=-\infty}^{0}$ in $Y$ satisfying $Sx_{n-1} = x_n$, for $n \leq 0$. We denote by $\overline{m}$ the associated $\overline{S}$-invariant measure on $\overline{Y}$. There is a canonical projection from $\overline{Y}$ to $Y$ defined by $\pi(\underline{x}) = x_0$. The $\sigma$-algebra $\mathcal{B}$ for $Y$ allows us to associate a natural $\sigma$-algebra $\mathcal{B}_0 = \pi^{-1}\mathcal{B}$ on the natural extension. Let us denote $\mathcal{B}_n := \overline{S}^n \mathcal{B}_0$, for $n \geq 0$.

*Definition.* Given a nested sequence of $\sigma$-algebras $\mathcal{B}_0 \subset \mathcal{B}_1 \subset \mathcal{B}_2 \subset \ldots$, a sequence of functions $\Phi_n : \overline{Y} \to \mathbb{R}$ is called an *increasing martingale* if $\Phi_n$ is $\mathcal{B}_n$-measurable, and $E(\Phi_n | \mathcal{B}_{n-1}) = \Phi_{n-1}$ (or, equivalently, $E(\Phi_n - \Phi_{n-1} | \mathcal{B}_{n-1}) = 0$).

The function $\chi : Y \to \mathbb{R}$ naturally extends to a function $\overline{\chi} : \overline{Y} \to \mathbb{R}$ by $\overline{\chi}\left((x_n)_{n=-\infty}^{0}\right) = \chi(x_0)$. We shall now denote

$$\overline{\chi}^n(\underline{x}) := \overline{\chi}(\overline{S}^{-n}\underline{x}) + \ldots + \overline{\chi}(\overline{S}^{-2}\underline{x}) + \overline{\chi}(\overline{S}^{-1}\underline{x}).$$

Since $\overline{\chi}$ is $\mathcal{B}_0$-measurable, it immediately follows that $\overline{\chi}^n$ is $\mathcal{B}_n$-measurable.

**Lemma 3.** *The sequence $\overline{\chi}^n$ is a martingale with respect to the increasing sequence of $\sigma$-algebras $\mathcal{B}_n$, $n \geq 1$.*

*Proof.* For $n \geq 1$, we can write

$$\begin{aligned} E(\overline{\chi}^n - \overline{\chi}^{n-1} | \mathcal{B}_{n-1}) &= E(\overline{\chi} \circ \overline{S}^{-n} | \overline{S}^{n-1}\mathcal{B}_0) \\ &= E(\overline{\chi} | \overline{S}^{-1}\mathcal{B}_0) \circ \overline{S}^n \\ &= E(\chi | S^{-1}\mathcal{B}) \circ S^n \\ &= U_S^{n+1} P\chi = 0, \end{aligned}$$

since $P\chi = 0$. In particular, the sequence $\overline{\chi}^n$ is a martingale, as required.

The function $\psi$ has variance $\widetilde{\sigma}^2$ defined by

$$\widetilde{\sigma}^2 = \int \psi(x)^2 dm + 2\sum_{n=1}^{\infty} \int \psi(S^n x)\psi(x) dm > 0.$$

If we replace $\psi$ by the cohomologous function $\chi$ then the variance is unchanged, i.e.,

$$\widetilde{\sigma}^2 = \int \chi(x)^2 dm + 2\sum_{n=1}^{\infty} \int \chi(S^n x)\psi(x) dm > 0.$$

This is readily seen from alternative characterizations of the variance [15, Chapter 4].

**Lemma 4.**

(1) *We may write $\sigma^2 = \left(\int R dm\right)^{-1} \widetilde{\sigma}^2$.*
(2) *We may write $\widetilde{\sigma}^2 = \int \chi(x)^2 dm$.*

*Proof.* For part (1) it is technically easier to work with the invertible natural extension $\overline{T} : \overline{X} \to \overline{X}$ of $T : X \to X$ which we can identify with $\{(\underline{x}, i) : \underline{x} \in \overline{Y}, 0 \leq$

$i \le R(\underline{x})-1\}$. Corresponding to $\phi$ and $\mu$ we have $\overline{\phi}$ and $\overline{\mu}$ on the natural extension. Observe that

$$\overline{\phi}(\overline{T}^m(\underline{x}, i)) = \sum_{n=-\infty}^{+\infty} \sum_{j=0}^{R(\overline{S}^n \underline{x})-1} \overline{\phi}(\overline{S}^n \underline{x}, j)\delta(i+m-j-R^n(\underline{x})),$$

where $\delta(\cdot)$ denotes the Dirac delta function. Substituting into the definition of $\sigma^2$ we obtain

$$\begin{aligned}
&\sum_{m=-\infty}^{+\infty} \int \overline{\phi}(\overline{T}^m(\underline{x}, i))\overline{\phi}(\underline{x}, i)d\overline{\mu} \\
&= \left(\int Rd\overline{m}\right)^{-1} \sum_{n=-\infty}^{+\infty} \int \left(\sum_{i=0}^{R(\underline{x})-1} \overline{\phi}(\underline{x}, i)\right)\left(\sum_{j=0}^{R(\overline{S}^n \underline{x})-1} \overline{\phi}(\overline{S}^n \underline{x}, j)\right) d\overline{m} \\
&= \left(\int Rd\overline{m}\right)^{-1} \sum_{n=-\infty}^{+\infty} \int \overline{\psi}(\underline{x})\overline{\psi}(\overline{S}^n \underline{x})d\overline{m},
\end{aligned}$$

where we understand $\overline{\psi}(\underline{x})$ as being defined on $\overline{Y}$, in a natural way.

For part (2) we observe that, for $n \ge 1$,

$$\begin{aligned}
\int \chi(S^n x)\chi(x)dm &= \int P^n\left(\chi(S^n x)\chi(x)\right)dm \\
&= \int \chi(x)(P^n\chi)(x)dm = 0.
\end{aligned}$$

## 2. The proof of part (a) of the Technical Theorem

For a.e.$(m)$ $x \in Y$ we can associate to each $n \ge 0$ the unique value $k = k(n)$ satisfying

$$R(x) + R(Sx) + \ldots + R(S^{k-1}x) \le n < R(x) + R(Sx) + \ldots + R(S^{k-1}) + R(S^k x).$$

By the Birkhoff ergodic theorem we know that

$$k(n) = n\left(\int_Y Rd\widetilde{m}\right)^{-1}(1+o(1))$$

and using the notation $R^k(x) := R(x) + R(Sx) + \ldots + R(S^{k-1}x)$ we have that $|R^k(x) - n| \le R(S^k x)$. In particular, we can bound

$$|\phi^n(x) - \psi^k(x)| = |\phi^n(x) - \phi^{R^k}(x)| \le ||\phi||_\infty R(S^k x).$$

Recall that $\psi$ and $\chi$ differ by a coboundary, i.e., $\chi^k = \psi^k + (w - U_S w)$, and thus we can uniformly bound $|\psi^k(x) - \chi^k(x)| \le 2||w||_\infty$. Therefore, using the triangle inequality, we can bound

$$|\phi^n(x) - \chi^k(x)| \le ||\phi||_\infty R(S^k x) + 2||w||_\infty.$$

Thus to complete the proof of part (a) of the Technical Theorem it suffices to show that there exists $\delta > 0$ such that $R(S^n x) = O(n^{1/2-\delta})$, a.e.$(m)$. The following estimate is well-known (cf. [11]).

**Lemma 5.** $m\{x \in Y \,:\, R(x) \geq n\} = O\left(n^{\left(1-\frac{1}{\alpha}\right)}\right)$.

For $\alpha < \frac{1}{3}$ and $\delta > 0$ sufficiently small we have that $(\frac{1}{2} - \delta)(\frac{1}{\alpha} - 1) > 1$ and we can bound

$$\sum_{n=1}^{\infty} m\{x \in Y \,:\, R(S^n x) \geq n^{\frac{1}{2}-\delta}\} \leq \sum_{n=1}^{\infty} n^{\left(\frac{1}{2}-\delta\right)\left(1-\frac{1}{\alpha}\right)} < \infty.$$

Thus by the Borel-Cantelli Lemma we see that $R(S^n x) = O(n^{1/2-\delta})$, a.e.$(m)$ and this completes the proof of part (a) the Technical Theorem.

## 3. The proof of parts (b) and (c) of the Technical Theorem

In this section we complete the proof of the Technical Theorem by establishing the last two parts. Let $(\Omega, \nu)$ be a probability space. Recall that a stochastic process $W : \Omega \to C^0(\mathbb{R}^+, \mathbb{R})$ is called a *Brownian motion* if

(a) $W(\omega)(0) = 0$, a.e. $(\nu)$;
(b) there exists $\sigma^2 > 0$ such that for each $t_0 > 0$ the values $\omega \mapsto W(\omega)(t_0) \in \mathbb{R}$ have a normal distribution with variance $t_0\sigma^2$;
(c) for times $t_0 < t_1 < \ldots < t_n$ the differences $\omega \mapsto W(\omega)(t_{i+1}) - W(\omega)(t_i) \in \mathbb{R}$ are independent random variables.

The following result is standard.

**Lemma 6.** *Brownian motion satisifies the law of the iterated logarithm, i.e.,*

$$\limsup_{t\to+\infty} \frac{|W(\omega)(t)|}{\sqrt{2\sigma^2 t \log\log t}} = 1, \textit{ a.e.}(\nu).$$

We shall follow the analysis of Field, Melbourne and Török, based on a treatment of Philipp and Stout, for the martingale $\overline{\chi}^k$. A key ingredient is the martingale version of the Skorokhod embedding theorem [9, Appendix I] which we now state.

**Proposition 1** [9, Appendix 1]**.** *There exists a Brownian motion $W^*(\cdot)$ on a probability space $(\Omega, \nu)$ such that $W^*(\cdot)(t)$ has variance $t$, an increasing sequence of $\sigma$-algebras $\mathcal{F}_k$, and sequences of random variables $\tau_k : \Omega \to \mathbb{R}^+$ such that*

(1) $\Phi_k := W^*(T_k)$, *where* $T_k := \sum_{l=0}^{k-1} \tau_l$, *has the same distribution as* $\overline{\chi}^k$;
(2) $\Phi_k$ *and* $T_k$ *are* $\mathcal{F}_k$*-measurable; and*
(3) $E(\tau_l|\mathcal{F}_{l-1}) = E([\Phi_l - \Phi_{l-1}]^2|\mathcal{F}_{l-1})$, *a.e.* $(\nu)$, *for each* $l \geq 1$.

The above result immediately implies part (b) of the Technical Theorem.

To obtain the estimate in part (c) of the Technical Theorem we need to replace $T_k$ by $\widetilde{\sigma}^2 k$, where $\widetilde{\sigma}^2 = \int_Y \chi^2 dm > 0$. Following [16, p.11], and using part (3) of Proposition 1, we can write

$$\begin{aligned} T_k - \widetilde{\sigma}^2 k = &\sum_{l=0}^{k-1} \{\tau_l - E(\tau_l|\mathcal{F}_{l-1})\} \\ &+ \sum_{l=0}^{k-1} \left\{E\left([\Phi_l - \Phi_{l-1}]^2|\mathcal{F}_{l-1}\right) - [\Phi_l - \Phi_{l-1}]^2\right\} \\ &+ \sum_{l=0}^{k-1} [\Phi_l - \Phi_{l-1}]^2 - \widetilde{\sigma}^2 k, \text{ a.e. } (\nu), \end{aligned} \tag{3.1}$$

where we set $\Phi_{-1} = 0$. Both the first and second terms on the Right Hand Side of (3.1) are martingales since

$$E\left(\tau_l - E(\tau_l|\mathcal{F}_{l-1})|\mathcal{F}_l\right) = 0, \text{ and}$$
$$E\left(E\left([\Phi_l - \Phi_{l-1}]^2|\mathcal{F}_{l-1}\right) - [\Phi_l - \Phi_{l-1}]^2|\mathcal{F}_l\right) = 0.$$

We can therefore invoke the strong law of large numbers for martingales [6, §VII.9, Theorem 3] for these terms in (3.1) to see that, for any $\delta > 0$,

$$T_k - \widetilde{\sigma}^2 k = \sum_{l=0}^{k-1} [\Phi_l - \Phi_{l-1}]^2 - \widetilde{\sigma}^2 k + O(k^{1/2+\delta}), \text{ a.e. } (\nu). \tag{3.2}$$

To estimate the summation in (3.2) we shall consider the following integral

$$I_\delta := \int_\Omega \left( \sum_{l=1}^{\infty} l^{-(1/2+\delta)} \left( [\Phi_l(\omega) - \Phi_{l-1}(\omega)]^2 - \widetilde{\sigma}^2 \right) \right)^2 d\nu(\omega),$$

for $\delta > 0$.

The next lemma relates $I_\delta$ to the function $\chi$.

**Lemma 7.** *We can write*

$$I_\delta = \int_Y \left( \sum_{l=1}^{\infty} l^{-(1/2+\delta)} \left( \left[\chi(S^l x)\right]^2 - \int_Y \chi^2 dm \right) \right)^2 dm(x).$$

*Proof.* By Proposition 1, $\Phi_k$ and $\overline{\chi}^k$ are equal in distribution. Moreover, since $S$ is the natural extension of $S$ we have that $\overline{\chi}^k$ and $\chi^k$ are equal in distribution. Given any measurable function $F : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ we have that

$$\int_Y F\left((\chi^k(x))_{k=0}^{\infty}\right) dm(x) = \int_\Omega F\left((\Phi_k(\omega))_{k=0}^{\infty}\right) d\nu(\omega)$$

[1, Proposition 2.39]. To obtain the required result we choose

$$F\left((x_l)_{l=0}^{\infty}\right) = \left( \sum_{l=1}^{\infty} l^{-(1/2+\delta)} \left( [x_{l+1} - x_l]^2 - \int_Y \chi^2 dm \right) \right)^2,$$

noting that $\chi(S^l x) = \chi^{l+1}(x) - \chi^l(x)$.

For convenience we introduce a function $\rho$ defined by $\rho(x) := \chi^2(x) - \int_Y \chi^2 dm$.

**Lemma 8.** *There exists $C > 0$ and $0 < \theta < 1$ such that $|\int_Y \rho \circ S^k \rho dm| \le C\theta^k$, for $k \ge 0$.*

*Proof.* By part (5) of Lemma 1, we can bound $|\int_Y \rho \circ \S^k \rho dm| = |\int_Y \rho(P^k \rho) dm| \le C\theta^k$.

Consider the expansion

$$
\begin{aligned}
I_\delta &:= \sum_{l=1}^{\infty}\sum_{p=1}^{\infty} l^{-(1/2+\delta)} p^{-(1/2+\delta)} \int_Y \rho\circ S^l \rho\circ S^p dm \\
&= \sum_{l=1}^{\infty} l^{-(1+2\delta)} \int_Y \rho^2 dm + 2\sum_{l=1}^{\infty}\sum_{d=1}^{\infty} l^{-(1/2+\delta)}(l+d)^{-(1/2+\delta)} \int_Y \rho\circ S^d \rho dm \\
&\le \|\rho^2\|_\infty \left(\sum_{l=1}^{\infty} l^{-(1+2\delta)}\right) + 2C\left(\sum_{l=1}^{\infty} l^{-(1+2\delta)}\right)\left(\sum_{d=1}^{\infty}\theta^d\right) < +\infty,
\end{aligned}
$$

where we have used Lemma 8. In particular, we deduce that

$$
\sum_{l=1}^{\infty} l^{-(1/2+\delta)}\left([\Phi_l(\omega)-\Phi_{l-1}(\omega)]^2 - \int_Y \chi^2 dm\right)
$$

is finite a.e.($\nu$), for any $\delta > 0$. Applying the Kronecker lemma [9, p.31] we can deduce that

$$
\sum_{l=1}^{k-1}\left[[\Phi_l(\omega)-\Phi_{l-1}(\omega)]^2 - \int_Y \chi^2 dm\right] = O(k^{1/2+\delta}). \tag{3.3}
$$

Comparing (3.2) and (3.3) shows that $T_k = \widetilde{\sigma}^2 k + O(k^{1/2+\delta})$ and so $\Phi_k(\cdot) = W^*(T_k) = W^*(\widetilde{\sigma}^2 k) + O(k^{1/4+\delta})$.

To complete the proof of part (c) of the Technical Theorem we define a rescaled Brownian motion by $W(\cdot)(t) = W^*(\cdot)(\sigma^2 t)$. Then, for $k = k(n)$,

$$
\begin{aligned}
W^*(T_k) &= W^*(\widetilde{\sigma}^2 k) + O(n^{1/4+\delta}) \\
&= W^*\left(\frac{\widetilde{\sigma}^2 n}{\int R dm} + o(n)\right) + O(n^{1/4+\delta}) \\
&= W^*(\sigma^2 n) + o(n^{1/2}) \\
&= W(n) + o(n^{1/2})
\end{aligned}
\tag{3.3}
$$

*Remarks.*

(1) The error term in part (c) of the Technical Theorem can be improved to give $\Phi_k(\cdot) = W(\cdot)(n) + O(n^{1/4+\delta})$, for any $\delta > 0$. More precisely, by comparing known results on the rate of mixing in [14] and [19] with estimates on the rate of convergence in the Birkhoff ergodic theorem in [12, Theorem 16, part 3] we have the stronger estimate $k(n) = n(\int R dm)^{-1} + O(n^{1/2+\delta})$, for any $\delta > 0$. This allows (3.3) to be improved to $W^*(T_k) = W(n) + O(n^{1/4+\delta})$, for any $\delta > 0$.

(2) The method above can be adapted to study other systems (e.g., higher dimensional analogues of the interval maps considered here [10] and rational maps). It also applies to certain types of abstract tower model, as introduced by Young [19] providing the return time map $R$ satisfies $\sum_{n=1}^{\infty}\mu\{x : R(x) \ge \sqrt{n}\} < \infty$. Another way in which one could generalise these results is to consider more general invariant Gibbs measures [2].

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