A NON-SYMMETRIC KESTEN CRITERION AND RATIO LIMIT THEOREM FOR RANDOM WALKS ON AMENABLE GROUPS

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Abstract. We consider random walks on countable groups. A celebrated result of Kesten says that the spectral radius of a symmetric walk (whose support generates the group as a semigroup) is equal to one if and only if the group is amenable. We give an analogue of this result for walks which are not symmetric. We also conclude a ratio limit theorem for amenable groups.

1. Introduction

Let G be a finitely generated countable group. Let μ be a probability measure on G, i.e. a function $\mu : G \to \mathbb{R}^+$ such that $\sum_{g\in G} \mu(g) = 1$. Let $S_\mu := \{g \in G : \mu(g) > 0\}$, the support of μ. We say that μ is *non-degenerate* if S_μ generates G as a semigroup. (We do not require S_μ to be finite.)

Let $|\cdot|$ be a word metric on G associated to some finite generating set. (We do not assume any connection between this set and S_μ.) We say that μ has finite first moment if

$$\sum_{g\in G} |g|\mu(g) < \infty$$

and that μ has finite exponential moment of order $c > 0$ if

$$\sum_{g\in G} e^{c|g|}\mu(g) < \infty.$$

The measure μ defines a random walk on G with transition probabilities $p(s,t) = \mu(s^{-1}t)$. The convolution $\mu * \nu$ of two functions $\mu, \nu : G \to \mathbb{R}^+$ is defined by

$$\mu * \nu(g) = \sum_{s\in G} \mu(s)\nu(s^{-1}g).$$

We will be interested in the *spectral radius* of this random walk, defined by

$$\lambda(G,\mu) := \limsup_{n\to\infty} (\mu^{*n}(e))^{1/n}.$$

Clearly, $0 \le \lambda(G,\mu) \le 1$. A celebrated theorem of Kesten (which does not even require G to be finitely generated) says that if μ is symmetric then $\lambda(G,\mu) = 1$ if and only if G is amenable [8]. (We recall that G is amenable if and only if it admits a Banach mean, i.e. a linear functional $M : \ell^\infty(G) \to \mathbb{R}$ such that $M(\mathbb{1}) = 1$, $\inf_{g\in G} f(g) \le M(f) \le \sup_{g\in G} f(g)$, and $M(f_g) = M(f)$, where $f_g(x) = f(gx)$. See the papers of Følner [5] and Day [2] for further discussion.) The aim of this note is to generalise Kesten's criterion to the non-symmetric case.

To state our generalisation, we need to consider the abelianisation of G. Since G is finitely generated, this has a finite rank $k \ge 0$. Let $G^{\mathrm{ab}} = G/[G,G]$ denote the abelianisation of G and let $G^{\mathrm{ab}}_{\mathrm{T}}$ denote the torsion subgroup of G^{ab}. Now set $\overline{G} = G^{\mathrm{ab}}/G^{\mathrm{ab}}_{\mathrm{T}} \cong \mathbb{Z}^k$, for some $k \ge 0$, (the torsion-free part of the abelianisation) and let $\pi : G \to \overline{G}$ be the natural projection homomorphism. Write $\bar{\mu} = \pi_*(\mu)$, i.e.

$$\bar{\mu}(m) = \sum_{\substack{g \in G \\ \pi(g)=m}} \mu(g).$$

Theorem 1.1 (Non-symmetric Kesten criterion)**.** *Let G be a finitely generated group and let μ be a non-degenerate probability measure on G. Then*

$$G \text{ amenable} \iff \lambda(G,\mu) = \lambda(\overline{G},\bar{\mu}).$$

The special case where μ has finite support originally appeared in Dougall–Sharp [4], where it is written in the language of subshifts of finite type and Gibbs measures.

The value of $\lambda(\overline{G},\bar{\mu})$ may be characterised in the following way. Define $\phi_\mu : \mathbb{R}^k \to \mathbb{R}^+ \cup \{+\infty\}$ by

$$\phi_\mu(v) = \sum_{g \in G} e^{\langle v, \pi(g)\rangle} \mu(g) = \sum_{m \in \mathbb{Z}^k} e^{\langle v, m\rangle} \bar{\mu}(m),$$

where $\langle v, m\rangle = v_1 m_1 + \cdots + v_k m_k$. By a result of Stone [15], [16], there is a unique $\xi \in \mathbb{R}^k$ at which $\phi_\mu(v)$ attains its minimum. Then $\lambda(\overline{G},\bar{\mu}) = \phi_\mu(\xi)$. This is discussed in more detail in Section 2.

A probability measure μ (with finite first moment) is said to be *centred* if for each homomorphism $\chi : G \to \mathbb{R}$, we have

$$\sum_{g \in G} \chi(g)\mu(g) = 0.$$

Any such homomorphism factors through $\overline{G}$ so it is easy to see that μ is centred if and only if either $k = 0$ or

$$\sum_{g \in G} \pi(g)\mu(x) = \sum_{m \in \bar{G}} m\bar{\mu}(m) = 0.$$

In particular, μ is centred if and only if $\bar{\mu}$ is centred.

If, in addition, μ has a finite exponential moment of some order then we have the following result.

Corollary 1.2. *Let G be a finitely generated group and let μ be a non-degenerate probability measure on G. Provided μ has a finite exponential moment of some order, we have $\lambda(G,\mu) = 1$ if and only if G is amenable and μ is centred.*

Remark 1.3. In fact, the "if" direction above, that G amenable and μ centred implies that $\lambda(G,\mu) = 1$, is true if μ (and hence $\bar{\mu}$) has finite first moment; while the "only if" direction uses the exponential moment condition.

Theorem 1.1 allows us to prove a ratio limit theorem for amenable groups with an explicit limit. To avoid any parity issues, it is convenient to restrict to aperiodic walks. We say that (G,μ) is *aperiodic* if there exists $n_0 \geq 1$ such that $\mu^{*n}(e) > 0$ for all $n \geq n_0$.

Theorem 1.4 (Ratio limit theorem)**.** *Suppose that G is a finitely generated amenable and that μ is a non-degenerate probability measure on G. Assume in addition that (G,μ) is aperiodic. Then, for each $g \in G$,*

$$\lim_{n\to\infty} \frac{\mu^{*n}(g)}{\mu^{*n}(e)} = e^{-\langle \xi, \pi(g)\rangle},$$

where $\xi \in \mathbb{R}^k$ is the unique value for which $\lambda(G,\mu) = \phi_\mu(\xi)$.

Remark 1.5. One should compare this with a theorem of Avez [1] that says that if G is amenable and μ is *symmetric*, non-degenerate and aperiodic then $\lim_{n\to\infty} \mu^{*n}(g)/\mu^{*n}(e) = 1$, for all $g \in G$.

Remark 1.6. It should be noted that there is no *a priori* mechanism to guarantee that the ratios do indeed have a limit. However, notice that if one has the ratio limits

$$\lim_{n\to\infty}\frac{\mu^{*n}(g)}{\mu^{*n}(e)} = e^{-\langle\xi,\pi(g)\rangle},$$

for all $g \in G$, for *some* ξ then G is necessarily amenable. We give the short demonstration. From the hypothesis we have for any $s \in G$,

$$\frac{\mu^{*n}(s^{-1})}{\mu^{*n}(e)} = e^{\langle\xi,\pi(s)\rangle}.$$

Now, since

$$\mu^{*(n+1)}(g) = \sum_{s\in G}\mu(s)\mu^{*n}(s^{-1}g),$$

we then have

$$\lim_{n\to\infty}\frac{\mu^{*(n+1)}(e)}{\mu^{*n}(e)} = \lim_{n\to\infty}\sum_{s\in S_\mu}\mu(s)\frac{\mu^{*n}(s^{-1})}{\mu^{*n}(e)} \geq \sum_{s\in S_\mu}\mu(s)e^{\langle\xi,\pi(g)\rangle} = \phi_\mu(\xi). \tag{1.1}$$

In particular $\phi_\mu(\xi) < \infty$. We proceed with the proof assuming that $\xi = 0$ and deduce the general case after.

Now using that

$$\frac{\mu^{*n}(e)}{\mu^{*1}(e)} = \prod_{m=2}^{n}\frac{\mu^{*m}(e)}{\mu^{*(m-1)}(e)},$$

we see that (1.1) with $\phi_\mu(0) = 1$ implies that $\limsup_{n\to\infty}(\mu^{*n}(e))^{1/n} = 1$. This contradicts Day's [2] generalisation of Kesten's criterion to the random walk operator spectral radius — a consequence of which is that, for any non-degenerate probability, we have that if G is non-amenable then $\limsup_{n\to\infty}(\mu^{*n}(e))^{1/n} < 1$.

For the general case $\xi \neq 0$, we have already shown that $\phi_\mu(\xi) < \infty$, and so

$$\hat{\mu}(g) = \frac{e^{\langle\xi,\pi(g)\rangle}}{\phi_\mu(\xi)}\mu(g)$$

is a well-defined probability measure on G with ratio limits equal to one, and we again conclude that G is amenable.

Let us now outline the contents of the rest of the paper. In Section 2, we recall results of Stone on random walks on $\mathbb{Z}^k$ that are essential to the formulation of our results, and the rather general results of Gerl. In Section 3, we give a proof of Corollary 1.2 assuming Theorem 1.1. We prove Theorem 1.1 in Sections 4 and 5. Theorem 1.4 is proved in Section 6, as a consequence of equidistribution results for countable state shifts.

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2. Results of Stone and Gerl

In this section, we recall classic results of Stone concerning random walks on $\mathbb{Z}^k$. Let ω be a non-degenerate aperiodic probability measure on $\mathbb{Z}^k$ and define $\phi_\omega : \mathbb{R}^k \to \mathbb{R}\cup\{+\infty\}$ by

$$\phi_\omega(v) = \sum_{m\in\mathbb{Z}^k} e^{\langle v,m\rangle}\omega(m).$$

Lemma 2.1 (Stone [15], [16])**.** *If ω is non-degenerate then there is a unique $\xi \in \mathbb{R}^k$ such that $\phi_\omega(\xi) = \inf_{v\in\mathbb{R}^k}\phi_\omega(v)$. Furthermore, $\lambda(\mathbb{Z}^k,\omega) = 1$ if and only if $\phi_\omega(\xi) = 1$, i.e. if and only if $\xi = 0$.*

We note that $\phi_\omega(\xi) = 1$ if and only if $\phi_\omega(v) \geq 1$ for all $v \in \mathbb{R}^k$.

Corollary 2.2. *$\lambda(\mathbb{Z}^k, \omega) = \phi_\omega(\xi)$.*

Proof. Suppose that $\phi_\omega(\xi) < 1$. Then we can define a new probability measure ω_ξ on $\mathbb{Z}^k$ by

$$\omega_\xi(m) = (\phi_\omega(\xi))^{-1} e^{\langle \xi, m \rangle} \omega(m).$$

Then ω_ξ has the same support as ω and

$$\omega_\xi^{*n}(m) = (\phi_\omega(\xi))^{-n} e^{\langle \xi, m \rangle} \omega^{*n}(m).$$

We have

$$\sum_{m \in \mathbb{Z}^k} e^{\langle v, m \rangle} \omega_\xi(m) = \frac{1}{\phi_\omega(\xi)} \sum_{m \in \mathbb{Z}^k} e^{\langle v+\xi, m \rangle} \omega(m) = \frac{\phi_\omega(v+\xi)}{\phi_\omega(\xi)} \geq 1.$$

Hence, $\lambda(\mathbb{Z}^k, \omega_\xi) = 1$ and

$$\lambda(\mathbb{Z}^k, \omega) = \phi_\omega(\xi) \lambda(\mathbb{Z}^k, \omega_\xi) = \phi_\omega(\xi).$$

□

We now state a ratio limit theorem due to Stone.

Proposition 2.3 (Stone [15])**.** *Suppose that ω is non-degenerate and aperiodic. Then, for each $m \in \mathbb{Z}^k$,*

$$\lim_{n \to \infty} \frac{\omega^{*n}(m)}{\omega^{*n}(0)} = e^{-\langle \xi, m \rangle}. \tag{2.1}$$

Ratio limit theorems are intimately related to the existence of harmonic functions. Given a random walk (G, μ), we define the random walk operator $P_\mu : \ell^1_\mu(G) \to \ell^1_\mu(G)$ by

$$P_\mu f(g) = \sum_{s \in G} p(g, s) f(s) = \sum_{s \in G} \mu(g^{-1}s) f(s).$$

This may be written as a convolution $P_\mu f = f * \check{\mu}$, where $\check{\mu}(s) = \mu(s^{-1})$. A function $f : G \to \mathbb{R}^+$ is called *λ-harmonic* if $P_\mu f = \lambda f$, i.e. if

$$\sum_{s \in G} \mu(g^{-1}s) f(s) = \lambda f(g).$$

(Some authors define f to be λ-harmonic if $\mu * f = \lambda f$.) If we write $h_\xi(m) = e^{-\langle \xi, m \rangle}$ for the limit in Proposition 2.3 then we see that the function $\check{h}_\xi(m) := h_\xi(-m) = e^{\langle \xi, m \rangle}$ is $\lambda(\mathbb{Z}^k, \omega)$-harmonic (for ω). Furthermore $\lambda = \lambda(\mathbb{Z}^k, \omega)$ is the smallest value for which there is a λ-harmonic function.

One may ask about ratio limit theorems on more general groups than $\mathbb{Z}^k$. Following earlier work by Avez [1] and Gerl [6], a rather general ratio limit theorem was proved by Gerl [7], where it is obtained as a corollary of the following limit theorem. A detailed account and discussion may be found in the recent note by Woess [18].

Proposition 2.4 (Gerl's fundamental theorem [7])**.** *Suppose that μ is a non-degenerate probability measure on G such that (G, μ) is aperiodic. Then we have*

$$\lim_{n \to \infty} \frac{\mu^{*(n+1)}(e)}{\mu^{*n}(e)} = \lambda(G, \mu).$$

Gerl used this to prove the following conditional ratio limit theorem.

Proposition 2.5 (Gerl's ratio limit theorem [7])**.** *Suppose that μ is a non-degenerate probability measure on G such that (G, μ) is aperiodic. Suppose there is a set $\mathfrak{F} \subset \{f : G \to \mathbb{R}^+ : f(e) = 1\}$ such that*

(1) if $f : G \to \mathbb{R}^+$ is defined by

$$f(g) = \lim_{j\to\infty} \frac{\mu^{*n_j}(g)}{\mu^{*n_j}(e)},$$

for some subsequence $(n_j)_{j=1}^{\infty}$, then $f \in \mathfrak{F}$;

(2) there exists a unique $h \in \mathfrak{F}$ satisfying the equation $\mu * h = \lambda(G,\mu)h$.

Then

$$\lim_{n\to\infty} \frac{\mu^{*n}(g)}{\mu^{*n}(e)} = h(g),$$

for all $g \in G$.

In particular, if we have uniqueness of a $\lambda(G,\mu)$-harmonic function for (G,μ) then we know the ratio limit theorem holds. The advantage of our Theorem 1.4, for amenable groups, is that we don't consider arbitrary harmonic functions instead we directly work with functions coming from the abelianisation.

3. Proof of Corollary 1.2

In this section we prove Corollary 1.2, assuming Theorem 1.1. We note that $\phi_\mu = \phi_{\bar{\mu}}$, so we can use Lemma 2.1.

Proof of Corollary 1.2. If G is not amenable then Theorem 1.1 tells us that $\lambda(G,\mu) < \lambda(\overline{G},\bar{\mu}) \le 1$; so it suffices to show that, if G is amenable, then $\lambda(\overline{G},\bar{\mu}) = 1$ if and only if μ is centred.

Since $\bar{\mu}$ is non-degenerate, $\phi_\mu(v)$ is strictly convex on the set where it is finite. The hypothesis of a finite exponential moment implies that $\phi_\mu(v)$ is finite and differentiable in some neighbourhood of $0 \in \mathbb{R}^k$. Therefore, $\phi_\mu(v)$ has its unique minimum at $v = 0$ if and only if $\nabla\phi_\mu(0) = 0$. Suppose that $\lambda(\overline{G},\bar{\mu}) = 1$; then, by Lemma 2.1, ϕ_μ has its minimum at 0 and so

$$0 = \nabla\phi_\mu(0) = \sum_{m\in\mathbb{Z}^k} m\bar{\mu}(m),$$

i.e. μ is centred. On the other hand, if $\lambda(\overline{G},\bar{\mu}) < 1$ then, again by Lemma 2.1, the unique minimum of ϕ_μ is not at 0 and so

$$0 \ne \nabla\phi_\mu(0) = \sum_{m\in\mathbb{Z}^k} m\bar{\mu}(m),$$

i.e. μ is not centred. □

4. Proof of Theorem 1.1 ($\Longleftarrow$)

In this section we show that if $\lambda(G,\mu) = \lambda(\overline{G},\bar{\mu})$ then G is amenable. (In Kesten's original theorem, this was the harder direction but in our case it is the easier implication.) Noting that $\phi_\mu = \phi_{\bar{\mu}}$, recall from Section 2 that there is a unique $\xi \in \mathbb{R}^k$ such that

$$\phi_\mu(\xi) = \inf_{v\in\mathbb{R}^k} \phi_\mu(v)$$

and

$$\phi_\mu(\xi) = \sum_{g\in G} \mu(g)e^{\langle \xi,\pi(g)\rangle} = \lambda(\overline{G},\bar{\mu}).$$

We define a new probability measure μ_ξ on G (analogous to the measure ω_ξ on $\mathbb{Z}^k$ in the proof of Corollary 2.2) by

$$\mu_\xi(g) = \phi_\mu(\xi)^{-1}e^{\langle \xi,\pi(g)\rangle}\mu(g).$$

Proof of Theorem 1.1 ($\Longleftarrow$). Assume that G is non-amenable. By Theorem 1 of Day [2] (see also Theorem 5.4 of Stadlbauer [13]), we see that the probability measure μ_ξ satisfies

$$\limsup_{n\to\infty}(\mu_\xi^{*n}(e))^{1/n} < 1.$$

Unpicking the definitions, $\mu_\xi^{*n}(e) = \phi_\mu(\xi)^{-n}\mu^{*n}(e)$. Hence $\limsup_{n\to\infty}(\mu^{*n}(e))^{1/n} < \phi_\mu(\xi)$. □

5. Proof of Theorem 1.1 ($\Longrightarrow$)

In this section we will show the harder implication that if G is amenable then $\lambda(\overline{G},\bar{\mu}) \le \lambda(G,\mu)$, and hence that $\lambda(\overline{G},\bar{\mu}) = \lambda(G,\mu)$. We remark that the proof given here is significantly easier and more direct than the one we gave in [4]. Following that proof would introduce a family of measures, indexed by $g \in G$, on the space $S_\mu^{\mathbb{N}} \times G$, each describing the paths that visit $S_\mu^{\mathbb{N}} \times \{g\}$. These measures (which are also analysed in [14]) are not required here.

Let us begin by emphasising the following: though we know that $\bar{\mu}$ has a $\lambda(\overline{G},\bar{\mu})$-harmonic function *it plays no role in this part of proof*! The first element of the proof is the following proposition. Subsequently, the rest of the section will be devoted to showing its hypothesis is satisfied with $\lambda = \lambda(G,\mu)$.

Proposition 5.1. *If there is a homomorphism $h : \overline{G} \to \mathbb{R}^{>0}$, the multiplicative group of positive real numbers, such that, for all $n \in \mathbb{N}$,*

$$\sum_{g\in G}\mu^{*n}(g)h(-\pi(g)) \le \lambda^n$$

then $\lambda(\overline{G},\bar{\mu}) \le \lambda$.

Proof. Suppose such a homomorphism h exists. Since h is positive so we can throw away the terms where $-\pi(g) \neq 0$ and obtain

$$\sum_{\substack{g\in G\\ \pi(g)=0}}\mu^{*n}(g) \le \lambda^n.$$

Hence, for any $\delta < 1$,

$$\sum_{n\in\mathbb{N}}\bar{\mu}^{*n}(0)(\delta^{-1}\lambda)^{-n} = \sum_{n\in\mathbb{N}}\sum_{\substack{g\in G\\ \pi(g)=0}}\mu^{*n}(g)\delta^n\lambda^{-n} < \infty.$$

This says that $\lambda(\overline{G},\bar{\mu}) \le \lambda\delta^{-1}$. Since we can take δ arbitrarily close to 1 we are done. □

We view $\lambda(G,\mu)$ as a convergence parameter for the series

$$\zeta(t) = \sum_{n\in\mathbb{N}}\mu^{*n}(e)t^{-n},$$

where $t > 0$, i.e.

$$\inf\left\{t\in\mathbb{R}^+ : \sum_{n\in\mathbb{N}}\mu^{*n}(e)t^{-n} < \infty\right\} = \limsup_{n\to\infty}(\mu^{*n}(e))^{1/n} = \lambda(G,\mu).$$

This formulation is reminiscent of the Poincaré series used in the construction of Patterson–Sullivan measures on the limit sets of Kleinian groups and more general limit sets and, indeed, we employ ideas from this theory. The most relevant reference here is Roblin [9], which covers the basic tools of Patterson-Sullivan theory and the amenability "trick" we will use in the proof of Proposition 5.6.

The series $\zeta(t)$ does not necessarily diverge at $t = \lambda(G,\mu)$ but one can modify the series, in a controlled way, to guarantee divergence at this critical parameter. The following appears as Lemma 3.2 in Denker and Urbanski [3] (see also [14]).

Lemma 5.2. *Let $(a_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}^+$ and let $\rho = \limsup_{n\to\infty} a_n^{1/n}$. Then there is a sequence $(b_n)_{n=1}^{\infty}$ of positive real numbers such that $\lim_{n\to\infty} b_{n+1}/b_n = 1$ for which $\sum_{n\in\mathbb{N}} a_n b_n t^{-n} < \infty$ for $t > \rho$ but*

$$\sum_{n\in\mathbb{N}} a_n b_n \rho^{-n} = \infty.$$

Let $(b_n)_{n=1}^{\infty}$ be the sequence given by Lemma 5.2 for the series with $a_n = \mu^{*n}(e)$. We prefer to use $c_n = 1/b_n$, a decreasing sequence. Note that we have $\lim_{n\to\infty} c_{n-r}/c_n = 1$ for all $r \in \mathbb{N}$. We will work with a modified series $\zeta_c^e(t)$ defined by

$$\zeta_c^e(t) = \sum_{n\in\mathbb{N}} \frac{\mu^{*n}(e)}{c_n} t^{-n},$$

and also, for each $g \in G$, the series

$$\zeta_c^g(t) = \sum_{n\in\mathbb{N}} \frac{\mu^{*n}(g)}{c_n} t^{-n}.$$

Lemma 5.3. *For each $g \in G$,*

$$0 < \inf_{\lambda(G,\mu)<t\le 2} \frac{\zeta_c^g(t)}{\zeta_c^e(t)} \le \sup_{\lambda(G,\mu)<t\le 2} \frac{\zeta_c^g(t)}{\zeta_c^e(t)} < \infty.$$

Proof. We begin by observing that, for every $g, h \in G$, we have

$$\mu^{*(n+k)}(g) \ge \mu^{*k}(gh^{-1})\mu^{*n}(h)$$

and we may choose $k \ge 1$ for which $\mu^{*k}(gh^{-1}) > 0$. This gives us the inequality

$$\begin{aligned}\zeta_c^g(t) &= \sum_{m\le k} \frac{\mu^{*m}(g)}{c_m} t^{-m} + \sum_{n\in\mathbb{N}} \frac{\mu^{*(n+k)}(g)}{c_{n+k}} t^{-(n+k)} \\ &\ge \sum_{m\le k} \frac{\mu^{*m}(g)}{c_m} t^{-m} + \mu^{*k}(gh^{-1})t^{-k} \sum_{n\in\mathbb{N}} \frac{\mu^{*n}(h)}{c_n} \frac{c_n}{c_{n+k}} t^{-n}.\end{aligned}$$

Since the numbers c_n are positive and, for a fixed k, $\lim_{n\to\infty} c_n/c_{n+k} = 1$, we have $\inf_n c_n/c_{n+k} > 0$. Hence, for $\lambda(G,\mu) < t \le 2$, we have

$$\zeta_c^g(t) \ge C_1(g,k) + \frac{C_2(g,h,k)}{C_3(k)} \zeta_c^h(t),$$

for positive C_1, C_2 and C_3. We conclude

$$0 < \inf_{\lambda(G,\mu)<t\le 2} \frac{\zeta_c^g(t)}{\zeta_c^h(t)}.$$

Since g, h are arbitrary the lemma follows. (The choice of 2 as an upper bound for t in the lemma is arbitrary; we could work with $\lambda(G,\mu) < t \le c$ for any fixed $c > \lambda(G,\mu)$.) □

By the previous lemma and a standard diagonal argument, there exists a sequence $t_n \to \lambda(G,\mu)$ for which the following limits are well-defined

$$H(g) = \lim_{n\to\infty} \frac{\zeta_c^g(t_n)}{\zeta_c^e(t_n)} \in (0,\infty),$$

for all $g \in G$. A crucial observation is the following.

Lemma 5.4. *For any $r \in \mathbb{N}$, we have*

$$\sum_{s\in G} \mu^{*r}(s) H(s^{-1}g) = \lambda^r H(g)$$

with $\lambda = \lambda(G,\mu)$.

Proof. Fix $r \in \mathbb{N}$ and let $\epsilon > 0$. Since $\lim_{n\to\infty} c_{n-r}/c_n = 1$, we can choose n_0 such that $1-\epsilon \le c_{n-r}/c_n \le 1+\epsilon$, for all $n > n_0$. We will also use that

$$\mu^{*n}(g) = \sum_{s\in G} \mu^{*r}(s)\mu^{*(n-r)}(s^{-1}g).$$

Then, for $n > n_0$,

$$\frac{1-\epsilon}{c_{n-r}} \sum_{s\in G} \mu^{*r}(s)\mu^{*(n-r)}(s^{-1}g) \le \frac{\mu^{*n}(g)}{c_n} \le \frac{(1+\epsilon)}{c_{n-r}} \sum_{s\in G} \mu^{*r}(s)\mu^{*(n-r)}(s^{-1}g).$$

Setting

$$C_1(g,t,n_0) = \sum_{n\le n_0} \frac{\mu^{*n}(g)}{c_n} t^{-n},$$

we have

$$\begin{aligned} t^r \sum_{n\in\mathbb{N}} \frac{\mu^{*n}(g)}{c_n} t^{-n} &\le C_1(g,t,n_0) + t^r(1+\epsilon) \sum_{s\in G} \mu^{*r}(s) \sum_{n>n_0} t^{-n} \frac{\mu^{*(n-r)}(s^{-1}g)}{c_{n-r}} \\ &\le C_1(g,t,n_0) + (1+\epsilon) \sum_{s\in G} \mu^{*r}(s) H(s^{-1}g)\zeta_c^e(t), \end{aligned}$$

using that the terms in the series are non-negative. This gives

$$\lambda^r H(g) \le \lim_{m\to\infty} \frac{C_1(g,t_m,n_0)}{\zeta_c^e(t_m)} + (1+\epsilon) \sum_{s\in G} \mu^{*r}(s)H(s^{-1}g) = (1+\epsilon) \sum_{s\in G} \mu^{*r}(s)H(s^{-1}g)$$

and, since ϵ is arbitrary,

$$\lambda^r H(g) \le \sum_{s\in G} \mu^{*r}(s)H(s^{-1}g).$$

For the lower bound, we have

$$\begin{aligned} t^r \sum_{n\in\mathbb{N}} \frac{\mu^{*n}(g)}{c_n} t^{-n} &\ge (1-\epsilon) \sum_{s\in G} \mu^{*r}(s) \sum_{n>n_0} t^{-(n-r)} \frac{\mu^{*(n-r)}(s^{-1}g)}{c_{n-r}} \\ &= (1-\epsilon) \sum_{s\in G} \mu^{*r}(s) \sum_{n\in\mathbb{N}} t^{-n} \frac{\mu^{*n}(s^{-1}g)}{c_n}. \end{aligned}$$

This gives

$$\lambda^r H(g) \ge \sum_{s\in G} \mu^{*r}(s)H(s^{-1}g).$$

□

Lemma 5.4 gives us the following corollary.

Corollary 5.5. *For any fixed $\gamma \in G$, we have*

$$0 < \inf_{g\in G} \frac{H(\gamma g)}{H(g)} \le \sup_{g\in G} \frac{H(\gamma g)}{H(g)} < \infty.$$

Proof. Given $\gamma \in G$, we can find $x_1,\dots,x_k \in S_\mu$, for some $k \ge 1$, such that $x_1\cdots x_k = \gamma^{-1}$. This gives us

$$\mu(x_1)\cdots\mu(x_k)H(\gamma g) \le \sum_{(s_1,\dots,s_k)\in G^k} \mu(s_1)\cdots\mu(s_k)H((s_1\cdots s_k)^{-1}g) = \lambda^k H(g),$$

and so $\sup_{g\in G} H(\gamma g)/H(g)$ is finite.

Now we put γg on the right hand side and choose $y_1,\dots,y_\ell \in S_\mu$ such that $y_1\cdots y_\ell = \gamma$ to get

$$\mu(y_1)\cdots\mu(y_\ell)H(g) \le \sum_{(s_1,\dots,s_\ell)\in G^\ell} \mu(s_1)\cdots\mu(s_\ell)H((s_1\cdots s_\ell)^{-1}\gamma g) = \lambda^\ell H(\gamma g),$$

and so $\inf_{g\in G} H(\gamma g)/H(g)$ is positive. ∎

We are now ready to use the amenability of G. We use the existence of a Banach mean on $\ell^\infty(G)$ to "average over g" in the last lemma.

Proposition 5.6. *There is a homomorphism $h : \overline{G} \to \mathbb{R}^{>0}$ such that, for all $n \in \mathbb{N}$,*

$$\sum_{s\in G} \mu^{*n}(s)h(-\pi(s)) \le \lambda^n,$$

with $\lambda = \lambda(G, \mu)$.

Proof. Since any homomorphism $h : G \to \mathbb{R}^{>0}$ factors through $\overline{G}$, it suffices to show that there is a homomorphism $h : G \to \mathbb{R}^{>0}$ such that, for all $n \in \mathbb{N}$, we have

$$\sum_{s\in G} \mu^{*n}(s)h(s^{-1}) \le \lambda^n. \tag{5.1}$$

Let M be a left G-invariant Banach mean on $\ell^\infty(G)$. Jensen's inequality says that if φ is convex then

$$M(\varphi(f)) \ge \varphi(M(f)).$$

(This is more familiar when the linear functional is integration with respect to a probability measure. One can check that we only need monotonicity, finite additivity, and the unit normalisation $M(1) = 1$.) We apply this to the function $g \mapsto (H(\gamma g)/H(g))$ in $\ell^\infty(G)$. (Note we have used Corollary 5.5 to know that $g \mapsto \log(H(\gamma g)/H(g))$ is in $\ell^\infty(G)$.) Thus we obtain

$$\begin{aligned} M\left(g \mapsto \frac{H(\gamma g)}{H(g)}\right) &= M\left(g \mapsto \exp\log\frac{H(\gamma g)}{H(g)}\right) \\ &\ge \exp M\left(g \mapsto \log\frac{H(\gamma g)}{H(g)}\right). \end{aligned}$$

Now set

$$h(\gamma) = \exp M\left(g \mapsto \log\frac{H(\gamma g)}{H(g)}\right).$$

We will show that h satisfies (5.1), recalling that M is only finitely additive. Let $\{g_k\}_{k=1}^\infty$ be an enumeration of G and, for $N \ge 1$, let $G_N = \{g_1, \dots, g_N\}$. Lemma 5.4 gives us that, for any $n \ge 1$ and any $N \ge 1$, we have

$$\begin{aligned} \lambda^n &= M\left(g \mapsto \sum_{s\in G} \mu^{*n}(s)\frac{H(s^{-1}g)}{H(g)}\right) \ge M\left(g \mapsto \sum_{s\in G_N} \mu^{*n}(s)\frac{H(s^{-1}g)}{H(g)}\right) \\ &= \sum_{s\in G_N} \mu^{*n}(s)M\left(g \mapsto \frac{H(s^{-1}g)}{H(g)}\right) \ge \sum_{s\in G_N} \mu^{*n}(s)\exp M\left(g \mapsto \log\frac{H(s^{-1}g)}{H(g)}\right) \\ &= \sum_{s\in G_N} \mu^{*n}(s)h(s^{-1}). \end{aligned}$$

Taking the supremum over N gives (5.1).

It remains to show that h is a homomorphism. Notice that

$$\begin{aligned} \log h(ab) &= M\left(g \mapsto \log\frac{H(abg)}{H(g)}\right) \\ &= M\left(g \mapsto \log\frac{H(abg)}{H(bg)}\right) + M\left(g \mapsto \log\frac{H(bg)}{H(g)}\right) \\ &= \log h(a) + \log h(b) \end{aligned}$$

and

$$\log h(\gamma^{-1}) = M\left(g \mapsto \log \frac{H(\gamma^{-1}g)}{H(g)}\right) = M\left(g \mapsto \log \frac{H(g)}{H(\gamma g)}\right)$$
$$= M\left(g \mapsto -\log \frac{H(\gamma g)}{H(g)}\right) = -\log h(\gamma).$$

using translation invariance of M. The conclusion follows. □

Remark 5.7. In the above proof, *any* positive $\lambda(G,\mu)$-harmonic function $H : G \to \mathbb{R}^{>0}$ could be used to construct the desired homomorphism. If μ has finite support then it is known that such a function exists, see Lemma 7.6 of [17].

Combining Proposition 5.1 and Proposition 5.6 shows that if G is amenable then $\lambda(\overline{G},\bar{\mu}) \leq \lambda(G,\mu)$.

6. Equidistribution and proof of the ratio limit theorem

In this section we use Theorem 1.1 to prove a ratio limit theorem for amenable groups, Theorem 1.4. Our arguments will also give a new proof of Proposition 2.4 in this setting. Our approach is based on weighted equidistribution results for countable state shift spaces. Suppose that G is amenable, that μ is non-degenerate and that (G,μ) is aperiodic. We let λ denote the common value

$$\lambda = \phi_\mu(\xi) = \lambda(\overline{G},\bar{\mu}) = \lambda(G,\mu),$$

given by Theorem 1.1. As above, $\mu_\xi(g) = \lambda^{-1}h(g)\mu(g)$, where $h(g) = e^{\langle \xi, \pi(g)\rangle}$.

We consider the sequence space $\Sigma = S_\mu^{\mathbb{N}}$ and let $\sigma : \Sigma \to \Sigma$ be the shift map: $\sigma((s_i)_{i=1}^\infty) = (s_{i+1})_{i=1}^\infty$. If $s = (s_1,\ldots,s_n) \in S_\mu^n$, we write $[s] = [s_1,\ldots,s_n]$ for the cylinder set

$$[s_1,\ldots,s_n] := \{(x_i)_{i=1}^\infty \in \Sigma : x_i = s_i \ \forall i = 1,\ldots,n\}.$$

We give Σ the topology generated by cylinder sets (which are both open and closed).

We denote by ν_ξ the Bernoulli measure on Σ given by

$$\nu_\xi([s_1,\ldots,s_n]) = \mu_\xi(s_1)\cdots\mu_\xi(s_n).$$

Let

$$\Lambda_n = \{s = (s_1,\ldots,s_n) \in S_\mu^n : s_1\cdots s_n = e\}$$

and define a sequence of probability measures m_n on Σ by

$$m_n := \frac{1}{\mu_\xi^{*n}(e)} \sum_{s=(s_1,\ldots,s_n)\in\Lambda_n} \mu_\xi(s_1)\cdots\mu_\xi(s_n)\delta_{s_\infty} = \frac{1}{\mu_\xi^{*n}(e)} \sum_{s=(s_1,\ldots,s_n)\in\Lambda_n} \nu_\xi([s])\delta_{s_\infty},$$

where we use the notation $s_\infty \in \Sigma$ to mean the one-sided infinite concatenation of $s = (s_1,\ldots,s_n)$ and δ_{s_∞} denotes the Dirac measure at this point. We remark that we also have

$$m_n = \frac{1}{\mu^{*n}(e)} \sum_{s=(s_1,\ldots,s_n)\in\Lambda_n} \mu(s_1)\cdots\mu(s_n)\delta_{s_\infty}$$

but we do not use this formula. We will need to explicitly evaluate the measures m_n on cylinder sets.

Lemma 6.1. *For a cylinder set $[u_1,\ldots,u_k]$ we have that, for $n > k$,*

$$m_n([u_1,\ldots,u_k]) = \frac{\mu_\xi(u_1)\cdots\mu_\xi(u_k)}{\mu_\xi^{*n}(e)}\mu_\xi^{*(n-k)}(u^{-1}),$$

where $u = u_1\cdots u_k$.

Proof. This is a straightforward calculation. For $n > k$,

$$\begin{aligned} m_n([u_1,\dots,u_k]) &= \frac{1}{\mu_\xi^{*n}(e)} \sum_{(s_1,\dots,s_n)\in\Lambda_n} \mu_\xi(s_1)\cdots\mu_\xi(s_n)\,\delta_{(s_1,\dots,s_n)_\infty}([u_1,\dots,u_k]) \\ &= \frac{1}{\mu_\xi^{*n}(e)} \sum_{\substack{(s_1,\dots,s_n)\in\Lambda_n \\ s_1=u_1,\cdots,s_k=u_k}} \mu_\xi(s_1)\cdots\mu_\xi(s_n) \\ &= \frac{\mu_\xi(u_1)\cdots\mu_\xi(u_k)}{\mu_\xi^{*n}(e)} \sum_{\substack{(s_{k+1},\dots,s_n)\in S_\mu^{n-k} \\ s_{k+1}\cdots s_n=u^{-1}}} \mu_\xi(s_{k+1})\cdots\mu_\xi(s_n) \\ &= \frac{\mu_\xi(u_1)\cdots\mu_\xi(u_k)}{\mu_\xi^{*n}(e)} \mu_\xi^{*(n-k)}(u^{-1}). \end{aligned}$$

□

We will show that, for each cylinder set $[u_1,\dots,u_k]$, $m_n([u_1,\dots,u_k])$ converges to $\nu_\xi([u_1,\dots,u_k])$, as $n\to\infty$. In order to have *convergence* (as opposed to an accumulation point) we need $\mu_\xi^{*n}(e)^{1/n}$ to have a limit. This is a consequence of aperiodicity, as the next lemma shows.

Lemma 6.2. *We have*

$$\lim_{n\to\infty}(\mu_\xi^{*n}(e)^{1/n}) = 1.$$

Proof. We know that $\limsup_{n\to\infty}(\mu_\xi^{*n}(e))^{1/n} = 1$. Since μ is aperiodic, we have $\mu^{*n}(e) > 0$ for all sufficiently large n. Recall also that $\mu_\xi^{*(n+m)}(e) \ge \mu_\xi^{*n}(e)\mu_\xi^{*m}(e)$. This tells us that $-\log\mu_\xi^{*n}(e)$ is sub-additive. Hence by Fekete's lemma

$$\lim_{n\to\infty}\frac{-\log\mu^{*n}(e)}{n} = \inf_{n\ge1}\frac{-\log\mu^{*n}(e)}{n},$$

in particular the limit exists and is equal to the limsup. □

In order to show the required convergence for the m_n, we introduce some ideas and terminology from thermodynamic formalism and large deviation theory. A function $\varphi:\Sigma\to\mathbb{R}$ is called locally Hölder continuous if

$$\sup_{s\in S_\mu^n}\sup_{x,y\in[s]}|\varphi(x)-\varphi(y)| \le C\theta^n, \tag{6.1}$$

for some $C > 0$ and $0 < \theta < 1$, for all $n \ge 1$. For a locally Hölder continuous function $\varphi:\Sigma\to\mathbb{R}$, we define the *Gurevič pressure* $P_{\mathrm{G}}(\varphi)$ by

$$P_{\mathrm{G}}(\varphi) = \lim_{n\to\infty}\frac{1}{n}\log\sum_{s\in S_\mu^n}\exp\sum_{j=0}^{n-1}\varphi(\sigma^j s_\infty) \in \mathbb{R}\cup\{+\infty\}.$$

(The original definition given by Sarig in [10] is somewhat different, and only requires, (6.1) to hold for $n \ge 2$, but, by Corollary 1 of [11], the above formula gives the Gurevič pressure in our setting.) We now fix

$$\varphi((s_i)_{i=1}^\infty) := \log\mu_\xi(s_1) = \log\nu_\xi([s_1]),$$

so that, in particular, $P_{\mathrm{G}}(\varphi) = 0$. Let χ be the indicator function of some cylinder. We can easily calculate from the definition that, for $t\in\mathbb{R}$, $P_{\mathrm{G}}(\varphi+t\chi) \le \max\{0,|t|\}$ for all $t\in\mathbb{R}$. Hence, by Corollary 4 of [11], $t\mapsto P_{\mathrm{G}}(\varphi+t\chi)$ is real analytic for $t\in\mathbb{R}$ and, by Theorems 6.12 and 6.5 of [12],

$$\left.\frac{dP(\varphi+t\chi)}{dt}\right|_{t=0} = \int\chi\,d\nu_\xi. \tag{6.2}$$

(The same discussion remains true if χ is replaced with any *bounded* locally Hölder function but indicator functions of cylinders are sufficient for our purposes.)

For $s \in S_\nu^n$, let $\tau_{s,n}$ denote the orbital measure

$$\tau_{s,n} := \frac{1}{n}\sum_{j=0}^{n-1} \delta_{\sigma^j(s_\infty)}.$$

Following, Theorem 7.4 of [12], we have the following large deviations bound.

Proposition 6.3. *Given $\epsilon > 0$, there exists $C > 0$ and $\eta(\epsilon) > 0$ such that*

$$\sum_{\substack{s\in S_\mu^n \\ |\int \chi\, d\tau_{s,n} - \int \chi d\nu_\xi| > \epsilon}} \nu_\xi([s]) \le Ce^{-\eta(\epsilon)n}.$$

Proof. The proof is standard but we include it for completeness. We consider $s \in S_\mu^n$ such that $\int \chi\, d\tau_{s,n} > \int \chi\, d\nu_\xi + \epsilon$ and $\int \chi\, d\tau_{s,n} < \int \chi\, d\nu_\xi - \epsilon$ separately. For $t > 0$, we have

$$\sum_{\substack{s\in S_\mu^n \\ \int \chi\, d\tau_{s,n} > \int \chi d\nu_\xi + \epsilon}} \nu_\xi([s]) \le \sum_{s \in S_\mu^n} e^{\varphi^n(s_\infty) + t\chi^n(s_\infty) - nt\int \chi\, d\nu_\xi - nt\epsilon},$$

so that,

$$\limsup_{n\to\infty} \frac{1}{n} \log \sum_{\substack{s\in S_\mu^n \\ \int \chi\, d\tau_{s,n} > \int \chi d\nu_\xi + \epsilon}} \nu_\xi([s]) \le P_{\mathrm{G}}(\varphi + t\chi) - t\int \chi\, d\nu_\xi - t\epsilon.$$

Using $P_{\mathrm{G}}(\varphi) = 0$ and (6.2), we see that, for sufficiently small $t > 0$, we have

$$P_{\mathrm{G}}(\varphi + t\chi) - t\int \chi\, d\nu_\xi - t\epsilon < 0.$$

Similarly, for $t < 0$,

$$\limsup_{n\to\infty} \frac{1}{n} \log \sum_{\substack{s\in S_\mu^n \\ \int \chi\, d\tau_{s,n} < \int \chi d\nu_\xi - \epsilon}} \nu_\xi([s]) \le P_{\mathrm{G}}(\varphi + t\chi) - t\int \chi\, d\nu_\xi + t\epsilon.$$

and, for sufficiently small $t < 0$, this upper bound is negative. Combing these estimates gives the result. $\square$

Since $\Lambda_n \subset S_\mu^n$, we have the following immediate corollary.

Corollary 6.4. *For $\epsilon > 0$, we have*

$$\sum_{\substack{s\in \Lambda_n \\ |\int \chi\, d\tau_{s,n} - \int \chi d\nu_\xi| > \epsilon}} \nu_\xi([s]) \le Ce^{-\eta(\epsilon)n}.$$

The following equidistribution result is now an easy consequence.

Proposition 6.5. *For any cylinder set $[u_1, \ldots, u_k]$, we have that*

$$\lim_{n\to\infty} m_n([u_1, \ldots, u_k]) = \mu_\xi(u_1)\cdots\mu_\xi(u_k).$$

Proof. Let $\chi : \Sigma \to \mathbb{R}$ be the indicator function of $[u_1, \ldots, u_k]$, then we need to show

$$\lim_{n\to\infty} \int \chi\, dm_n = \int \chi\, d\nu_\xi.$$

We have

$$\int \chi \, dm_n - \int \chi \, d\nu_\xi = \frac{1}{\mu_\xi^{*n}(e)} \sum_{\substack{s \in \Lambda_n \\ |\int \chi \, d\tau_{s,n} - \int \chi d\nu_\xi| > \epsilon}} \nu_\xi([s]) \int \chi \, d\tau_{s,n}$$

$$+ \frac{1}{\mu_\xi^{*n}(e)} \sum_{\substack{s \in \Lambda_n \\ |\int \chi \, d\tau_{s,n} - \int \chi d\nu_\xi| \leq \epsilon}} \nu_\xi([s]) \int \chi \, d\tau_{s,n} - \int \chi \, d\nu_\xi.$$

By Lemma 6.2 and Corollary 6.4, the first term on the right hand side tends to zero exponentially fast. Also,

$$\left| \frac{1}{\mu_\xi^{*n}(e)} \sum_{\substack{s \in \Lambda_n \\ |\int \chi \, d\tau_{s,n} - \int \chi d\nu_\xi| \leq \epsilon}} \nu_\xi([s]) \int \chi \, d\tau_{s,n} - \int \chi \, d\nu_\xi \right|$$

$$\leq \epsilon + \frac{\left| \int \chi \, d\nu_\xi \right|}{\mu_\xi^{*n}(e)} \sum_{\substack{s \in \Lambda_n \\ |\int \chi \, d\tau_{s,n} - \int \chi d\nu_\xi| > \epsilon}} \nu_\xi([s]) \leq \epsilon + C e^{-\eta(\epsilon) n},$$

which, since ϵ is arbitrary, gives the result. □

Combining Proposition 6.5 with Lemma 6.1, we see that for $u = u_1 \cdots u_k$ we have

$$\lim_{n \to \infty} \frac{\mu_\xi^{*(n-k)}(u^{-1})}{\mu_\xi^{*n}(e)} = 1. \tag{6.3}$$

Most of the work is done. We prove Proposition 2.4 for amenable groups.

Proof of Proposition 2.4. It suffices to show that

$$\lim_{n \to \infty} \frac{\mu_\xi^{*(n+1)}(e)}{\mu_\xi^{*n}(e)} = 1.$$

By the aperiodicity of μ, there exists k_0 such that $\mu^{*k}(e) > 0$ for all $k \geq k_0$. In particular, for all $k \geq k_0$, we can choose $(u_1, \ldots, u_k) \in S_\mu^k$ with $u_1 \cdots u_k = e$. Then equation (6.3) gives that $\lim_{n \to \infty} \mu_\xi^{*(n-k)}(e)/\mu_\xi^{*n}(e) = 1$, so that

$$\lim_{n \to \infty} \mu_\xi^{*(n+k)}(e)/\mu_\xi^{*n}(e) = 1,$$

for all $k \geq k_0$. Applying this with $k = k_0$ and $k = k_0 + 1$, we have that both

$$\frac{\mu_\xi^{*(n+k_0)}(e)}{\mu^{*n}(e)} = \frac{\mu_\xi^{*(n+k_0)}(e)}{\mu_\xi^{*(n+k_0-1)}(e)} \cdots \frac{\mu_\xi^{*(n+1)}(e)}{\mu_\xi^{*n}(e)}$$

and

$$\frac{\mu_\xi^{*(n+k_0+1)}(e)}{\mu^{*n}(e)} = \frac{\mu_\xi^{*(n+k_0+1)}(e)}{\mu_\xi^{*(n+k_0)}(e)} \cdots \frac{\mu_\xi^{*(n+1)}(e)}{\mu_\xi^{*n}(e)}$$

converge to 1, as $n \to \infty$. Dividing the second expression by the first, we conclude that

$$\lim_{n \to \infty} \frac{\mu_\xi^{*(n+k_0+1)}(e)}{\mu_\xi^{*(n+k_0)}(e)} = 1,$$

which is equivalent to the required limit. □

We can now establish the ratio limit theorem for amenable groups.

Proof of Theorem 1.4. Let $g \in G$ be arbitrary. Choosing $(u_1, \ldots, u_k)$ with $u_1 \cdots u_k = g^{-1}$ and applying (6.3) gives that $\lim_{n\to\infty} \mu_\xi^{*(n-k)}(g)/\mu_\xi^{*n}(e) = 1$ and hence that

$$\lim_{n\to\infty} \frac{\mu^{*(n-k)}(g)}{\mu^{*n}(e)} = \frac{\lambda^{-k}}{h(g)}.$$

Now, applying Proposition 2.4,

$$\frac{\mu^{*n}(g)}{\mu^{*n}(e)} = \frac{\mu^{*n}(g)}{\mu^{*(n+k)}(e)} \frac{\mu^{*(n+k)}(e)}{\mu^{*n}(e)} \to e^{-\langle \xi, \pi(g) \rangle},$$

as $n \to \infty$. □

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