LARGE DEVIATIONS, FLUCTUATIONS AND SHRINKING INTERVALS

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Abstract. This paper concerns the statistical properties of hyperbolic diffeomorphisms. We obtain a large deviation result with respect to slowly shrinking intervals for a large class of Hölder continuous functions. In case case of time reversal symmetry, we obtain a corresponding version of the Fluctuation Theorem.

0. Introduction

In statistical mechanics, the second law of thermodynamics states that the entropy of a system increases until it reaches an equilibrium. However, since this is a statistical law, away from thermodynamic equilibrium the entropy may increase or decrease over a given amount of time and the Cohen-Gallavotti fluctuation theorem implies that the relative probability that entropy will flow in a direction opposite to that given by the second law of thermodynamics decreases exponentially.

To formulate a mathematical model for these results, let $T : \Lambda \to \Lambda$ be a mixing hyperbolic diffeomorphism, let μ be an equilibrium state for a Hölder continuous function and let $\Psi : \Lambda \to \mathbb{R}$ be a Hölder continuous function such that $\int \Psi d\mu > 0$. Let $\mathcal{M}_T$ denote the space of all T-invariant Borel probability measures on Λ and write

$$\mathcal{I}_\Psi = \left\{ \int \Psi \ dm : m \in \mathcal{M}_T \right\}.$$

If we denote $\Psi^n(x) = \Psi(x) + \Psi(Tx) + \cdots + \Psi(T^{n-1}x)$ then, by standard large deviation estimates, we can deduce that if $(-p, p) \subset \mathcal{I}_\Psi$ then the limit

$$\lim_{n \to +\infty} \frac{1}{n} \log \left(\frac{\mu \left\{ x : \frac{1}{n}\Psi^n(x) \in (p - \delta, p + \delta) \right\}}{\mu \left\{ x : \frac{1}{n}\Psi^n(x) \in (-p - \delta, -p + \delta) \right\}} \right) \tag{0.1}$$

exists. The limit takes a particularly simple form if we assume that $T : \Lambda \to \Lambda$ has a time reversal symmetry (i.e., an involution $i : \Lambda \to \Lambda$ such that $i \circ T \circ i = T^{-1}$) and we consider functions of the special form $\Psi = \Phi - \Phi \circ i \circ T$, for a given function $\Phi : \Lambda \to \mathbb{R}$. Let μ_Φ and μ_Ψ be the equilibrium states of Φ and Ψ, respectively. A version of the following theorem was formulated by Gallavotti in 1995 [4], [5] and particularly nice treatments appear in the work of Ruelle [19] and Maes and Verbitsky [11], and in the book [8]. For a more abstract formulation, see Wojtkowski [20]. (See also Gentile [6] for the case of Anosov flows.)

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Fluctuation Theorem. *Suppose that μ_Ψ is not the measure of maximal entropy for T. Then*

(i) *we have that $\int \Psi \, d\mu_\Phi > 0$;*
(ii) *there exists $p^* > 0$ such that, if $|p| < p^*$, then*

$$\lim_{\delta \to 0} \lim_{n \to +\infty} \frac{1}{n} \log \frac{\mu_\phi(\{x : \frac{1}{n}\Psi^n(x) \in (p-\delta, p+\delta)\})}{\mu_\phi(\{x : \frac{1}{n}\Psi^n(x) \in (-p-\delta, -p+\delta)\})} = p.$$

The physical interpretation corresponds to the particular choice of functions $\Phi(x) = -\log \|DT|_{E^u(x)}\|$, for which the corresponding equilibrium state μ_Φ is the Sinai-Ruelle-Bowen measure, and $\Psi(x) = -\log|\det(D_x f)|$. The quantity $\int \log|\det(D_x f)| d\mu_\Phi(x)$ is then the entropy production originally introduced by Ruelle [18].

The existence of the limit in (0.1) and part (ii) of the Fluctuation Theorem follows from a basic large deviation result. Note that, under the assumption that the equilibrium state for Ψ is not the measure of maximal entropy, we have $\mathrm{int}(\mathcal{I}_\Psi) \neq \varnothing$.

In the following, we adopt the convention that $\inf \varnothing = -\infty$.

Large Deviation Theorem [9],[13]**.** *Suppose that μ_Ψ is not the measure of maximal entropy for T. There is a real analytic rate function $I : \mathrm{int}(\mathcal{I}_\Psi) \to \mathbb{R}^+$, with $I(p) = 0$ if and only in $p = \int \Psi \, d\mu_\Phi$, such that, for an interval $J \subset \mathbb{R}$, we have*

$$\lim_{n \to +\infty} \frac{1}{n} \log \mu_\Phi \left(\left\{ x : \frac{\Psi^n(x)}{n} \in J \right\} \right) = -\inf\{I(p) : p \in J \cap \mathrm{int}(\mathcal{I}_\Psi)\}.$$

In particular, we have that for $p \in \mathrm{int}(\mathcal{I}_\Psi)$,

$$\lim_{\delta \to 0} \lim_{n \to +\infty} \frac{1}{n} \log \mu_\Phi \left(\left\{ x : \frac{\Psi^n(x)}{n} \in (p-\delta, p+\delta) \right\} \right) = -I(p).$$

These theorems lead to the following natural question.

Question. Can we obtain similar results where δ is allowed to shrink as a function of n (and we only need to take a single limit as $n \to +\infty$)? More precisely, if δ_n decreases to zero sufficiently slowly, do we have

$$\lim_{n \to +\infty} \frac{1}{n} \log \mu_\Phi \left(\left\{ x : \frac{\Psi^n(x)}{n} \in (p-\delta_n, p+\delta_n) \right\} \right) = -I(p)?$$

We shall show that, subject to a modest condition on the function Ψ, the answer to the question is always in the affirmative provided δ_n^{-1} grows no faster than $n^{1+\kappa}$, for some $\kappa = \kappa(\Psi) > 0$. The condition on Ψ that we require is the following.

Diophantine Condition. We say that a function Ψ satisfies the Diophantine condition (with respect to a transformation T) if there are periodic orbits $T^{n_i} x_i = x_i$ $(i = 1, 2, 3)$ such that

$$\alpha = \frac{\Psi^{n_3}(x_3) - \Psi^{n_1}(x_1)}{\Psi^{n_2}(x_2) - \Psi^{n_1}(x_1)}$$

is a diophantine number (i.e., there exists $c > 0$ and $\beta > 1$ such that $|q\alpha - p| \geq cq^{-\beta}$, for all $p \in \mathbb{Z}$ and $q \in \mathbb{N}$).

In section 1, we shall see that if Ψ satisfies the Diophantine condition then μ_Ψ is not the measure of maximal entropy for T.

An answer the above question is given by the following version of the large deviation theorem with shrinking intervals.

Theorem 1. *Let $T : \Lambda \to \Lambda$ be a mixing hyperbolic diffeomorphism and let μ_Φ be the equilibrium state of a Hölder continuous function $\Phi : \Lambda \to \mathbb{R}$. Let $\Psi : \Lambda \to \mathbb{R}$ be a Hölder continuous function which satisfies the Diophantine condition. Then there exists $\kappa > 0$ such that, if $\delta_n > 0$ decreases to zero and $\delta_n^{-1} = O(n^{1+\kappa})$, as $n \to +\infty$, we have, for $p \in \mathrm{int}(\mathcal{I}_\Psi)$,*

$$\lim_{n\to+\infty} \frac{1}{n} \log \mu_\Phi \left(\left\{ x : \frac{\Psi^n(x)}{n} \in (p-\delta_n, p+\delta_n) \right\} \right) = -I(p).$$

In fact, it follows from the Large Deviation Theorem that an upper bound holds for all Hölder continuous $\Psi : \Lambda \to \mathbb{R}$ and sequences $\delta_n \to 0$ without assuming any further condition.

Proposition 0.1. *Suppose that $\Psi : \Lambda \to \mathbb{R}$ is a Hölder continuous function such that μ_Ψ is not the measure of maximal entropy. Let $\delta_n > 0$ be any sequence converging to zero. Then*

$$\limsup_{n\to+\infty} \frac{1}{n} \log \mu_\Phi \left(\left\{ x : \frac{\Psi^n(x)}{n} \in (p-\delta_n, p+\delta_n) \right\} \right) \le -I(p).$$

Theorem 1 leads to a version of the fluctuation theorem for shrinking intervals.

Theorem 2. *Let $T : \Lambda \to \Lambda$ be a mixing hyperbolic diffeomorphism with time reversal symmetry $i : \Lambda \to \Lambda$. Let $\Phi : \Lambda \to \mathbb{R}$ be Hölder continuous and let $\Psi = \Phi - \Phi \circ i \circ T$. Suppose that Ψ satisfies the Diophantine condition then there exists $\kappa > 0$ such that, if $\delta_n > 0$ decreases to zero and $\delta_n^{-1} = O(n^{1+\kappa})$, as $n \to +\infty$, we have, for $|p| < p^*$,*

$$\lim_{n\to+\infty} \frac{1}{n} \log \frac{\mu_\Phi(\{x : \Psi^n(x)/n \in (p-\delta_n, p+\delta_n)\})}{\mu_\Phi(\{x : \Psi^n(x)/n \in (-p-\delta_n, -p+\delta_n)\})} = p. \tag{0.2}$$

In section 1 we recall some basic results about hyperbolic diffeomorphisms and the thermodynamic formalism associated to them. In section 2, we discuss the corresponding properties of shifts of finite type. In section 3, we describe some examples related to our results. In section 4, we prove a large deviations result with shrinking intervals in the context of subshifts of finite type and deduce Theorem 1. In section 5, we restrict to systems with time reversal symmetry and prove Theorem 2.

1. Hyperbolic Diffeomorphisms

In this section we recall some basic definitions and results. Let M be a compact C^∞ Riemannian manifold and let $T : M \to M$ be a C^∞ diffeomorphism. We call a compact T-invariant set Λ hyperbolic if:

1. $T : \Lambda \to \Lambda$ is transitive;
2. the periodic orbits for $T : \Lambda \to \Lambda$ are dense in Λ;
3. there exists an open set $U \supset \Lambda$ such that $\Lambda = \cap_{n=-\infty}^{\infty} T^{-n}U$;
4. there exists $C > 0$, $0 < \lambda < 1$ and a splitting $T_\Lambda M = E^u \oplus E^s$ such that

$$\|DT^n v\| \le C\lambda^n \|v\| \text{ where } v \in E^s$$
$$\|DT^{-n} v\| \le C\lambda^n \|v\| \text{ where } v \in E^u,$$

for $n \ge 0$.

We call the restriction $T : \Lambda \to \Lambda$ a hyperbolic diffeomorphism. In the case that $\Lambda = M$ we call T a transitive Anosov diffeomorphism.

We write $\mathcal{M}_T$ for the space of T-invariant measures on Λ. For a continuous function $\Psi : \Lambda \to \mathbb{R}$, we define its pressure by the variational principle:

$$P(\Psi) = \sup \left\{ h_m(T) + \int \Psi \, dm : m \in \mathcal{M}_T \right\},$$

where $h_m(T)$ denotes the measure theoretic entropy. A function of the form $u \circ T - u$, where $u : X \to \mathbb{R}$ is continuous, is called a coboundary. We have $P(\Psi + u \circ T - u + c) = P(\Psi) + c$, where $c \in \mathbb{R}$ is a constant. Assume from now on that Ψ is Hölder continuous. We write μ_Ψ for the equilibrium state of Ψ, i.e., the unique $\mu_\Psi \in \mathcal{M}_T$ such that $P(\Psi) = h_{\mu_\Psi}(T) + \int \Psi \, d\mu_\Psi$. This measure is unchanged by adding a coboundary and a constant to Ψ.

The pressure of Ψ may also be characterized in terms of periodic points, as in the following proposition.

Proposition 1.1. *Suppose that $\Psi : \Lambda \to \mathbb{R}$ is Hölder continuous. Then*

$$P(\Psi) = \lim_{n \to +\infty} \frac{1}{n} \log \sum_{T^n x = x} e^{\Psi^n(x)}.$$

Recall that we defined $\mathcal{I}_\Psi = \left\{ \int \Psi \, dm : m \in \mathcal{M}_T \right\}$. If Ψ is cohomologous to a constant c then $\mathcal{I}_\Psi = \{c\}$; otherwise, $\mathcal{I}_\Psi$ is a non-trivial closed interval. The equilibrium state of the function which is identically zero is called the measure of maximal entropy. In view of the above discussion, μ_Ψ is the measure of maximal entropy if and only if Ψ is cohomologous to a constant.

If Ψ satisfies the Diophantine condition then, in particular, there are two probability measures, ν_1 and ν_2, supported on periodic orbits, for which $\int \Psi \, d\nu_1 \neq \int \Psi \, d\nu_2$, so $\mathcal{I}_\Psi$ is not a single point. Hence, the Diophantine condition for Ψ implies that μ_Ψ is not the measure of maximal entropy for T.

We shall now concentrate on a fixed Hölder continuous function $\Psi : \Lambda \to \mathbb{R}$ and a measure μ_Φ, the equilibrium state of another Hölder continuous function $\Phi : \Lambda \to \mathbb{R}$. To avoid a degenerate situation, we suppose that Ψ is not cohomologous to a constant (i.e., μ_Ψ is not the measure of maximal entropy). Then the function $q \mapsto P(\Phi + q\Psi)$ is strictly convex and real analytic. Furthermore,

$$\frac{dP(\Phi + q\Psi)}{dq} = \int \Psi \, d\mu_{\Phi + q\Psi}, \quad \mathrm{int}(\mathcal{I}_\Psi) = \left\{ \int \Psi \mu_{\Phi + q\Psi} : q \in \mathbb{R} \right\}$$

and the endpoints of $\mathcal{I}_\Psi$ are

$$\lim_{q \to \pm\infty} \int \Psi \, d\mu_{\Phi + q\Psi}.$$

Let $I(p) : \mathrm{int}(\mathcal{I}_\Psi) \to \mathbb{R}$ denote the (real analytic) Legendre transform of $P(\Phi + q\Psi) - P(\Phi)$, i.e.,

$$-I(p) = \inf\{P(\Phi + q\Psi) - P(\Phi) - qp : q \in \mathbb{R}\}.$$

(Since we can always add a constant to Φ without affecting $I(p)$ or μ_Φ we can assume without loss of generality that $P(\Phi) = 0$.) We also have that $-I(p) = P(\Phi + \xi_p\Psi) - \xi_p p$, where ξ_p is the unique real number with

$$\left.\frac{dP(\Phi + q\Psi)}{dq}\right|_{q=\xi_p} = \int \Psi \, d\mu_{\Phi+\xi_p\Psi} = p.$$

Theorem 1 is relatively straightforward to prove in the case where δ_n decreases more slowly than n^{-1}. Indeed, in this case it holds assuming only that Ψ is non-lattice, i.e., that if, for $a, b \in \mathbb{R}$, $\{a\Psi^n(x) + bn : T^n x = x, n \in \mathbb{N}\} \subset \mathbb{Z}$ then $a = b = 0$. A function which satisfies the Diophantine condition is automatically non-lattice. To see this, recall that local limit theorems for hyperbolic diffeomorphisms [7], [10] imply that if Ψ is non-lattice then

$$\begin{aligned} -I(p) &= \lim_{n\to+\infty} \frac{1}{n} \log \mu_\Phi \left(\{x : (\Psi - p)^n(x) \in (-\delta, \delta)\}\right) \\ &= \lim_{n\to+\infty} \frac{1}{n} \log \mu_\phi \left(\left\{x : \frac{\Psi^n(x)}{n} \in \left(p - \frac{\delta}{n}, p + \frac{\delta}{n}\right)\right\}\right), \end{aligned}.$$

This shows that the required growth rate holds if $\delta_n = \delta/n$ or decreases more slowly than this. However, to prove the full version of the Theorem 1 we need to study a symbolic model for $T : \Lambda \to \Lambda$.

2. Subshifts of Finite Type

Let A be a $k \times k$ matrix with entries 0 or 1, which is aperiodic (i.e., there exists $n \geq 1$ such that A^n has all entries positive). We let

$$X = \left\{x \in \prod_{n=-\infty}^{\infty} \{1, \cdots, k\} : A(x_n, x_{n+1}) = 1 \text{ for all } n \in \mathbb{Z}\right\},$$

the space of two-sided sequences with adjacent entries allowed by A. Let $\sigma : X \to X$ be the two-sided subshift of finite type defined by $(\sigma x)_n = x_{n+1}$. We make X into a compact metric space by defining $d(x, y) = \sum_{n=-\infty}^{\infty}(1 - \delta_{x_n y_n})2^{-|n|}$, where δ_{ij} is the Kronecker symbol. Aperiodicity of A is equivalent to topological mixing for σ.

Similarly, let

$$X^+ = \left\{x \in \prod_{n=0}^{\infty} \{1, \cdots, k\} : A(x_n, x_{n+1}) = 1 \text{ for all } n \in \mathbb{Z}^+\right\},$$

the corresponding space of one-sided sequences. Let $\sigma : X^+ \to X^+$ be the one-sided subshift of finite type defined by $(\sigma x)_n = x_{n+1}$. As above, we make X^+ into a compact metric space by defining $d(x, y) = \sum_{n=0}^{\infty}(1 - \delta_{x_n y_n})2^{-n}$. Again, aperiodicity of A is equivalent to topological mixing for σ.

By analogy with the previous section we write $\mathcal{M}_\sigma$ for the space of σ-invariant measures on X and for a continuous function $\psi : \Sigma \to \mathbb{R}$, we define its pressure to be

$$P(\psi) = \sup\left\{h_m(\sigma) + \int \psi \, dm : m \in \mathcal{M}_\sigma\right\},$$

where $h_m(\sigma)$ denotes the measure theoretic entropy. A function of the form $u\circ\sigma-u$, where $u : X \to \mathbb{R}$ is continuous, is called a coboundary. We have $P(\psi+u\circ\sigma-u+c) = P(\psi)+c$, where $c \in \mathbb{R}$ is a constant. Assume from now on that ψ is Hölder continuous. We write μ_ψ for the equilibrium state of ψ, i.e., the unique $\mu_\psi \in \mathcal{M}_\sigma$ such that $P(\psi) = h_{\mu_\psi}(T) + \int \psi \, d\mu_\psi$. This measure is unchanged by adding a coboundary and a constant to ψ. Exactly the same results hold for $\sigma : X^+ \to X^+$.

A particularly useful property of subshifts of finite type, which is relevant for our analysis, is that they serve as models of hyperbolic diffeomorphisms, in the following precise sense.

Proposition 2.1. *Given a mixing hyperbolic diffeomorphism $T : \Lambda \to \Lambda$. there exists a mixing (two-sided) subshift of finite type $\sigma : X \to X$ and a Hölder continuous surjective map $\pi : X \to \Lambda$ such that*

(1) *$T \circ \pi = \pi \circ \sigma$;*
(2) *π one-to-one almost everywhere with respect to the equilibrium states of $f\circ\pi$ and f, for any Hölder continuous function $f : \Lambda \to \mathbb{R}$* [1].

Given Hölder continuous functions $\Phi, \Psi : \Lambda \to \mathbb{R}$, we can use Proposition 2.1 to define $\phi = \Phi \circ \pi : X \to \mathbb{R}$ and $\psi = \Psi \circ \pi :\to \mathbb{R}$, which are also Hölder continuous, and ψ satisfies the Diophantine condition if and only if Ψ does. Furthermore, $P(\Phi + q\Psi) = P(\phi + q\psi)$, for $q \in \mathbb{R}$.

To shorten our subsequent notation, we shall write $\psi_p = \psi - p$. Notice that ψ_p satisfies the Diophantine condition if and only if ψ does. With this notation

$$-I(p) = \inf_{q\in\mathbb{R}} P(\phi + q\psi_p) = P(\phi + \xi_p\psi_p)$$

and

$$\left.\frac{dP(\phi + q\psi_p)}{dq}\right|_{q=\xi_p} = \int \psi_p \, d\mu_{\phi+\xi_p\psi} = 0.$$

Our analysis makes use of transfer operators and thus it is necessary to work initially with one-sided shifts of finite type.

Definition. For functions $\psi, \phi \in C^\alpha(X^+, \mathbb{R})$, define the family of transfer operators $\mathcal{L}_{\phi+(\xi+iu)\psi} : C^\alpha(X^+,\mathbb{C}) \to C^\alpha(X^+,\mathbb{C})$ by

$$\mathcal{L}_{\phi+(\xi+iu)\psi} k(x) = \sum_{\sigma y = x} e^{\phi(y)+(\xi+iu)\psi(y)} k(y),$$

for ξ and $u \in \mathbb{R}$.

By adding a coboundary and a constant to ϕ, we may assume that ϕ is normalized, i.e., that $\mathcal{L}_\phi 1 = 1$. In particular, if ϕ is normalized then $P(\phi) = 0$. Normalization leaves the equilibrium state μ_ϕ unchanged. From now on, we shall write $\xi = \xi_p$.

Proposition 2.2.

(1) *The operator $\mathcal{L}_{\phi+\xi\psi_p}$ has a simple eigenvalue $\lambda_\xi = e^{P(\phi+\xi\psi_p)}$ and the rest of the spectrum is contained in disk of smaller radius.*
(2) *For $u \in \mathbb{R}$, the operator $\mathcal{L}_{\phi+(\xi+iu)\psi_p}$ has spectral radius $\le \lambda_\xi$.*

(3) *There exists $a > 0$ such that, for $|u| < a$, $\mathcal{L}_{\phi+(\xi+iu)\psi_p}$ has a simple eigenvalue $e^{P(\phi+(\xi+iu)\psi_p)}$, depending analytically on u, with $|e^{P(\phi+(\xi+iu)\psi_p)}| < \lambda_\xi$ for $u \neq 0$. Furthermore, the rest of the spectrum of $\mathcal{L}_{\phi+(\xi+iu)\psi_p}$ is contained in a disk of radius $\theta\lambda_\xi$, for some $\theta < 1$.*

(4)
$$\left.\frac{d^2P(\phi + (\xi + iu)\psi_p)}{du^2}\right|_{u=0} = -\sigma^2 < 0.$$

The following identity will be important in subsequent calculations.

Lemma 2.1. *If ϕ is normalized then*

$$\int e^{(\xi_p+iu)\psi_p^n(x)} d\mu_\phi(x) = \int \mathcal{L}^n_{\phi+(\xi_p+iu)\psi_p} 1(x) d\mu_\phi(x).$$

Later, we shall need to bound iterates of $\mathcal{L}_{\phi+(\xi+iu)\psi_p}$. Estimates of the kind we require were developed in [15], following the ideas of Dolgopyat [3].

Lemma 2.2. *Assume that ψ satisfies the Diophantine condition. Then there exists $\gamma > 0$, $D > 0$ and $C, c > 0$ such that, for $|u| \geq a$, we have that*

$$\|\mathcal{L}^{2Nm}_{\phi+(\xi+iu)\psi_p} 1\|_\infty \leq C\lambda_\xi^n \left(1 - \frac{c}{|u|^\gamma}\right)^m, \text{ for } m \geq 1, \tag{2.1}$$

where $N = [D \log |u|]$.

Proof. Since we are assuming the Diophantine condition, the hypotheses of Proposition 2 in [15] hold. This gives the inequality (2.1). □

3. Examples

In this section, we discuss some examples related to our theorems. We begin by considering two examples for subshifts of finite type. These can easily be adapted to Axiom A diffeomorphisms [2]. In particular, the show that the Diophantine condition is necessary for Theorem 1.

Example 1. Consider the case of a (two-sided) full shift on two symbols $\sigma : X \to X$, where $X = \{0, 1\}^{\mathbb{Z}}$, and a function $\psi : X \to \mathbb{R}$ defined by

$$\psi(x) = \begin{cases} \beta & \text{if } x_0 = 0 \\ -1 & \text{if } x_0 = 1. \end{cases}$$

By a judicious choice of β we can arrange for the Diophantine condition to hold. An example of a hyperbolic diffeomorphism with the same behaviour is given by suitable horseshoe.

However, as the next example shows, results such as Theorem 1 cannot hold without some assumption on the Hölder continuous function. In particular, it is necessary that a non-lattice condition is satisfied. It is not clear if shrinking interval results hold for functions which are non-lattice but fail to satisfy the Diophantine condition.

Example 2. Consider again a (two-sided) full shift on two symbols $\sigma : X \to X$. Now define a function $\psi : X \to \mathbb{R}$ by

$$\psi(x) = \begin{cases} \beta - 1 & \text{if } x_0 = 0 \\ -1 & \text{if } x_0 = 1, \end{cases}$$

with $\beta > 1$. For any $\sigma^n x = x$, $\beta^{-1}\psi^n(x) - \beta^{-1}n \in \mathbb{Z}$, so ψ is not a non-lattice function (and hence, in particular, does not satisfy the Diophantine condition), regardless of the value of β. It is easy to see that $\mathcal{I}_\psi = [-1, \beta - 1]$ and that $p_0 := \int \psi \, d\mu_0 = -1 + \beta/2$, where μ_0 is the $(\frac{1}{2}, \frac{1}{2})$-Bernoulli measure. Of course, $p_0 \in \mathrm{int}(\mathcal{I}_\psi)$.

For any $x \in X$, we have that $\psi^n(x)/n = -1 + (m\beta/n)$, where $0 \le m \le n$. Thus, for a sequence $\delta_n > 0$, $\psi^n(x)/n \in (p_0 - \delta_n, p_0 + \delta_n)$ is equivalent to

$$-\delta_n < \left(\frac{m}{n} - \frac{1}{2} \right) \beta < \delta_n.$$

If we restrict to odd values of n, then this fails for large n as soon as we take $\delta_n = O(n^{-(1+\epsilon)})$, for $\epsilon > 0$. Thus, for large odd values of n, we see that

$$\{x : \psi^n(x) \in (p_0 - \delta_n, p_0 + \delta_n)\} = \varnothing.$$

In particular, for any choice of measure μ_ϕ, the exponential growth rate is less than the exponent $-I(p_0)$. As in Example 1, a suitable horseshoe gives a smooth version.

We next give two examples of Anosov diffeomorphisms for which there is a time-reversing involution.

Example 3. Let $T : M \to M$ be an Anosov diffeomorphism and consider the product diffeomorphism $\widetilde{T} :=: M \times M \to M \times M$ defined by $\widetilde{T}(x, y) = (Tx, T^{-1}y)$. This is again Anosov and satisfies $i \circ \widetilde{T} \circ i = \widetilde{T}^{-1}$ where $i(x, y) = (y, x)$.

Example 4. Consider the map $T(x, y) = (y, -x + Cx + f(x))$ on the 2-torus, where C is an integer and f is a small perturbation. If $f(\cdot)$ is C^1 small then this is a perturbation of the linear map associated to $\left(\begin{smallmatrix} 0 & 1 \\ -1 & C \end{smallmatrix}\right)$ and if $|C| > 2$ then this is Anosov. Let $S(x, y) = (y, x)$ then S^2 is the identity and $STS = T^{-1}$.

4. Large Deviations and Shrinking Intervals

We will first consider large deviations results for a one-sided subshift of finite type. The two-sided result can then be deduced from this and results for hyperbolic diffeomorphisms can be derived from this using symbolic dynamics (using Proposition 1.1).

Let us consider a mixing one-sided subshift of finite type $\sigma : X^+ \to X^+$ and Hölder continuous functions $\psi, \phi : X^+ \to \mathbb{R}$.

In this context, our shrinking large deviations result takes the following form.

Proposition 4.1. *Suppose that $\psi : X^+ \to \mathbb{R}$ satisfies the Diophantine condition (with respect to σ). Then there exists $\kappa > 0$ such that, if $\delta_n > 0$ decreases to zero and $\delta_N^{-1} = O(n^{1+\kappa})$, as $n \to +\infty$, we have, for $p \in \mathrm{int}(\mathcal{I}_\psi)$,*

$$\lim_{n \to \infty} \frac{1}{n} \log \mu_\phi \left(\left\{ x : \frac{\psi^n(x)}{n} \in (p - \delta_n, p + \delta_n) \right\} \right) = -I(p).$$

We shall prove the theorem under the additional assumption that δ_n goes to zero faster than $1/n$, so

$$\epsilon_n := n\delta_n \to 0.$$

As we have seen, the result holds automatically if $\delta_n^{-1} = O(n)$.

This assumption means that we can convert the problem to a local limit-type problem. However, matters are simplified by our only wanting weaker information, i.e., the exponential growth rate rather than an asymptotic. Recall that $\psi_p := \psi - p$. Then

$$\left\{x : \frac{\psi^n(x)}{n} \in (p-\delta_n, p+\delta_n)\right\} = \{x : \psi_p^n(x) \in (-\epsilon_n, \epsilon_n)\}.$$

We shall first prove a modified result, where the interval $(-\epsilon_n, \epsilon_n)$ is replaced by a sequence of smooth test functions. Let $\chi : \mathbb{R} \to \mathbb{R}$ be a compactly supported C^k function (where k will be chosen later). We shall write $\chi_n(y) = \chi(\epsilon_n^{-1} y)$ and we note that the Fourier transform satisfies $\widehat{\chi}_n(u) = \epsilon_n \widehat{\chi}(\epsilon_n u)$. Let us define

$$\rho(n) := \int \chi_n(\psi_p^n(x)) d\mu_\phi(x).$$

Proposition 4.2. *There exists $\kappa > 0$ such that, provided $\epsilon_n^{-1} = O(n^\kappa)$, we have*

$$\lim_{n\to\infty} \frac{1}{n} \log \rho(n) = P(\phi + \xi_p \psi_p) = -I(p).$$

For technical reasons it is useful to modify χ_n to $\omega_n(y) = e^{-\xi y}\chi_n(y)$ (where $\xi = \xi_p$). Then

$$\rho(n) = \int e^{\xi \psi_p^n(x)} \omega_n(\psi_p^n(x)) d\mu_\phi(x).$$

To prove Proposition 4.1 we first use the inverse Fourier transform and Fubini's Theorem to write

$$\begin{aligned}
\rho(n) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int e^{iu\psi_p^n(x)} d\mu_\phi(x) \right) \widehat{\chi}_n(u) du \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int e^{(\xi + iu)\psi_p^n(x)} d\mu_\phi(x) \right) \widehat{\omega}_n(u) du \qquad (4.1)\\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int \mathcal{L}^n_{\phi + (\xi+iu)\psi_p} 1(x) d\mu_\phi(x) \right) \widehat{\omega}_n(u) du,
\end{aligned}$$

using Lemma 2.1 for the last equality. We shall estimate $\rho(n)$ by splitting the outer integral over $\mathbb{R}$ into two pieces.

4.1 u close to zero. If we choose $a > 0$ sufficiently small, we can change coordinates on $(-a, a)$ to $v = v(u)$ and write $e^{P(\phi + (\xi + iu)\psi_p)} = \lambda_\xi(1 - v^2 + iQ(v))$, for $|v| < a$, say. If $\mathcal{P}_{\psi + (\xi + iu)\psi_p}$ is the associated one dimensional eigenprojection, then by perturbation theory $\mathcal{P}_{\phi + (\xi + iu)\psi_p}(1) = 1 + O(|v|)$.

Using the formula

$$\mathcal{L}^n_{\phi + (\xi + iu)\psi_p} 1 = e^{nP(\phi + (\xi + iu)\psi_p)}(1 + O(|v|)) + O(\theta^n)$$

$(0 < \theta < 1)$, we may write

$$\begin{aligned}
&\int_{-a}^{a} \left(\int \mathcal{L}^n_{\phi+(\xi+iu)\psi_p} 1(x) d\mu_g(x) \right) \widehat{\omega}_n(u) du \\
&= \lambda_\xi^n \int_{-a}^{a} (1 - v^2 + iQ(v))^n (1 + O(|v|)) \widehat{\omega}_n(u(v)) \frac{du}{dv} dv + O(\lambda_\xi^n \theta^n) \\
&= \frac{\epsilon_n \widehat{\chi}(0)\sqrt{2}\lambda_\xi^n}{\sigma} \int_{-a}^{a} (1 - v^2 + iQ(v))^n (1 + O(|v|)) dv + O\left(\frac{\epsilon_n \lambda_\xi^n}{n}\right) + O(\lambda_\xi^n \theta^n),
\end{aligned} \tag{4.2}$$

where the $O(\epsilon_n n^{-1})$ estimate follows from a simple calculation in [14, p.409]. Using another calculation in [14, pp.408-409], we see that the principle term in the last line of (2) is asymptotic to

$$\lambda_\xi^n \int_{-a}^{a} (1 - v^2)^n dv;$$

by making the substitution $w = v^2$, we may estimate this as λ_ξ^n multiplied by the factor

$$\begin{aligned}
\frac{\epsilon_n \widehat{\chi}(0)\sqrt{2}}{\sigma} \int_{-a}^{a} (1 - v^2)^n dv &= 2\frac{\epsilon_n \widehat{\chi}(0)\sqrt{2}}{\sigma} \int_{0}^{a} (1 - v^2)^n dv \\
&= \frac{\epsilon_n \widehat{\chi}(0)\sqrt{2}}{\sigma} \int_{0}^{a^2} \frac{(1 - w)^n}{w^{1/2}} dw \\
&= \frac{\epsilon_n \widehat{\chi}(0)\sqrt{2}}{\sigma} \int_{0}^{1} \frac{(1 - w)^n}{w^{1/2}} dw + O((1 - a^2)^n) \\
&\sim \sqrt{2\pi} \frac{\widehat{\chi}(0)}{\sigma} \frac{\epsilon_n}{\sqrt{n}},
\end{aligned} \tag{4.3}$$

as $n \to +\infty$. Moreover, the term rising from the $O(|v|)$ term in the integrand is of order

$$\lambda_\xi^n \int_{-a}^{a} (1 - v^2)^n |v| dv = \lambda_\xi^n \int_{0}^{a^2} (1 - w)^n dw = O\left(\frac{\lambda_\xi^n}{n}\right).$$

So, in particular, we have shown that

$$\int_{-a}^{a} \left(\int \mathcal{L}^n_{\phi+(\xi+iu)\psi_p} 1(x) d\mu_\phi(x) \right) \widehat{\omega}_n(u) du = \sqrt{2\pi} \frac{\widehat{\chi}(0)}{\sigma} \frac{\epsilon_n}{\sqrt{n}} + O\left(\frac{\epsilon_n \lambda_\xi^n}{n}\right). \tag{4.4}$$

4.2 Away from zero. It remains to estimate the integral in (4.1) over $|u| \geq a$ and, in particular, to show that its contribution is smaller than the above. To do this we shall use a bound on the transfer operators $\mathcal{L}_{\phi+(\xi+iu)\psi_p}$. We shall also use the following simple lemma.

Lemma 4.2. *If $\chi : \mathbb{R} \to \mathbb{R}$ is C^k and compactly supported then the Fourier transform $\widehat{\chi}(u)$ satisfies $\widehat{\chi}(u) = O(|u|^{-k})$, as $|u| \to \infty$.*

Using Lemma 2.2, we have the bound

$$\begin{aligned}
&\int_{|u|\geq a}\left(\int \mathcal{L}^n_{\phi+(\xi+iu)\psi_p}1(x)d\mu_\phi(x)\right)\widehat{\omega}_n(u)\\
&= \epsilon_n\int_{|u|\geq a} e^{izu}\left(\int \mathcal{L}^n_{\phi+(\xi+iu)\psi_p}1(x)d\mu_\phi(x)\right)\widehat{\chi}(\epsilon_n u)du \qquad (4.5)\\
&= O\left(\frac{1}{\epsilon_n^{k-1}}\int_a^\infty \left(1-\frac{c}{u^\gamma}\right)^{n/2[D\log|u|]} u^{-k}du\right).
\end{aligned}$$

We need to show that this quantity tends to zero more quickly than $\epsilon_n n^{-1/2}$. To see this we shall split the integral in (4.5) into two parts:

$$\begin{aligned}
&\int_a^\infty \left(1-\frac{c}{u^\gamma}\right)^{n/2[D\log|u|]} u^{-k}du\\
&= \int_a^{n^{\delta'}} \left(1-\frac{c}{u^\gamma}\right)^{n/2[D\log|u|]} u^{-k}du + \int_{n^{\delta'}}^\infty \left(1-\frac{c}{u^\gamma}\right)^{n/2[D\log|u|]} u^{-k}du,
\end{aligned}$$

where we choose $\delta<\delta'<1/\gamma$. The first integral may be bounded by

$$\int_a^{n^{\delta'}} \left(1-\frac{c}{u^\gamma}\right)^{n/2[D\log|u|]} u^{-k}du = O\left(n^{\delta'}\left(1-\frac{c}{n^{\delta'\gamma}}\right)^{n/2D\delta'\log n}\right)$$

and, since $\delta'\gamma<1$, this tends to zero faster than the reciprocal of any polynomial. The second integral may be bounded by

$$\int_{n^{\delta'}}^\infty \left(1-\frac{c}{u^\gamma}\right)^n u^{-k}du = O(n^{(1-k)\delta'}).$$

Combining these estimates we see that

$$\begin{aligned}
\int_{|u|\geq a}\left(\int \mathcal{L}^n_{\phi+(\xi+iu)\psi_p}1(x)d\mu_\phi(x)\right)\widehat{\chi}_n(u)du &= O(\epsilon_n^{-(k-1)}n^{(1-k)\delta'})\\
&= O(n^{(k-1)(\delta-\delta')}).
\end{aligned}$$

We obtain the required bound by choosing k sufficiently large that $(k-1)(\delta-\delta'),(k-1)\delta'>1$. Together with (4.4), this completes the proof of Proposition 4.2.

Proposition 4.1 follows by an approximation argument. Choose smooth functions $\chi^+,\chi^-:\mathbb{R}\to[0,1]$ such that $\chi^-\leq\chi_{(-1,1)}\leq\chi^+$. Then

$$\int \chi_n^-(\psi_p^n(x))d\mu_\phi \leq \mu_\phi\left(\left\{x:\frac{\psi^n(x)}{n}\in(p-\delta_n,p+\delta_n)\right\}\right)\leq\int\chi_n^+(\psi_p^n(x))d\mu_\phi,$$

which gives the required estimate.

Now let $\sigma:X\to X$ be the corresponding two-sided subshift of finite type. In [16, §3], it was shown how the analogue of Proposition 4.2 (and hence Proposition 4.1) maybe deduced for $\sigma:X\to X$, given the result for $\sigma:X^+\to X^+$. Thus the analogue of Proposition 4.1 holds in the two-sided case.

As a consequence of the two-sided version of Proposition 4.1,

$$\mu_\Phi\left(\left\{x \in \Lambda : \frac{\Psi^n(x)}{n} \in (p-\delta_n, p+\delta_n)\right\}\right)$$
$$= \mu_\phi\left(\left\{x \in X : \frac{\psi^n(x)}{n} \in (p-\delta_n, p+\delta_n)\right\}\right).$$

This shows that the limit in Theorem 1 exists and is equal to

$$\begin{aligned} P(\phi + \xi_p \psi_p) &= \inf\{P(\phi + q\psi) - P(\phi) - qp : q \in \mathbb{R}\} \\ &= \inf\{P(\Phi + q\Psi) - P(\Phi) - qp : q \in \mathbb{R}\} = -I(p). \end{aligned}$$

This completes the proof of Theorem 1.

5. Fluctuation Theorems

In this final section, we derive our generalization of the Fluctuation Theorem for slowly shrinking intervals. In fact, Theorem 2 will follow directly from Theorem 1 once we show the limit has the desired form. This follows from standard arguments and is included for completeness.

Let $T : \Lambda \to \Lambda$ be a mixing hyperbolic diffeomorphism with a time-reversing involution $i : \Lambda \to \Lambda$, $i \circ T \circ i = T^{-1}$. For a Hölder continuous function $\Phi : \Lambda \to \mathbb{R}$, let $\Psi = \Phi - \Phi \circ i \circ T$. Observe that $P(\Phi) = P(\Phi \circ i \circ T)$ and thus by the variational principle, $P(\Phi) = h(\mu_\Phi) + \int \Phi \, d\mu_\Phi \geq h(\mu_\Phi) + \int \Phi \circ i \circ T \, d\mu_\Phi$, i.e., $\int \Psi \, d\mu_\Phi = \int (\Phi - \Phi \circ i \circ T) \, d\mu_\Phi \geq 0$. Moreover, equality occurs only when μ_Φ is the equilibrium state $\Phi \circ i \circ T$ (which is equivalent to μ_Ψ being the measure of maximal entropy). This explains part (i) of the Fluctuation Theorem.

Suppose that Ψ satisfies the Diophantine condition and that $\delta_n > 0$ decreases to zero such that $\delta_n^{-1} = O(n^{1+\kappa})$, where $\kappa > 0$ is chosen so that Theorem 1 holds. By applying Theorem 1 to the numerator and denominator in (0.2), we obtain

$$\lim_{n \to +\infty} \frac{1}{n} \log \frac{\mu_\Phi(\{x : \Psi^n(x)/n \in (p-\delta_n, p+\delta_n)\})}{\mu_\Phi(\{x : \Psi^n(x)/n \in (-p-\delta_n, -p+\delta_n)\})} = I(-p) - I(p)$$

(provided $-p, p \in \mathrm{int}(\mathcal{I}_\Psi)$).

We need to exploit some special symmetries of the pressure function $P(\Phi + q\Psi) - P(\Phi)$. More precisely, we have the following.

Lemma 5.1. $P(\Phi + q\Psi) = P(\Phi - (1+q)\Psi)$.

Proof. Let $T^n x = x$ be a periodic point. Then since $(\Phi \circ T^{-1})^n(x) = \Phi^n(x)$ we have that

$$\begin{aligned} (\Phi \circ i)^n(x) + q(\Psi \circ i)^n(x) &= (\Phi \circ i)^n(x) + q((\Phi \circ i)^n(x) - (\Phi \circ i \circ T \circ i)^n(x)) \\ &= (\Phi \circ i)^n(x) + q((\Phi \circ i)^n(x) - (\Phi \circ T^{-1})^n(x)) \\ &= (\Phi \circ i)^n(x) + q((\Phi \circ i)^n(x) - \Phi^n(x)) \\ &= \Phi^n(x) - (1+q)(\Phi^n(x) - (\Phi \circ i)^n(x)) \\ &= \Phi^n(x) - (1+q)\Psi^n(x). \end{aligned}$$

Since i acts as a bijection on the set of periodic orbits of period n, we obtain

$$\sum_{T^n x = x} e^{\Phi^n(x) + q\Psi^n(x)} = \sum_{T^n x = x} e^{\Phi^n(x) - (1+q)\Psi^n(x)}.$$

In particular, the two sums have the same exponential growth rate and hence the result follows from Proposition 1.1. $\square$

Lemma 5.2. *There exists* $p^* > 0$ *such that* $\mathcal{I}_\Psi = [-p^*, p^*]$.

Proof. Since Ψ is not cohomologous to a constant, $\mathcal{I}_\Psi$ is a non-trivial interval. By Lemma 5.1,

$$\int \Psi \, d\mu_{\Phi+q\Psi} = \frac{dP(\Phi+q\Psi)}{dq} = \frac{dP(\Phi-(1+q)\Psi)}{dq} = -\int \Psi \, d\mu_{\Psi-(1+q)\Psi}.$$

Thus

$$\lim_{t\to+\infty} \int \Psi \, d\mu_{\Phi+t\Psi} = -\lim_{t\to-\infty} \int \Psi \, d\mu_{\Phi+t\Psi},$$

which shows that $\mathcal{I}_\Psi$ has the required form. □

Lemma 5.3. *For* $|p| < p^*$, $I(-p) - I(p) = p$.

Proof. We have

$$\begin{aligned} I(-p) - I(p) &= \inf_{q\in\mathbb{R}} (P(\Phi+q\Psi)+qp) - \inf_{q\in\mathbb{R}} (P(\Phi+q\Psi)-qp) \\ &= \inf_{q\in\mathbb{R}} (P(\Phi-(1+q)\Psi)+qp) - \inf_{q\in\mathbb{R}} (P(\Phi+q\Psi)-qp) \\ &= \inf_{r\in\mathbb{R}} (P(\Phi+r\Psi)-rp+p) - \inf_{q\in\mathbb{R}} (P(\Phi+q\Psi)-qp) \\ &= p, \end{aligned}$$

as required.

Combining equation (5.1) and Lemmas 5.2 and 5.3 completes the proof of Theorem 2.

Remark. Some fluctuation theorems and large deviation results for Young towers (and thus for Billiards and Hénon attractors) have already been proved by Young and Rey-Bellet [17]. For shrinking intervals, the analogue of the upper bound presumably follows easily. The proof of the lower bound should follow from suitable properties of the corresponding transfer operator. In particular, [12] extended Dolgopyat's results on transfer operators to Hölder functions satisfying a Diophantine condition in terms of four periodic points.

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