LOCAL LIMIT THEOREMS FOR FREE GROUPS

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Abstract. In this paper we obtain a local limit theorem for elements of a free group G under the abelianization map $[\cdot] : G \to G/[G,G]$. This is obtained via an analysis involving subshifts of finite type, where we obtain a result of independent interest. The case of fundamental groups of compact surfaces of genus ≥ 2 is also discussed.

0. Introduction

Let G denote the free group on $k \geq 2$ generators $\{a_1, \ldots, a_k\}$. For $g \in G$, let $|g|$ denote its word length, i.e., $|g| = \inf\{n \geq 0 : g = g_1 \cdots g_n,\ g_i \in \{a_1^{\pm 1}, \ldots, a_k^{\pm 1}\}\}$, and let $[g]$ denote the image of g under the abelianization map $[\cdot] : G \to G/[G,G] \cong \mathbb{Z}^k$. Let $\mathcal{W}(n) = \{g \in G : |g| = n\}$ and observe that $\#\mathcal{W}(n) = 2k(2k-1)^{n-1}$. In this paper, we shall be interested in the distribution of the elements of $\mathcal{W}(n)$ in $\mathbb{Z}^k$ via the mapping $[\cdot]$, as $n \to \infty$. In particular, defining $\mathcal{W}(n,\alpha) = \{g \in \mathcal{W}(n) : [g] = \alpha\}$, we wish to examine the dependence of $\#\mathcal{W}(n,\alpha)$ on α as well as on n.

Our approach is to regard $\#\mathcal{W}(n,\alpha)/\#\mathcal{W}(n)$ as a probability distribution on $\mathbb{Z}^k$ and to ask about its limiting behaviour as $n \to \infty$. Rivin has shown that a central limit theorem is satisfied, i.e., for $A \subset \mathbb{R}^k$,

$$\lim_{n\to\infty} \frac{1}{\#\mathcal{W}(n)} \#\{g \in \mathcal{W}(n) : [g]/\sqrt{n} \in A\} = \frac{1}{(2\pi)^{k/2}\sigma^k} \int_A e^{-||x||^2/2\sigma^2} dx,$$

where $||\cdot||$ denotes the Euclidean norm and where

$$\sigma^2 = \frac{1}{\sqrt{2k-1}} \left[1 + \left(\frac{k+\sqrt{2k-1}}{k-\sqrt{2k-1}} \right)^{1/2} \right] \tag{0.1}$$

[18]. (In fact, this result is similar in spirit to earlier results for subshifts of finite type, hyperbolic diffeomorphisms, and interval maps [1], [4], [5], [10], [12], [17], [19], [20], [23].)

Here, we shall establish a more precise local limit theorem. First we note a combinatorial restriction. We shall say that $\alpha = (\alpha_1, \ldots, \alpha_k)$ is even if $\alpha_1 + \cdots + \alpha_k$ is even, and odd otherwise. It is clear that if $[g] = \alpha$ then α has the same parity as $|g|$. Thus, in particular,

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either $\#\mathcal{W}(n,\alpha)$ or $\#\mathcal{W}(n+1,\alpha)$ is equal to zero and we are led to consider the behaviour of the sum

$$\frac{\#\mathcal{W}(n,\alpha)}{\#\mathcal{W}(n)} + \frac{\#\mathcal{W}(n+1,\alpha)}{\#\mathcal{W}(n+1)}.$$

Theorem 1. *Let G be the free group on $k \geq 2$ generators. Then we have that*

$$\lim_{n\to\infty} \left| \sigma^k n^{k/2} \left(\frac{\#\mathcal{W}(n,\alpha)}{\#\mathcal{W}(n)} + \frac{\#\mathcal{W}(n+1,\alpha)}{\#\mathcal{W}(n+1)} \right) - \frac{2}{(2\pi)^{k/2}} e^{-||\alpha||^2/2\sigma^2 n} \right| = 0,$$

uniformly in $\alpha \in \mathbb{Z}^k$.

In the case where $\alpha = 0$, the asymptotic behaviour of $\#\mathcal{W}(n,\alpha)$, as $n \to \infty$, has been studied as a means of analysing the *relative growth series* $\xi(z)$ defined by

$$\xi(z) = \sum_{n=0}^{\infty} \#\mathcal{W}(n,0) z^n.$$

Estimates on the growth of $\#\mathcal{W}(n,0)$ allow one to deduce that $\xi(z)$ cannot be the series of a rational function [8], [16], [22]. More generally, Theorem 1 implies the following result for fixed values of α.

Corollary 1.1.
For fixed $\alpha \in \mathbb{Z}^k$,

$$\#\mathcal{W}(2n+\delta_\alpha, \alpha) \sim \frac{2}{(2\pi)^{k/2}\sigma^k} \frac{\#\mathcal{W}(2n+\delta_\alpha)}{n^{k/2}}, \text{ as } n \to \infty,$$

where $\delta_\alpha = 0$ if α is even and $\delta_\alpha = 1$ if α is odd.

Remark. For given functions A and B, we shall write $A(n) \sim B(n)$, as $n \to \infty$, if $\lim_{n\to\infty} A(n)/B(n) = 1$, and $A(n) = O(B(n))$ if $|A(n)| \leq CB(n)$, for some constant $C > 0$.

We see from Corollary 1.1 that the asymptotic behaviour of $\#\mathcal{W}(n,\alpha)$ is independent of α. However, Theorem 1 enables us to make comparisons as α varies.

Corollary 1.2. *Suppose that $\alpha, \beta \in \mathbb{Z}^k$ have the same parity. If $||\alpha|| < ||\beta||$ then we have that $\#\mathcal{W}(n,\alpha) > \#\mathcal{W}(n,\beta)$ for all sufficiently large n with the same parity as α and β.*

We say that a word $g_1 \cdots g_n$ in the generators $\{a_1, \ldots, a_k\}$ is *reduced* if $g_{i+1} \neq g_i^{-1}$, $i = 1, \ldots, n-1$. It is clear that there is a one-to-one correspondence between reduced words of length n and elements of $\mathcal{W}(n)$ (and we abuse notation by letting g denote both a word and the corresponding group element). We say that a reduced word $g_1 \cdots g_n$ is *cyclically reduced* if we also have that $g_n \neq g_1^{-1}$. Let $\mathcal{C}(n)$ denote the set of cyclically reduced words of length n and let $\mathcal{C}(n,\alpha) = \{g \in \mathcal{C}(n) : [g] = \alpha\}$. The above theorem still holds if we replace $\#\mathcal{W}(n)$ and $\#\mathcal{W}(n,\alpha)$ by $\#\mathcal{C}(n)$ and $\#\mathcal{C}(n,\alpha)$, respectively. (Notice that the map $[\cdot] : \mathcal{C}(n) \to \mathbb{Z}^k$ is well-defined.)

The paper is organized as follows. Section 1 consists of some preliminary material concerning subshifts of finite type and thermodynamic formalism. In section 2, we introduce a family of twisted matrices used in subsequent calculations and analyse their spectra. In section 3, we prove a local limit theorem associated to periodic points in a subshift of finite type using arguments adapted from [19] (see also [1]). In section 4 we see that this corresponds directly to the local limit theorem for $\mathcal{C}(n)$ and we give the amendments necessary to obtain Theorem 1. In the final section we sketch how our results may be extended to the fundamental groups of compact oriented surfaces of genus ≥ 2.

1. Preliminaries

Let A be a $l \times l$ matrix with entries zero and one and define the associated shift space X_A by

$$X_A = \{x \in \{0, 1, \dots, l-1\}^{\mathbb{Z}^+} : A(x_n, x_{n+1}) = 1 \forall n \in \mathbb{Z}^+\}.$$

The subshift of finite type $\sigma : X_A \to X_A$ is defined by $(\sigma x)_n = x_{n+1}$.

We shall always assume that A is aperiodic, i.e., that there exists $N > 0$ such that A^N has all its entries positive. This is equivalent to the map $\sigma : X_A \to X_A$ being topologically mixing. Then, by the Perron-Frobenius Theorem, A will have a simple positive eigenvalue $\lambda > 1$ which is strictly maximal in modulus and the topological entropy h of σ is equal to $\log \lambda$.

Let $\mathcal{M}$ denote the set of σ-invariant probability measures on X_A. For $m \in \mathcal{M}$, we will write $h(m)$ for its measure theoretic entropy and we have that $h(m) \leq h$. There is a unique measure $\mu \in \mathcal{M}$, called the measure of maximal entropy, for which $h(\mu) = h$. Given a continuous function $\varphi : X_A \to \mathbb{R}$, we define the pressure $P(\varphi)$ by $P(\varphi) = \sup_{m \in \mathcal{M}} \left\{ h(m) + \int \varphi dm \right\}$. If φ is Hölder continuous then there is a unique measure $\mu_\varphi \in \mathcal{M}$ for which the supremum is attained and we call μ_φ the equilibrium state of φ. Clearly, $\mu_0 = \mu$.

Set $\mathrm{Fix}_n = \{x \in X_A : \sigma^n x = x\}$. It is well-known and easy to prove that $\#\mathrm{Fix}_n = \mathrm{trace} A^n \sim \lambda^n$, as $n \to \infty$. We shall be interested in the asymptotics of certain subsets of Fix_n.

Fix a function $f : X_A \to \mathbb{Z}^k$, such that $f(x)$ depends on only finitely many co-ordinates of x. Without loss of generality, we may suppose that $f(x)$ depends on only the first two co-ordinates, i.e., that $f(x) = f(x_0, x_1)$. Write $f^n(x) = f(x) + f(\sigma x) + \cdots + f(\sigma^{n-1} x)$. For $\alpha \in \mathbb{Z}^k$, consider the subset $\{x \in \mathrm{Fix}_n : f^n(x) = \alpha\}$ of Fix_n; we shall be interested in the asymptotics of the cardinality of this set as n and α vary.

In order to make progress, we need to assume that f satisfies the following two natural conditions.

(A1) The set $\bigcup_{n=1}^\infty \{f^n(x) : x \in \mathrm{Fix}_n\}$ generates $\mathbb{Z}^k$ (i.e. it is not contained in a proper subgroup of $\mathbb{Z}^k$).

(A2) $\int f dm = 0$, where m is some fully supported σ-invariant measure.

If condition (A2) holds then it was shown in [15] that we may choose m to be equal to $\mu_{\langle \xi, f \rangle}$, for some (unique) $\xi \in \mathbb{R}^k$. Furthermore, in this case we have

$$0 < h^* := h(\mu_{\langle \xi, f \rangle}) = P(\langle \xi, f \rangle) = \sup \left\{ h(m) : \int f dm = 0, \ m \in \mathcal{M} \right\}.$$

A subgroup of $\mathbb{Z}^k$, familiar from the coding theory of subshifts of finite type, will play an important rôle in our subsequent analysis. We define

$$\Delta_f = \bigcup_{n=1}^{\infty} \{f^n(x) - f^n(y) : z, y \in \mathrm{Fix}_n\}.$$

Choose $x \in \mathrm{Fix}_n$ and $y \in \mathrm{Fix}_{n+1}$ (for some fixed n) and set $c_f = f^{n+1}(x) - f^n(y)$. Then the coset $\Delta_f + c_f$ is well-defined and $\mathbb{Z}^k/\Delta_f$ is the cyclic group generated by $\Delta_f + c_f$ [14]. Conditions (A1) and (A2) ensure that $\mathbb{Z}^k/\Delta_f$ is finite and we write $d = |\mathbb{Z}^k/\Delta_f|$ [13].

Remark. At first sight, it is not clear that Δ_f is a group or, more precisely, that it is closed under addition: we shall give a proof of this fact. It is convenient to consider the directed graph with vertices $\{0, 1, \ldots, l-1\}$ and an edge joining i to j if and only if $A(i,j) = 1$. Then elements of Fix_n correspond to cycles in the graph and $f^n(x)$ to the sum of f around the edges. For a cycle γ, we shall denote this sum by $f(\gamma)$ and the length of γ by $l(\gamma)$. Since A is aperiodic there exists $N \geq 1$ such that, for each pair of vertices (i,j), we can choose a path $\delta(i,j)$ of length N joining i to j. Now choose a vertex i_0 and, for every cycle γ, a vertex $i_\gamma \in \gamma$. For each cycle γ form a new cycle $\overline{\gamma}$ passing through i_0 by $\overline{\gamma} = \delta(i_0, i_\gamma)\gamma\delta(i_\gamma, i_0)$. Let $f(\gamma) - f(\gamma')$ and $f(\eta) - f(\eta')$ be two arbitrary elements of Δ_f, where $\gamma, \gamma', \eta, \eta'$ are cycles with $l(\gamma) = l(\gamma')$ and $l(\eta) = l(\eta')$. Then $\overline{\gamma}\overline{\eta}$ and $\overline{\gamma}'\overline{\eta}'$ are cycles, $l(\overline{\gamma}\overline{\eta}) = l(\overline{\gamma}'\overline{\eta}')$ and

$$(f(\gamma) - f(\gamma')) + (f(\eta) - f(\eta')) = f(\overline{\gamma}\overline{\eta}) - f(\overline{\gamma}'\overline{\eta}').$$

This shows that Δ_f is closed under addition.

In this context (and in closely related situations) a variety of central limit theorems have been established (see the references cited in the introduction). In particular, in [4], a central limit theorem over periodic points is obtained and the rate of convergence is estimated. In this paper, however, we concentrate on local limit theorems; more precisely we seek to obtain estimates on

$$\sum_{j=0}^{d} \frac{e^{(h-h^*)n} n^{k/2}}{\#\mathrm{Fix}_{n+j}} \#\{x \in \mathrm{Fix}_{n+j} : f^{n+j}(x) = \alpha\},$$

as $n \to \infty$, which are uniform in $\alpha \in \mathbb{Z}^k$. (The summation is required since $\{x \in \mathrm{Fix}_{n+j} : f^{n+j}(x) = \alpha\} \neq \varnothing$ for a unique $j \in \{0, 1, \ldots, d-1\}$, depending on the coset of α in $\mathbb{Z}^k/\Delta_f$.) This kind of problem has been addressed in [11] (following an idea of Sinai) and [19] (see also [1]) but the conditions imposed there are too stringent for our purposes.

2. Twisted Matrices

In order to analyse the behaviour of $\#\{x \in \mathrm{Fix}_n : f^n(x) = \alpha\}$, we shall introduce a family of twisted $l \times l$ matrices A_t, indexed by $t \in \mathbb{R}^k/2\pi\mathbb{Z}^k$. Define A_t by

$$A_t(i,j) = A(i,j)e^{i\langle t, f(i,j)\rangle + \langle \xi, f\rangle},$$

where the Right Hand Side is understood to be zero when $A(i,j)=0$. In particular, A_0 is an aperiodic positive matrix. An easy calculation shows that

$$\mathrm{trace} A_t^n = \sum_{x \in \mathrm{Fix}_n} e^{i\langle t, f^n(x)\rangle + \langle \xi, f^n(x)\rangle}.$$

In order to estimate this quantity, we need to analyse the eigenvalues of A_t.

The matrix A_t will have l eigenvalues which we denote by $\widetilde{\lambda}_1(t), \ldots, \widetilde{\lambda}_l(t)$ with $|\widetilde{\lambda}_1(t)| \geq |\widetilde{\lambda}_2(t)| \geq \cdots \geq |\widetilde{\lambda}_l(t)|$. The classical Perron-Frobenius Theorem ensures that $\lambda_\xi = \widetilde{\lambda}_1(0)$ is simple and positive and that the remaining eigenvalues of A_0 are strictly smaller in modulus than λ_ξ. Furthermore, $P(\langle \xi, f\rangle) = \log \lambda_\xi$ and $\lambda_\xi < \lambda$ unless $\xi = 0$. In subsequent calculations it will prove more convenient to work with the quantities $\lambda_j(t) = \widetilde{\lambda}_j(t)/\lambda_\xi$, $j = 1, \ldots, l$. We will need to understand when $|\lambda_1(t)|$ is maximised.

Proposition 1.

(i) *We have that* $|\lambda_1(t)| \leq 1$ *for all* $t \in \mathbb{R}^k/2\pi\mathbb{Z}^k$. *Furthermore, if* $|\lambda_1(t)| = 1$ *then* $\widetilde{\lambda}_1(t)$ *is simple and* $|\lambda_j(t)| < 1$, $j = 2, \ldots, l$.

(ii) *We have the two identities*

$$\{e^{2\pi i\langle t, \cdot\rangle} : |\lambda_1(t)| = 1\} = \Delta_f^\perp,$$

$$\{\lambda_1(t) : e^{2\pi i\langle t,\cdot\rangle} \in \Delta_f^\perp\} = \{e^{2\pi i r/d} : r = 0, 1, \ldots, d-1\}.$$

Proof. Part (i) is part of Wielandt's Theorem [6, p. 57]. Part (ii) is proved in [15].

We shall write $t^{(r)}$ for the unique value of t satisfying $\lambda_1(t^{(r)}) = e^{2\pi i r/d}$. For (small) $\delta > 0$, we define a neighbourhood of $t^{(0)} = 0 \in \mathbb{R}^k/2\pi\mathbb{Z}^k$ by $U_0(\delta) = \{t : ||t|| \leq \delta\}$ and let $U_r(\delta) = U_0(\delta) + t^{(r)}$ for $r = 1, 2, \ldots, d-1$. A simple calculation shows that, for $t \in U_r(\delta)$,

$$\lambda_1(t) = e^{2\pi i r/d}\lambda_1(t - t^{(r)}) \tag{2.1}$$

([15]). In particular, for $r = 1, 2, \ldots, d-1$ and $n \geq 1$,

$$\sum_{j=0}^{d-1} \lambda_1(t^{(r)})^{n+j} = 0. \tag{2.2}$$

If w_t is the right eigenvector for A_t corresponding to the eigenvalue $\widetilde{\lambda}_1(t)$ then, for $t \in U_r(\delta)$, we also have $w_t = w_{t-t^{(r)}}$. Since $\widetilde{\lambda}_1(t^{(r)})$ is an isolated simple eigenvalue of $A_{t^{(r)}}$, eigenvalue perturbation theory ensures that $\lambda_1(t)$ and w_t depend analytically on t in $U_r(\delta)$ [9].

In view of the above discussion, we have the following estimates on $\lambda_j(t)$. For all sufficiently small $\delta > 0$ there exists $0 < \theta < 1$ such that

(i) $|\lambda_j(t)| \leq \theta$ for all $t \in \bigcup_{r=0}^{d-1} U_r(\delta)$, $j = 2, \ldots, l$;

(ii) $|\lambda_j(t)| \leq \theta$ for all $t \notin \bigcup_{r=0}^{d-1} U_r(\delta)$, $j = 2, \ldots, l$.

The following result is standard (cf. [15] for example).

Lemma 1. *Assume that f satisfies (A1) and (A2). Then the gradient $\nabla\lambda_1(0) = 0$ and the Hessian matrix $\nabla^2\lambda_1(0)$ is real and strictly negative definite.*

From now on, we shall write $\mathcal{D}_\xi = -\nabla^2\lambda_1(0)$, so that $\mathcal{D}_\xi$ is strictly positive definite. In particular, $\det \mathcal{D}_\xi > 0$ and we define $\sigma_\xi > 0$ by $\sigma_\xi^{2k} = \det \mathcal{D}_\xi$. The following result on the limiting behaviour of $\lambda_1(t)$ appears in several places, e.g. [4], [19].

Proposition 2. *There exists $\delta > 0$ such that, for $t \in U_0(\delta\sigma_\xi\sqrt{n})$,*

$$\lim_{n\to\infty} \lambda_1\left(\frac{t}{\sigma_\xi\sqrt{n}}\right)^n = e^{-\langle t,\mathcal{D}_\xi t\rangle/2\sigma_\xi^2}.$$

Furthermore,

$$\left|\lambda_1\left(\frac{t}{\sigma_\xi\sqrt{n}}\right)^n - e^{-\langle t,\mathcal{D}_\xi t\rangle/2\sigma_\xi^2}\right| \leq 2e^{-\langle t,\mathcal{D}_\xi t\rangle/4\sigma_\xi^2}.$$

Proof. Recall that $\nabla\lambda_1(0) = 0$. Since, in a neighbourhood of 0, $\lambda_1(t)$ depends analytically on t, we may apply Taylor's Theorem to write

$$\lambda_1\left(\frac{t}{\sigma_\xi\sqrt{n}}\right) = 1 - \frac{\langle t,\mathcal{D}_\xi t\rangle}{2\sigma_\xi^2 n} + O(||t||^3/n^{3/2}).$$

The first part of the result now follows from the standard formula $\lim_{n\to\infty}(1-x/n)^n = e^{-x}$.

For the second part, notice that, provided δ is sufficiently small, for $||u|| \leq \delta$ we have

$$\frac{\langle u,\mathcal{D}_\xi u\rangle}{2} + O(||u||^3) \geq \frac{\langle u,\mathcal{D}_\xi u\rangle}{4}.$$

Applying the triangle inequality and the inequality $(1 - x/n)^n < e^{-x}$, we have

$$\begin{aligned}\left|\lambda_1\left(\frac{t}{\sigma_\xi\sqrt{n}}\right)^n - e^{-\langle t,\mathcal{D}_\xi t\rangle/2\sigma_\xi^2}\right| &\leq e^{-\langle t,\mathcal{D}_\xi t\rangle/4\sigma\xi^2} + e^{-\langle t,\mathcal{D}_\xi t\rangle/2\sigma_\xi^2} \\ &\leq 2e^{-\langle t,\mathcal{D}_\xi t\rangle/4\sigma_\xi^2}.\end{aligned}$$

3. A Local Limit Theorem for Subshifts

In this section we shall obtain a local limit theorem for the function $f : X_A \to \mathbb{Z}^k$ with respect to the periodic points of $\sigma : X_A \to X_A$. We shall examine the quantity

$$\mathcal{S}(n,\alpha) = \sum_{j=0}^{d-1} \frac{e^{-\langle\xi,\alpha\rangle}\sigma_\xi^k n^{k/2}(\lambda/\lambda_\xi)^{n+j}}{\#\mathrm{Fix}_{n+j}} \#\{x \in \mathrm{Fix}_{n+j} \,:\, f^{n+j}(x) = \alpha\}.$$

For $a > 0$, write $I(a) = [-a,a]^k$. Using the orthogonality relationship

$$\frac{1}{(2\pi)^k}\int_{I(\pi)} e^{-i\langle t,\alpha\rangle}e^{i\langle t,y\rangle}dt = \begin{cases} 1 \text{ if } y = \alpha \\ 0 \text{ otherwise}\end{cases},$$

we have that

$$\mathcal{S}(n,\alpha) = \frac{1}{(2\pi)^k} \sum_{j=0}^{d-1} \frac{\sigma_\xi^k n^{k/2} (\lambda/\lambda_\xi)^{n+j}}{\#\mathrm{Fix}_{n+j}} \int_{I(\pi)} e^{-i\langle t,\alpha\rangle} \sum_{x\in \mathrm{Fix}_{n+j}} e^{i\langle t, f^{n+j}(x)\rangle} dt.$$

Making the substitution $t \mapsto t/\sigma_\xi\sqrt{n}$, we obtain

$$\mathcal{S}(n,\alpha) = \frac{1}{(2\pi)^k} \sum_{j=0}^{d-1} \int_{I(\pi\sigma_\xi\sqrt{n})} e^{-i\langle t,\alpha\rangle/\sigma_\xi\sqrt{n}} \frac{(\lambda/\lambda_\xi)^{n+j}}{\#\mathrm{Fix}_{n+j}} \sum_{x\in \mathrm{Fix}_{n+j}} e^{i\langle t, f^{n+j}(x)\rangle/\sigma_\xi\sqrt{n}} dt.$$

We are now in a position to prove the following theorem.

Theorem 2. *Suppose that $f : X_A \to \mathbb{Z}^k$ satisfies conditions (A1) and (A2). Then*

$$\lim_{n\to\infty} \left| \sum_{j=0}^{d-1} \frac{\sigma_\xi^k n^{k/2} (\lambda/\lambda_\xi)^{n+j}}{\#\mathrm{Fix}_{n+j}} \#\{x \in \mathrm{Fix}_{n+j} \,:\, f^{n+j}(x) = \alpha\} - \frac{d e^{\langle \xi,\alpha\rangle}}{(2\pi)^{k/2}} e^{-\langle \alpha, \mathcal{D}_\xi^{-1}\alpha\rangle/2n} \right| = 0,$$

uniformly in $\alpha \in \mathbb{Z}^k$.

Proof. Using the identity (valid for any positive definite Hermitian matrix $\mathcal{D}_\xi$),

$$e^{-\langle \alpha, \mathcal{D}_\xi^{-1}\alpha\rangle/2n} = \frac{1}{(2\pi)^{k/2}} \int_{\mathbb{R}^k} e^{-i\langle t,\alpha\rangle/\sigma_\xi\sqrt{n}} e^{-\langle t, \mathcal{D}_\xi t\rangle/2\sigma_\xi^2} dt,$$

we have established the bound

$$\begin{aligned}
&(2\pi)^k \left| \sum_{j=0}^{d-1} \frac{e^{-\langle \xi,\alpha\rangle} \sigma_\xi^k n^{k/2} \gamma^{n+j}}{\#\mathrm{Fix}_{n+j}} \#\{x \in \mathrm{Fix}_{n+j} \,:\, f^{n+j}(x) = \alpha\} - \frac{d e^{-\langle \alpha, \mathcal{D}_\xi^{-1}\alpha\rangle/2n}}{(2\pi)^{k/2}} \right| \leq \\
&\left| \int_{U_0(\delta\sigma_\xi\sqrt{n})} e^{-i\langle t,\alpha\rangle/\sigma_\xi\sqrt{n}} \left\{ \sum_{j=0}^{d-1} \frac{\gamma^{n+j}}{\#\mathrm{Fix}_{n+j}} \sum_{x\in \mathrm{Fix}_{n+j}} e^{i\langle t, f^{n+j}(x)\rangle/\sigma_\xi\sqrt{n}} - d e^{-\langle t, \mathcal{D}_\xi t\rangle/2\sigma_\xi^2} \right\} dt \right| \\
&+ \left| \int_{I(\pi\sigma_\xi\sqrt{n})\setminus U_0(\delta\sigma_\xi\sqrt{n})} e^{-i\langle t,\alpha\rangle/\sigma_\xi\sqrt{n}} \sum_{j=0}^{d-1} \frac{\gamma^{n+j}}{\#\mathrm{Fix}_{n+j}} \sum_{x\in \mathrm{Fix}_{n+j}} e^{i\langle t, f^{n+j}(x)\rangle/\sigma_\xi\sqrt{n}} \right| \\
&+ \left| \int_{\mathbb{R}^k\setminus U_0(\delta\sigma_\xi\sqrt{n})} d e^{-i\langle t,\alpha\rangle/\sigma_\xi\sqrt{n}} e^{-\langle t, \mathcal{D}_\xi t\rangle/2\sigma_\xi^2} dt \right| \\
&= A_1(n,\alpha) + A_2(n,\alpha) + A_3(n,\alpha),
\end{aligned}$$

where $\gamma = \lambda/\lambda_\xi$. An easy calculation shows that $\lim_{n\to\infty} \sup_{\alpha\in\mathbb{Z}^k} A_3(n,\alpha) = 0$, so it remains to consider A_1 and A_2.

For $t \in U_0(\delta\sigma_\xi\sqrt{n})$, we have that

$$\sum_{j=0}^{d-1} \frac{\gamma^{n+j}}{\#\mathrm{Fix}_{n+j}} \sum_{x\in \mathrm{Fix}_{n+j}} e^{i\langle t, f^{n+j}(x)\rangle/\sigma_\xi\sqrt{n}} = \lambda_1 \left(\frac{t}{\sigma_\xi\sqrt{n}}\right)^n \sum_{j=0}^{d-1} \lambda_1 \left(\frac{t}{\sigma_\xi\sqrt{n}}\right)^j + O(\theta^n).$$

and that

$$\left|\sum_{j=0}^{d-1} \lambda_1\left(\frac{t}{\sigma_\xi\sqrt{n}}\right)^j - d\right| \le C\delta^2,$$

for some constant $C > 0$. By Proposition 2, we know that $\lambda_1(t/\sigma_\xi\sqrt{n})^n$ converges uniformly to $e^{-\langle t,\mathcal{D}_\xi t\rangle/2\sigma_\xi^2}$, as $n \to \infty$. Furthermore, we have the estimates

$$\left|d\lambda_1\left(\frac{t}{\sigma_\xi\sqrt{n}}\right)^n - de^{-\langle t,\mathcal{D}_\xi t\rangle/2\sigma_\xi^2}\right| \le 2de^{-\langle t,\mathcal{D}_\xi t\rangle/4\sigma_\xi^2}$$

and

$$\left|\lambda_1\left(\frac{t}{\sigma_\xi\sqrt{n}}\right)^n\left\{\sum_{j=0}^{d-1} \lambda_1\left(\frac{t}{\sigma_\xi\sqrt{n}}\right)^j - d\right\}\right| \le Ce^{-\langle t,\mathcal{D}_\xi t\rangle/4\sigma_\xi^2}\delta^2.$$

Thus, by the Dominated Convergence Theorem, we obtain

$$\limsup_{n\to\infty}\sup_{\alpha\in\mathbb{Z}^k} A_1(n,\alpha) \le C\left\{\int_{\mathbb{R}^k} e^{-\langle t,\mathcal{D}_\xi t\rangle/4\sigma_\xi^2}dt\right\}\delta^2.$$

Finally, we consider A_2. If $t \notin \bigcup_{r=1}^{d-1} U_r(\delta\sigma_\xi\sqrt{n})$, then

$$\sum_{j=0}^{d-1}\frac{\gamma^{n+j}}{\#\mathrm{Fix}_{n+j}}\sum_{x\in\mathrm{Fix}_{n+j}} e^{i\langle t,f^{n+j}(x)\rangle/\sigma_\xi\sqrt{n}} = O(\theta^n).$$

On the other hand, if $t \in \bigcup_{r=1}^{d-1} U_r(\delta\sigma_\xi\sqrt{n})$, then

$$\begin{aligned}
&\left|\sum_{j=0}^{d-1}\frac{\gamma^{n+j}}{\#\mathrm{Fix}_{n+j}}\sum_{x\in\mathrm{Fix}_{n+j}} e^{i\langle t,f^{n+j}(x)\rangle/\sigma_\xi\sqrt{n}}\right| \\
=&\left|\sum_{j=0}^{d-1} e^{2\pi ir(n+j)/d}\lambda_1\left(\frac{t}{\sigma_\xi\sqrt{n}} - t^{(r)}\right)^{n+j}\right| + O(\theta^n) \\
\le& C'e^{-\langle t',\mathcal{D}_\xi t'\rangle/4\sigma^2}\delta^2 + O(\theta^n),
\end{aligned}$$

for some constant $C' > 0$ and where $t' = t - \sigma_\xi\sqrt{n}t^{(r)}$, the last estimate following from (2.2), the analyticity of λ_1 and the vanishing of its first derivatives. This gives us

$$\limsup_{n\to\infty}\sup_{\alpha\in\mathbb{Z}^k} A_2(n,\alpha) \le C'\left\{\int_{\mathbb{R}^k} e^{-\langle t,\mathcal{D}_\xi t\rangle/4\sigma_\xi^2}dt\right\}\delta^2.$$

Combining the above estimates we have

$$\begin{aligned}
&\limsup_{n\to\infty}\sup_{\alpha\in\mathbb{Z}^k}\left|\sum_{j=0}^{d-1}\frac{e^{-\langle\xi,\alpha\rangle}\sigma_\xi^k n^{k/2}\gamma^{n+j}}{\#\mathrm{Fix}_{n+j}}\#\{x \in \mathrm{Fix}_{n+j} : f^{n+j}(x) = \alpha\} - \frac{de^{-\langle\alpha,\mathcal{D}_\xi^{-1}\alpha\rangle/2n}}{(2\pi)^{k/2}}\right| \\
\le& \frac{(C+C')}{(2\pi)^k}\left\{\int_{\mathbb{R}^k} e^{-\langle t,\mathcal{D}_\xi t\rangle/4\sigma_\xi^2}dt\right\}\delta^2.
\end{aligned}$$

Since this holds for all sufficiently small $\delta > 0$, the proof of the theorem is complete.

We state the special case where $\xi = 0$ as a corollary. Here we write $\mathcal{D}_0 = \mathcal{D}$ and $\sigma_0 = \sigma$.

Corollary 2.1. *Suppose that $f : X_A \to \mathbb{Z}^k$ satisfies condition (A1) and $\int f d\mu = 0$, where μ is the measure of maximal entropy. Then*

$$\lim_{n\to\infty} \left| \sum_{j=0}^{d-1} \frac{\sigma^k n^{k/2}}{\#\mathrm{Fix}_{n+j}} \#\{x \in \mathrm{Fix}_{n+j} \,:\, f^{n+j}(x) = \alpha\} - \frac{d}{(2\pi)^{k/2}} e^{-\langle \alpha, \mathcal{D}^{-1}\alpha\rangle/2n} \right| = 0,$$

uniformly in $\alpha \in \mathbb{Z}^k$

Remark. In particular, we have recovered the main result of [15], namely that $\#\{x \in \mathrm{Fix}_{dn} : f^{dn}(x) = 0\} \sim C\lambda_\xi^{dn}/n^{k/2}$, as $n \to \infty$, for some constant $C > 0$. However, the above method does not allow us to estimate the error term in this approximation. (The $O(n^{-1/2})$ error estimate claimed there is erroneous and needs to be corrected to $O(n^{-1/2+\epsilon})$. Conjecturally, the optimal error estimate is $O(n^{-1})$.)

4. Free Groups

In this section we shall deduce Theorem 1 from Theorem 2 and give an explicit expression for the matrix $\mathcal{D}$. Let G be the free group on $k \geq 2$ generators. Define a $(2k+1)\times(2k+1)$ matrix A, indexed by $\{*, 1, 2, \ldots, 2k\}$, by $A(*, *) = 0$, $A(*, j) = 1$ for all $j = 1, 2, \ldots, 2k$, $A(i, *) = 0$ for all $i = 1, 2, \ldots, 2k$, and, for $i, j = 1, 2, \ldots, 2k$,

$$A(i,j) = \begin{cases} 1 \text{ if } j \neq i + k \pmod{2k} \\ 0 \text{ if } j = i + k \pmod{2k} \end{cases}.$$

Then the maximal eigenvalue λ of A is equal to $2k-1$. Let B denote the $2k \times 2k$ submatrix of A indexed by $\{1, 2, \ldots, 2k\}$; it is easy to check that B is aperiodic and that $\bigcup_{n\geq 1} \mathrm{Fix}_n \subset X_B$. If we index the generators of G by $\{a_1, \ldots, a_k, a_{k+1} = a_1^{-1}, \ldots, a_{2k} = a_k^{-1}\}$, then it is clear that there is a natural bijection between cyclically reduced words of length n in G and elements of Fix_n, and between reduced words of length n and all sequences of the form $(x_0, x_1, \ldots, x_n)$ with $x_0 = *$ and $A(x_m, x_{m+1}) = 1$, $m = 1, \ldots, n-1$. In particular, $\#\mathcal{W}(n) = \langle u, A^n v\rangle$, where $u = (1, 0, \ldots, 0)$ (with the 0 occurring in the $*$ position) and $v = (1, 1, \ldots, 1)$, and that $\#\mathcal{C}(n) = \mathrm{trace} A^n$.

If we define a function $f : X_A \to \mathbb{Z}^k$ by $f(i, j) = [a_j]$ then it is easy to see that the element of $\mathbb{Z}^k$ corresponding to the cyclically reduced word associated to $x \in \mathrm{Fix}_n$ is $f^n(x)$. In particular, $\#\mathcal{C}(n, \alpha) = \#\{x \in \mathrm{Fix}_n \,:\, f^n(x) = \alpha\}$ and

$$\bigcup_{n\geq 1} \{f^n(x) \,:\, x \in \mathrm{Fix}_n\} = \bigcup_{n\geq 1} \{[g] \,:\, g \in \mathcal{C}(n)\} = \mathbb{Z}^k.$$

This last identity implies that the restriction $f : X_B \to \mathbb{Z}^k$ satisfies condition (A1).

If μ denotes the measure of maximal entropy on X_B then it is well-known that the periodic points of $\sigma : X_B \to X_B$ are equidistributed with respect to μ. More precisely, we have the identity

$$\int f d\mu = \lim_{n\to\infty} \frac{1}{\#\mathrm{Fix}_n} \sum_{x \in \mathrm{Fix}_n} \frac{f^n(x)}{n}.$$

The symmetry $[g^{-1}] = -[g]$ then shows that we have $\int f d\mu = 0$. A simple calculation shows that Δ_f is the subgroup of $\mathbb{Z}^k$ consisting of all even elements, so that $d = |\mathbb{Z}^k/\Delta_f| = 2$.

The following result now follows immediately from Corollary 2.1. A simple symmetry argument shows that the covariance matrix $\mathcal{D}$ is diagonal, $\mathcal{D} = \mathrm{diag}(\sigma^2, \ldots, \sigma^2)$, say, and the explicit formula for σ^2 given by (0.1) is due to Rivin [18].

Proposition 3.

$$\lim_{n\to\infty} \sup_{\alpha\in\mathbb{Z}^k} \left| \sigma^k n^{k/2} \left(\frac{\#\mathcal{C}(n,\alpha)}{\#\mathcal{C}(n)} + \frac{\#\mathcal{C}(n+1,\alpha)}{\#\mathcal{C}(n+1)} \right) - \frac{2}{(2\pi)^{k/2}} e^{-||\alpha||^2/2\sigma^2 n} \right| = 0.$$

Proof of Theorem 1. We shall now discuss the modifications necessary to prove the result for $\mathcal{W}(n)$. For $t \in \mathbb{R}^k/2\pi\mathbb{Z}^k$, we introduce matrices A_t, B_t defined by $A_t(i,j) = A(i,j)e^{i\langle t, f(i,j)\rangle}$ and $B_t(i,j) = B(i,j)e^{i\langle t, f(i,j)\rangle}$. A simple calculation shows that A_t has the same non-zero spectrum as B_t. Since B is aperiodic and $f : X_B \to \mathbb{Z}^k$ satisfies (A1) and (A2), the maximal eigenvalue $\widetilde{\lambda}_1(t)$ continues to enjoy the properties described in Section 2.

We note that $\#\mathcal{W}(n) = 2k\lambda^{n-1}$ and that

$$\begin{aligned}\#\mathcal{W}(n,\alpha) = \sum_{g\in\mathcal{W}(n)} \frac{1}{(2\pi)^k} \int_{I(\pi)} e^{-i\langle t,\alpha\rangle} e^{i\langle t,[g]\rangle} dt &= \sum_{j=1}^{2k} \frac{1}{(2\pi)^k} \int_{I(\pi)} e^{-i\langle t,\alpha\rangle} A_t^n(*,j) dt \\ &= \frac{1}{(2\pi)^k} \int_{I(\pi)} e^{-i\langle t,\alpha\rangle} \langle u, A_t^n v\rangle dt.\end{aligned}$$

For $t \in U_r(\delta)$, we have

$$\langle u, A_t^n v\rangle = (-1)^r \widetilde{\lambda}_1(t - t^{(r)})^n \langle u, w_{t-t^{(r)}}\rangle + O((\theta\lambda)^n),$$

where w_t is the eigenprojection of v for A_t associated to the eigenvalue $\widetilde{\lambda}_1(t)$. It is easy to see that $w_0 = (2k/(2k-1), 1, \ldots, 1)$.

Applying the analysis of the preceding section to

$$\sigma^k n^{k/2} \left(\frac{\#\mathcal{W}(n,\alpha)}{\#\mathcal{W}(n)} + \frac{\#\mathcal{W}(n+1,\alpha)}{\#\mathcal{W}(n+1)} \right),$$

we obtain

$$\begin{aligned}&(2\pi)^k \left| \sigma^k n^{k/2} \left(\frac{\#\mathcal{W}(n,\alpha)}{\#\mathcal{W}(n)} + \frac{\#\mathcal{W}(n+1,\alpha)}{\#\mathcal{W}(n+1)} \right) - \frac{2e^{-||\alpha||^2/2\sigma^2 n}}{(2\pi)^{k/2}} \right| \\ &\le \left| \int_{U_0(\delta\sigma\sqrt{n})} e^{-i\langle t,\alpha\rangle/\sigma\sqrt{n}} \left\{ \frac{\langle u, A^n_{t/\sigma\sqrt{n}} v\rangle}{\#\mathcal{W}(n)} + \frac{\langle u, A^{n+1}_{t/\sigma\sqrt{n}} v\rangle}{\#\mathcal{W}(n+1)} - 2e^{-||t||^2/2} \right\} dt \right| \\ &+ \left| \int_{I(\pi\sigma\sqrt{n})\setminus U_0(\delta\sigma\sqrt{n})} e^{-i\langle t,\alpha\rangle/\sigma\sqrt{n}} \left\{ \frac{\langle u, A^n_{t/\sigma\sqrt{n}} v\rangle}{\#\mathcal{W}(n)} + \frac{\langle u, A^{n+1}_{t/\sigma\sqrt{n}} v\rangle}{\#\mathcal{W}(n+1)} \right\} dt \right| \\ &+ \left| \int_{\mathbb{R}^k\setminus U_0(\delta\sigma\sqrt{n})} 2e^{-i\langle t,\alpha\rangle/\sigma\sqrt{n}} e^{-||t||^2/2} dt \right|.\end{aligned}$$

Now, for $t \in U_0(\delta\sigma\sqrt{n})$,

$$\frac{1}{\#\mathcal{W}(n)} \sum_{g\in\mathcal{W}(n)} e^{i\langle t,[g]\rangle/\sigma\sqrt{n}} + \frac{1}{\#\mathcal{W}(n+1)} \sum_{g\in\mathcal{W}(n+1)} e^{i\langle t,[g]\rangle/\sigma\sqrt{n}}$$
$$= \lambda_1\left(\frac{t}{\sigma\sqrt{n}}\right)^n \left(1+\lambda_1\left(\frac{t}{\sigma\sqrt{n}}\right)\right)\langle u, w_{t/\sigma\sqrt{n}}\rangle + O(\theta^n)$$

and for $t \in U_1(\delta\sigma\sqrt{n})$,

$$\frac{1}{\#\mathcal{W}(n)} \sum_{g\in\mathcal{W}(n)} e^{i\langle t,[g]\rangle/\sigma\sqrt{n}} + \frac{1}{\#\mathcal{W}(n+1)} \sum_{g\in\mathcal{W}(n+1)} e^{i\langle t,[g]\rangle/\sigma\sqrt{n}}$$
$$= (-1)^n\lambda_1\left(\frac{t}{\sigma\sqrt{n}} - t^{(1)}\right)^n \left(1+\lambda_1\left(\frac{t}{\sigma\sqrt{n}} - t^{(1)}\right)\right)\langle u, w_{t/\sigma\sqrt{n}}\rangle + O(\theta^n)$$

Thus we may repeat the arguments in the proof of Theorem 2 to obtain the estimate

$$\limsup_{n\to\infty} \sup_{\alpha\in\mathbb{Z}^k} \left|\sigma^k n^{k/2}\left(\frac{\#\mathcal{W}(n,\alpha)}{\#\mathcal{W}(n)} + \frac{\#\mathcal{W}(n+1,\alpha)}{\#\mathcal{W}(n+1)}\right) - \frac{2e^{-||\alpha||^2/2\sigma^2 n}}{(2\pi)^{k/2}}\right|$$
$$\le C\left\{\int_{\mathbb{R}^k} e^{-\langle t,\mathcal{D}t\rangle/4\sigma^2} dt\right\}\delta,$$

for some constant $C > 0$. (The only additional feature being that $\langle u, w_t\rangle = \langle u, w_0\rangle + O(||t||)$.) Since this holds for all sufficiently small $\delta > 0$, Theorem 1 is proved.

5. Strongly Markov Groups

In this final section we shall sketch the generalizations necessary to extend our results to certain groups G satisfying the following *strong Markov* property: for any finite symmetric generating set S, there exists

(i) a finite directed graph consisting of vertices V and edges $E \subset V \times V$;
(ii) a distinguished vertex $* \in V$, with no edges terminating at $*$;
(iii) a labeling map $\rho : E \to S$;

such that

(a) there is a bijection between finite paths in the graph starting at $*$ and passing through the consecutive edges $e_1, \ldots, e_n$, say and elements $g \in G$ given by the correspondence $g = \rho(e_1)\cdots\rho(e_n)$ (where the empty path corresponds to the identity element);
(b) the word length $|g|$ is equal to the path length n.

In particular, this condition is satisfied by all (Gromov) hyperbolic groups [3],[7].

Write $|V| = l + 1$. Let A denote the incidence matrix of the graph (V, E), i.e., A is a $(l+1)\times(l+1)$ matrix, indexed by V, with entries $A(i,j) = 1$ if $(i,j) \in E$ and 0 otherwise. Let B denote the $l \times l$ submatrix of A obtained by deleting the row and column corresponding to $*$. We shall assume that B is aperiodic with maximal eigenvalue $\lambda > 1$..

The abelianization of G takes the form $G/[G,G] \cong \mathbb{Z}^k \oplus \text{torsion}$. We suppose that $k > 0$ and write $[\cdot] : G \to \mathbb{Z}^k$ for the natural homomorphism. As in the case of free groups, we define a function $f : X_A \to \mathbb{Z}^k$ by $f(x) = [\rho(x_0, x_1)]$. A new feature here is that it is not clear that the group Γ_f generated by $\{f^n(x) : x \in \mathrm{Fix}_n\}$ is not necessarily equal to $\mathbb{Z}^k$. However, we still have that Γ_f/Δ_f is a finite cyclic group and it was shown in [22] that $\mathbb{Z}^k/\Gamma_f$ is finite; we set $d_0 = |\Gamma_f/\Delta_f|$ and $d_1 = |\mathbb{Z}^k/\Gamma_f|$.

As before, for $t \in \mathbb{R}^{2\mathfrak{g}}/2\pi\mathbb{Z}^{2\mathfrak{g}}$, define matrices A_t,B_t by $A_t(i,j) = A(i,j)e^{i\langle t, f(i,j)\rangle}$ and $B_t(i,j) = B(i,j)e^{i\langle t, f(i,j)\rangle}$, and note that again A_t has the same non-zero spectrum as B_t. There are $d = d_0 d_1$ values, $t^{(0)} = 0, \dots, t^{(d-1)}$, of t for which A_t has an eigenvalue of maximum modulus $\widetilde{\lambda}_1(t^{(r)})$ with $|\widetilde{\lambda}_1(t^{(r)})| = \lambda$. Furthermore, $\widetilde{\lambda}_1(t^{(r)}) = e^{2\pi i r/d_0}\lambda$. (Note that each $e^{2\pi i r/d_0}\lambda$ occurs for d_1 values of t.)

One can show that $f : X_B \to \mathbb{Z}^k$ satisfies that $\int f d\mu = 0$, where μ is the measure of maximal entropy on X_B or, equivalently, that A_t and B_t have spectral radius λ [22].

From the definition it is easy to see that we have the identities

$$\#\mathcal{W}(n) = \sum_{j \in V} A^n(*, j) = \langle u, A^n v\rangle$$

and

$$\#\mathcal{W}(n,\alpha) = \frac{1}{(2\pi)^k} \int_{I(\pi)} e^{-i\langle t, \alpha\rangle} \langle u, A_t^n v\rangle dt,$$

where $u = (1, 0, \dots, 0)$ (with the 1 occurring in the $*$ position) and $v = (1, 1, \dots, 1)$. Furthermore, for $t \in U_r(\delta)$, $r = 0, 1, \dots, d-1$, we still have

$$\langle u, A_t^n v\rangle = e^{2\pi i n r/d_0} \widetilde{\lambda}_1(t - t^{(r)})^n \langle u, w_t\rangle + O((\theta\lambda)^n),$$

where w_t is the eigenprojection of v for A_t associated to the eigenvalue $\widetilde{\lambda}_1(t)$ and $0 < \theta < 1$. Mimicing the proof of Theorem 1, we obtain the following result, where, as in Corollary 2.1, $\mathcal{D} = -\nabla^2 \lambda_1(0)$. (It is worthwhile noting that it is possible to have $\mathcal{W}(n+j, \alpha) \neq \varnothing$ for several values of $j \in \{0, 1, \dots, d_0 - 1\}$.)

Theorem 3. *Let G be a strongly Markov group such that $G/[G,G] \cong \mathbb{Z}^k \oplus \text{torsion}$ with $k \geq 1$. Let S be finite symmetric generating set and suppose that the associated matrix B defined above is aperiodic. Then there exists a symmetric positive definite real matrix $\mathcal{D}$ such that*

$$\lim_{n \to \infty} \left| \sigma^k n^{k/2} \sum_{j=0}^{d_0} \frac{\#\mathcal{W}(n+j, \alpha)}{\#\mathcal{W}(n+j)} - \frac{d_0}{(2\pi)^{k/2} \langle u, w_0\rangle} \sum_{r=0}^{d_1 - 1} \langle u, w_{t^{(d_0 r)}}\rangle \; e^{-\langle \alpha, \mathcal{D}^{-1}\alpha\rangle/2n} \right| = 0,$$

uniformly in $\alpha \in \mathbb{Z}^k$.

Remark. A similar analysis can be made in the case where B is irreducible, i.e., when, for each pair (i,j), there exists $n(i,j) > 0$ such that $B^{n(i,j)}(i,j) > 0$. In this case, the maximum modulus eigenvalues of B are the q-th roots of the maximum modulus eigenvalues of a certain aperiodic matrix, where $q = \mathrm{hcf}\{n(i,i) : i \in V \backslash \{*\}\}$ is called the period of B.

One can then obtain the following more complicated formulae along the subsequence nq, $n \geq 1$.
If d_0 does not divide q then

$$\lim_{n\to\infty} \left| \sigma^k (nq)^{k/2} \sum_{j=0}^{d_0} \frac{\#\mathcal{W}(nq+jq,\alpha)}{\#\mathcal{W}(nq+jq)} - \frac{d_0 \sum_{m=0}^{q-1} \sum_{r=0}^{d_1-1} \langle u, w_{t^{(d_0 r)}}^{(m)} \rangle}{(2\pi)^{k/2} \sum_{m=0}^{q-1} \langle u, w_0^{(m)} \rangle} e^{-\langle \alpha, \mathcal{D}^{-1}\alpha \rangle / 2nq} \right| = 0,$$

uniformly in $\alpha \in \mathbb{Z}^k$.
If d_0 divides q then

$$\lim_{n\to\infty} \left| \sigma^k (nq)^{k/2} \sum_{j=0}^{d_0} \frac{\#\mathcal{W}(nq+jq,\alpha)}{\#\mathcal{W}(nq+jq)} - \frac{d_0 \sum_{m=0}^{q-1} \sum_{r=0}^{d-1} \langle u, w_{t^{(r)}}^{(m)} \rangle}{(2\pi)^{k/2} \sum_{m=0}^{q-1} \langle u, w_0^{(m)} \rangle} e^{-\langle \alpha, \mathcal{D}^{-1}\alpha \rangle / 2nq} \right| = 0,$$

uniformly in $\alpha \in \mathbb{Z}^k$.
(Here, the terms $w_{t^{(r)}}^{(m)}$ are certain eigenvectors, associated to eigenvalues $e^{2\pi i m/q}\widetilde{\lambda}_1(t^{(r)})$, $m = 0, \ldots, q-1$, of B.)

A particular group presentation satisfying our hypotheses is the fundamental group G of a compact orientable surface of genus $\mathfrak{g} \geq 2$ given the standard one-relator presentation

$$G = \left\langle a_1, \ldots, a_{\mathfrak{g}}, b_1, \ldots, b_{\mathfrak{g}} : \prod_{i=1}^{\mathfrak{g}} a_i b_i a_i^{-1} b_i^{-1} = 1 \right\rangle. \tag{5.1}$$

(Note that $G/[G,G] \cong \mathbb{Z}^{2\mathfrak{g}}$.) This is an example of a hyperbolic group and thus is strongly Markov; however, in this case the result follows from earlier explicit constructions due to Cannon [2] and Series [21]. In particular, B is aperiodic. A nice feature of this construction is that closed loops in the directed graph (V,E) correspond precisely to conjugacy classes in G, from which one can deduce that $\Gamma_f = \mathbb{Z}^{2\mathfrak{g}}$. One can also see that Δ_f is the set of even elements of $\mathbb{Z}^{2\mathfrak{g}}$, so that $d = 2$. The following result now follows immediately from Theorem 3.

Theorem 4. *Let G be the fundamental group of a compact surface of genus $\mathfrak{g} \geq 2$ equipped with the presentation (5.1). Then there exists a symmetric positive definite real matrix $\mathcal{D}$ such that*

$$\lim_{n\to\infty} \left| \sigma^k n^{\mathfrak{g}} \left(\frac{\#\mathcal{W}(n,\alpha)}{\#\mathcal{W}(n)} + \frac{\#\mathcal{W}(n+1,\alpha)}{\#\mathcal{W}(n+1)} \right) - \frac{2}{(2\pi)^{\mathfrak{g}}} e^{-\langle \alpha, \mathcal{D}^{-1}\alpha \rangle / 2n} \right| = 0,$$

uniformly in $\alpha \in \mathbb{Z}^{2\mathfrak{g}}$.

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