CORRELATIONS FOR PAIRS OF CLOSED GEODESICS

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Abstract. In this article we consider natural counting problems for closed geodesics on negatively curved surfaces. We present asymptotic estimates for pairs of closed geodesics, the differences of whose lengths lie in a prescribed family of shrinking intervals. Related pair correlation problems have been studied in both Quantum Chaos and number theory.

0. Introduction

One of the most striking properties of negatively curved surfaces is the regularity of the distribution of the lengths of their closed geodesics. This is shown by the well-known prime geodesic theorem. More precisely, let V denote a compact surface with a C^∞ Riemannian metric of strictly negative curvature. Given any closed geodesic γ we denote its length by $\lambda(\gamma)$. By a remarkable result of Margulis, there exists $h > 0$ such that

$$\#\{\gamma : \lambda(\gamma) \le T\} \sim \frac{e^{hT}}{hT}, \text{ as } T \to +\infty \tag{0.1}$$

[15], where $A(T) \sim B(T)$ denotes that $\lim_{T\to+\infty} A(T)/B(T) = 1$. The quantity h is the topological entropy of the associated geodesic flow. In the special case of surfaces with constant curvature $\kappa < 0$, this was proved earlier by Huber, with $h = \sqrt{-\kappa} > 0$ [9]. Let $\pi_1(V)$ denote the fundamental group of V. If V has genus $\mathfrak{g} \ge 2$ then we consider the standard presentation $\pi_1(V) = \langle a_1, \dots, a_{\mathfrak{g}}, b_1, \dots, b_{\mathfrak{g}} : \prod_{i=1}^{2\mathfrak{g}} [a_i, b_i] = 1\rangle$ and set $S = \{a_1^{\pm 1}, \dots, a_{\mathfrak{g}}^{\pm 1}, b_1^{\pm 1}, \dots, b_{\mathfrak{g}}^{\pm 1}\}$. We can associate to each element $g \in \pi_1(V) - \{1\}$ the word length $|g|$, i.e., the smallest number of elements from S needed to write g. Every conjugacy class in $\pi_1(V)$ contains a unique closed geodesic. Given a closed geodesic γ we define $|\gamma| = \inf\{|g| : g \in \langle\gamma\rangle\}$, where the infimum is over the elements in the conjugacy class $\langle\gamma\rangle$ associated to γ. An analogue of (0.1) for word lengths is the following

$$\#\{\gamma : |\gamma| \le n\} \sim \frac{e^{h_0}}{e^{h_0} - 1}\frac{e^{h_0 n}}{n}, \text{ as } n \to +\infty. \tag{0.2}$$

The quantity h_0 is the logarithm of an algebraic integer and depends on the choice of generators S. Given $a < b$ we denote

$$\pi(n, [a,b]) = \#\{(\gamma, \gamma') : |\gamma|, |\gamma'| \le n,\ a \le \lambda(\gamma) - \lambda(\gamma') \le b\}.$$

We shall prove the following result.

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Theorem 1. *There exists $\sigma > 0$ such that, for any $a < b$,*

$$\pi(n, [a, b]) \sim \frac{(b-a)e^{2h_0}}{(2\pi)^{1/2}\sigma(e^{h_0}-1)^2} \frac{e^{2h_0 n}}{n^{5/2}}, \text{ as } n \to +\infty.$$

Related questions have been studied numerically [2],[3]. The proof of Theorem 1 depends on techniques developed in [20]. However, using a more delicate analysis, incorporating Dolgopyat's estimates on transfer operators [8], it is possible to prove a version of Theorem 1 which is stronger in two respects. Firstly, we will allow the interval in which the differences lie to shrink as the word length n tends to infinity and, secondly, we will obtain a result uniform in the positioning of the interval. To make this more precise, it is convenient to write the initial interval in the form $[z+a, z+b]$, for some choice of $z \in \mathbb{R}$. We will then consider a sequence of intervals of the form $I_n(z) := [z + \epsilon_n a, z + \epsilon_n b]$, where $\epsilon_n > 0$ tends to zero. We will also allow z to vary over $\mathbb{R}$ and obtain a uniform result. We say that a sequence of positive numbers ϵ_n tends to zero subexponentially if $\limsup_{n\to+\infty} |\log \epsilon_n|/n = 0$.

Theorem 2. *There exists $\sigma > 0$ such that for any $a < b$ and a sequence $\epsilon_n > 0$ which tends to zero at a subexponential rate, we have that*

$$\lim_{n\to+\infty} \sup_{z\in\mathbb{R}} \left| \frac{\sigma n^{5/2}}{\epsilon_n e^{2h_0 n}} \pi(n, I_n(z)) - \frac{(b-a)e^{2h_0}}{(2\pi)^{1/2}(e^{h_0}-1)^2} e^{-z^2/2\sigma^2 n} \right| = 0.$$

In particular, this implies that, for any fixed $z \in \mathbb{R}$,

$$\pi(n, I_n(z)) \sim \frac{(b-a)e^{2h_0}\epsilon_n}{(2\pi)^{1/2}\sigma(e^{h_0}-1)^2} \frac{e^{2h_0 n}}{n^{5/2}}, \text{ as } n \to +\infty.$$

Clearly, we cannot expect the result to hold if ϵ_n tends to zero too quickly. In particular, counting the diagonal terms (γ, γ) guarantees that, provided $0 \in I_n$, $\pi(n, I_n) \geq ce^{h_0 n}$, for some $c > 0$ and sufficiently large $n \geq 1$. Thus the asymptotics in Theorem 2 cannot hold if $\epsilon_n = O(e^{-\eta n})$, with $\eta > h_0$. Theorem 1 and Theorem 2 may be compared with pair correlation results in other contexts [4], [13], [17], [18], [24]. Similar asymptotic formula do not seem to be available if we order the pairs by their geometric length. For example, we might consider ordering the pairs (γ, γ) by the sum of their lengths $\lambda(\gamma) + \lambda(\gamma')$ but this apparently only leads to weaker estimates of the form

$$\sum_{\substack{\lambda(\gamma)+\lambda(\gamma')\leq T \\ 0\leq\lambda(\gamma)-\lambda(\gamma')\leq\epsilon}} \lambda(\gamma)\lambda(\gamma')e^{-h(\lambda(\gamma)+\lambda(\gamma'))} \sim AT, \text{ as } T \to +\infty, \tag{0.3}$$

for some $A > 0$. We shall now outline the contents of the paper. Section 1 contains some preliminary results on subshifts of finite type and the coding of closed geodesics by periodic orbits. Section 2 presents an estimate on certain sums over periodic points of subshifts of finite type in terms of pressure functions. The proof of Theorem 1 is presented in section 3. The proof of Theorem 2 is presented in section 5, following additional estimates on transfer operators, given in section 4. Section 6 gives asymptotic estimates related to (0.3). In section 7, we describe a generalization to higher dimensional correlation functions and analogous results for the action of the fundamental group of V on its universal cover. We would like to thank Jon Keating and Jens Marklof for interesting discussions.

1. Symbolic dynamics

We shall study closed geodesics on V via the geodesic flow $\phi_t : SV \to SV$. Note that there is a one-to-one correspondence between closed geodesics on V and periodic orbits for the geodesic flow. An essential ingredient in our analysis will be a symbolic model for the geodesic flow. Given a $k \times k$ aperiodic matrix A with entries 0 or 1, we define a space

$$\Sigma = \{x = (x_n)_{n=0}^{\infty} : A(x_n, x_{n+1}) = 1, \forall n \in \mathbb{Z}^+\}$$

and a shift $\sigma : \Sigma \to \Sigma$ given by $(\sigma x)_n = x_{n+1}$. There is a metric on Σ given by $d(x,y) = \sum_{n=0}^{\infty} 2^{-n}(1 - \delta(x_n, y_n))$. A shift is mixing if the matrix A is aperiodic, i.e., there exists $N \geq 1$ such that $A^N(i,j) \geq 1$ for any $1 \leq i, j \leq k$. Given $n \geq 1$, we denote $r^n(x) := r(x) + r(\sigma x) + \cdots + r(\sigma^{n-1}x)$.

Lemma 1.1. *We can associate to the geodesic flow a mixing subshift $\sigma : \Sigma \to \Sigma$ and a Hölder continuous function $r : \Sigma \to \mathbb{R}$ such that, with at most a finite number of exceptions, prime periodic orbits $\{x, \sigma x, \ldots, \sigma^{n-1}x\}$ correspond to prime closed geodesics γ whose word length is given by $|\gamma| = n$ and whose length is $\lambda(\gamma) = r^n(x)$.*

Proof. This essentially follows from the work of Series [28, 29, 30, 31] (or alternatively Adler and Flatto [1]) and is now folklore in this subject. We shall give a sketch of the main points. Consider first the case of a constant curvature surface. In this case we may identify the universal cover $\widetilde{V}$ with the Poincaré disk $\mathbb{D}^2 = \{z \in \mathbb{C} : |z| < 1\}$ equipped with the Poincaré metric $ds^2 = \frac{1}{4}(dx^2 + dy^2)/(1-(x^2+y^2))^2$ and regard the covering group Γ as a discrete group of isometries. Consider a finite symmetric generating set S. In particular, S is a set of side pairing transformations for a fundamental domain $R \subset \mathbb{D}^2$ whose boundary ∂R is a finite union of geodesic arcs. We first assume for convenience that R satisfies the *even corners* condition. More precisely, this condition means that if any side of R is extended to a complete geodesic in $\mathbb{D}^2$ then this geodesic is contained in $\cup_{g\in\Gamma} g(\partial R)$. The importance of this condition lies in the following. The lift of any geodesic γ on V to $\mathbb{D}^2$ cuts a sequence of sides $\ldots, s_{-1}, s_0, s_1, \ldots, s_k, \ldots$ of $\cup_{g\in\Gamma} g(\partial R)$. The associated sequence of labels $\ldots, g_{-1}, g_0, g_1, \ldots, g_k, \ldots$ (of side identifications, where we take the label on the far side of each s_i) is called the cutting sequence of γ. If the even corners condition is satisfied, we have that $g = g_0 \cdots g_{n-1}$ has word length equal to n (i.e, the cutting sequence gives an expression for g which is shortest with respect to the word metric) [5]. (In the exceptional case γ passes through a vertex of $\cup_{g\in\Gamma} g(\partial R)$ then one needs to make a slight technical modification, but this contribution is negligible for closed geodesics.) We can now outline the construction of an associated expanding Markov map $f : S^1 \to S^1$ on the boundary of $\mathbb{D}^2$ (see [31, p.134-138] for full details). Given $g \in S$, let $H(g)$ denote the half-space in $\mathbb{D}^2 \cup S^1$ (i.e., the closed unit disk) obtained by extending the corresponding side of R and the requirement that $H(g)$ does not contain R. For each non-empty intersection of pairs of such sets $H(g_i) \cap H(g_j)$, $g_i, g_j \in S$, one makes a consistent choice of $g_m \in \{g_i, g_j\}$ and defines $f|_{H(g_i)\cap H(g_j)} = g_m^{-1}$. On the remaining part of $H(g_i)$, $H(g_j)$, one defines $f = g_i^{-1}$, $f = g_j^{-1}$, respectively. We set $B(g_i) = \{x \in H(g_i) : f(x) = g_i^{-1}x\}$ and define $I(g_i) \subset S^1$ to be the intersection of the closure of $B(g_i)$ with S^1. A key property of $f : \mathbb{D}^2 \backslash R \to \mathbb{D}^2 \backslash R$ is that if $f(gR) = hR$ then $|h| = |g| - 1$ [31, Theorem 5.10]. It is useful to consider the alternative construction using the Cayley graph of (Γ, S) in

[29]. More precisely, using the relations in Γ (which correspond geometrically to the side pairings) Series presented an algorithm for (uniquely) writing each $g \in \Gamma - \{e\}$ of word length $|g| = n$, say, as a composition $g = g_{i_0} \dots g_{i_{n-1}}$ of elements from S [29, Theorem 3.5]. The key idea was to look at paths in the Cayley graph and specify how closed loops associated to the relations are negotiated. This presentation of elements of $\Gamma - \{e\}$ is called *admissible* [29, p.356]. In this construction, each interval $I(g_i)$ in the partition of the boundary is associated to a generator $g_i \in S$, and is the union of limit points of the sequences $g0 = g_i g_{i_2} \dots g_{i_n} 0$. Distinct intervals $I(g_i)$ can only intersect at end points [29, p.351] and one can define $f : \coprod_i I(g_i) \to \coprod_i I(g_i)$ by $f|I(g_i) = g_i^{-1}$ [29, p.351]. The collection of closed intervals (or, more accurately, arcs) $I(g)$, $g \in S$, arising the above constructions can be refined by intersections with their preimages to give a collection $\mathcal{J} = \{J_1, \dots, J_k\}$ of closed intervals with the crucial property that each image $f(J_i)$ is the union of intervals in $\mathcal{J}$ [31, p. 138]. In particular, this property makes the expanding map $f : \coprod J_i \to \coprod J_i$ on the disjoint union of closed intervals into a Markov mapping, and thus by a standard construction it is semi-conjugate to a subshift of finite type $\sigma : \Sigma \to \Sigma$ [31, Corollary 5.11]. More precisely, there exists a continuous map $\pi : \Sigma \to S^1$, which is bijective except possibly at a countable number of points, with the property that $f \circ \pi = \pi \circ \sigma$. The transition matrix A is given by the condition

$$A(i,j) = \begin{cases} 1 & \text{if } f^{-1}(\mathrm{int}(J_j)) \cap \mathrm{int}(J_i) \neq \varnothing \\ 0 & \text{otherwise.} \end{cases}$$

To code geodesics on V we need to consider the associated two-sided subshift of finite type. Let $\sigma : \Sigma^* \to \Sigma^*$ correspond to Σ, i.e.,

$$\Sigma^* = \{x = (x_n)_{n=-\infty}^{\infty} : A(x_n, x_{n+1}) = 1, \forall n \in \mathbb{Z}\}.$$

The sequence $(x_n)_{n=0}^{\infty} \in \Sigma$ codes a point $\pi(x) = \widetilde{\gamma}(+\infty) \in S^1$. A complementary map $\overline{f}$ on the boundary similarly allows $(x_n)_{n=-\infty}^{-1}$ to code a second point $\widetilde{\gamma}(-\infty)$ (cf. [31, §5.3.2]). Let $\widetilde{\gamma}$ be the corresponding geodesic on $\mathbb{D}^2$ with these endpoints at times $t = +\infty$ and $t = -\infty$, respectively. Finally, let γ be the projection onto a geodesic on V. Let Σ^{**} be the subset of Σ^* defined on page 617 of [30] (where it is called Σ) of uniquely coded geodesics. Note that Σ^{**} contains all the periodic points in Σ^*. Let $\mathcal{R}$ denote the set of geodesics on $\mathbb{H}^2$ which intersect the interior of R. Define $\tau : \mathcal{R} \to \mathcal{R}$ by defining $\tau(\gamma)$ to be the geodesic equivalent to γ which enters R at a point equivalent to the point where γ leaves R. One of the main results of [30] is that there are large subsets of Σ^{**} and $\mathcal{R}$ on which σ and τ are conjugate. We can identify $\mathcal{R}$ with Markov sections corresponding to inwardly directed tangent vectors based on the sides of $\mathcal{R}$ (and τ with a Poincaré map). Once this coding is established we can easily dispense with the even corners condition (cf. [28], or by using structural stability [11, Corollary 18.2.2]). In the particular case of a closed geodesic γ, we can associate a finite number of lifts to the universal cover $\widetilde{V}$ which intersect the interior of $\mathcal{R}$. Let $\widetilde{\gamma}$ be one of these lifts and let $\widetilde{\gamma}(+\infty) \in S^1$ again denote its endpoint at time $t = +\infty$. At the symbolic level, if $\widetilde{\gamma}$ corresponds to a closed geodesic then $\tau^n(\gamma) = \gamma$, for some $n \geq 1$, and it is coded by a periodic orbit in Σ . Provided $\widetilde{\gamma}(+\infty)$ does not lie in the intersection of two of the intervals in $\{J_i\}_{i=1}^k$, there is a unique sequence $x \in \Sigma$ such that $\pi(x) = \widetilde{\gamma}(+\infty)$. (There are at most $2k$ exceptional closed geodesics.) The shift $\sigma : \Sigma \to \Sigma$ corresponds to the action of f

on $\pi(x)$. Since γ is closed there exists $g \in \Gamma$ such that $g\widetilde{\gamma}(+\infty) = \widetilde{\gamma}(+\infty)$. The f-orbit of the endpoint $\widetilde{\gamma}(+\infty)$ gives a word $x_0, \ldots, x_{n-1}$ where $f^i(\widetilde{\gamma}(+\infty)) \in J_{x_i}$, for $i = 0, \ldots, n-1$, and we can write $g = g_{x_0} \cdots g_{x_{n-1}}$. Since f and σ are semi-conjugate, $x \in \Sigma$ is the associated σ-periodic point (of period n). Thus we have a correspondence between closed geodesics γ and periodic orbits for σ, with at most finitely many exceptions (cf. also [29, proof of Theorem 4.12]). Moreover, by construction $|g| = n$, and since (cyclically reduced) conjugate elements have the same word length [5, Theorem 2.12], we can write $|\gamma| = n$. We can take $r : \Sigma \to \mathbb{R}$ to be the map $r(x) := \log |f'(\pi(x))|$ (cf. [29, §5]). For a periodic point $\sigma^n x = x$ we have that $r^n(x) = \log |(f^n)'(\pi(x))|$, which is easily checked to be the expansion rate at $\widetilde{\gamma}(-\infty)$ of the element $g \in \Gamma$ with $g\widetilde{\gamma}(-\infty) = \widetilde{\gamma}(-\infty)$. In particular, in the constant curvature case this is exactly the length of the closed geodesic. In the case of variable negative curvature, every metric is conformally equivalent to a metric of constant curvature (cf. [10, p.139]). We can carry out the above construction for the constant curvature metric, and then the coding persists for the metric of variable curvature by structural stability [11, Corollary 18.2.2]. Furthermore, the word length of closed geodesics still corresponds to the period of periodic σ-orbits. However, the function r will be replaced by a Hölder continuous function arising from the standard (Hölder) reparametrization of the geodesic flow. $\square$

The finite number of possible exceptional geodesics which may be inaccurately counted clearly does not affect the asymptotics we consider later. Since the geodesic flow is mixing, the function r has the additional property that it is not cohomologous to a constant, i.e., it is not possible to find a continuous function $\psi : \Sigma \to \mathbb{R}$ and $c \in \mathbb{R}$ such that $r = \psi \circ \sigma - \psi + c$. Theorem 1 involves estimates on numbers of closed geodesics ordered by word length. In order to obtain these we need to consider periodic orbits for the subshift of finite type Σ. By the Perron-Frobenius Theorem, A has a unique simple positive maximal eigenvalue $\lambda > 1$ which is a algebraic integer. It is clear that $\#\{x \in \Sigma : \sigma^n x = x\} = \mathrm{Trace} A^n$ and from this it is easy to deduce that $\#\{x \in \Sigma : \sigma^n x = x\} = \lambda^n(1 + O(e^{-\epsilon n}))$, for some $\epsilon > 0$. By Lemma 1.1, the number of closed geodesics of word length n is equal, up to a bounded error, to the number of periodic orbits for σ of period n. In particular, we see that $\log \lambda$ is equal to the growth rate h_0 defined in (0.2). Moreover, as is well known, $h_0 = \log \lambda$ is equal to the topological entropy of $\sigma : \Sigma \to \Sigma$. Let $\mathcal{M}_\sigma$ denote the set of all σ-invariant probability measures on Σ. For $\mu \in \mathcal{M}_\sigma$, we let $h(\mu)$ denote the measure theoretic entropy of σ with respect to μ. By the variational principle, $h_0 = \sup \{h(\mu) : \mu \in \mathcal{M}_\sigma\}$ and there is a unique measure μ_0, called the measure of maximal entropy, for which $h(\mu_0) = h_0$. For the standard presentation of a surface of genus $\mathfrak{g}$ described in the introduction, it can be shown that $\log(4\mathfrak{g} - 2) \le h_0 \le \log(4\mathfrak{g} - 1)$. For a continuous function $g : \Sigma \to \mathbb{R}$, we define the pressure $P(g)$ by

$$P(g) = \sup \left\{ h(m) + \int g dm : m \in \mathcal{M}_\sigma \right\}.$$

If g is a Hölder continuous function then the supremum is attained at a unique measure μ_g, called the equilibrium state of g. We shall be particularly interested in the function $s \mapsto P(sr)$, $s \in \mathbb{R}$. This function is real analytic and has an analytic extension to a neighbourhood of the real line, furthermore, at $s = 0$, it takes the value h_0,

$$\frac{d}{ds}P(sr)\Big|_{s=0} = \int r d\mu_0 \quad \text{and} \quad \frac{d^2}{ds^2}P(sr)\Big|_{s=0} > 0$$

(where the last statement uses that r is not cohomologous to a constant) [25]. In order to study pairs of closed geodesics we shall consider a product space. Let $\overline{\Sigma} = \Sigma \times \Sigma$, and consider the product transformation $\overline{\sigma} = \sigma \times \sigma$ on $\overline{\Sigma}$ defined by $\overline{\sigma}(x, y) = (\sigma x, \sigma y)$. This is again a mixing subshift of finite type. The symbols for $\overline{\Sigma}$ are pairs (i, j), where i, j and the associated matrix $\overline{A}$ is given by $\overline{A}\left((i,j),(i',j')\right) = A(i,j)A(i',j')$. The topological entropy of $\overline{\sigma}$ is equal to $2h_0$. Given the Hölder continuous function $r : \Sigma \to \mathbb{R}$ we can associate a function $R : \overline{\Sigma} \to \mathbb{R}$ defined by $R(x, y) = r(x) - r(y)$. The following lemma illustrates the usefulness of introducing $\overline{\Sigma}$ to studying problems on pairs of closed geodesics.

Lemma 1.2. *Periodic points $\overline{\sigma}^n(x, y) = (x, y) \in \overline{\Sigma}$ project to periodic points $\sigma^n x = x, \sigma^n y = y \in \Sigma$ which correspond to closed geodesics γ and γ', say, with $|\gamma| = |\gamma'| = n$. Moreover, $R^n(x, y) = \lambda(\gamma) - \lambda(\gamma')$.*

Proof. This follows from the above definitions and Lemma 1.1. □

The following technical result will be prove important later.

Lemma 1.3. *R is not cohomologous to the sum of a continuous function taking values in a discrete subgroup of $\mathbb{R}$ and a constant, i.e., there are no continuous $\psi : \overline{\Sigma} \to \mathbb{R}$, $M : \overline{\Sigma} \to a\mathbb{Z}$ (with $a > 0$) and $c \in \mathbb{R}$ such that $R = \psi \circ \overline{\sigma} - \psi + M + c$.*

Proof. Assume for a contradiction that R is cohomologous to such a $M + c$, as above, then $R^n(x, y) - nc = r^n(x) - r^n(y) - nc = M^n(x, y)$ whenever $\sigma^n x = x$ and $\sigma^n y = y$. Suppose that γ_0 is a closed geodesic with $|\gamma_0| = 1$ and let $\sigma y = y$ be the associated fixed point. Then, for any closed geodesic γ with $|\gamma| = n$ and associated periodic point $\sigma^n x = x$, we have $l(\gamma) - n(l(\gamma_0) + c) = r^n(x) - nr(y) - nc = M^n(x, y)$. In other words, $r : \Sigma \to \mathbb{R}$ is cohomologous to $r(y) + c + M(\cdot, y)$, i.e., the sum of a constant and a function valued in $a\mathbb{Z}$. However, it is easy to see that this cannot be the case. More precisely, we can find sequences of closed geodesics γ_n, γ_n' with $|\gamma_n| = |\gamma_n'|$, but such that the lengths satisfy $l(\gamma_n) \neq l(\gamma_n')$ and $l(\gamma_n) - l(\gamma_n') \to 0$, as $n \to +\infty$. □

As for $\sigma : \Sigma \to \Sigma$, given a continuous function $G : \overline{\Sigma} \to \mathbb{R}$, we define the pressure $P(G)$ by

$$P(G) = \sup\left\{h(m) + \int G dm \,:\, m \in \mathcal{M}_{\overline{\sigma}}\right\}.$$

If G is a Hölder continuous function then the supremum is attained at a unique measure m_G, called the equilibrium state of G. In particular, m_0 is the measure of maximal entropy for $\overline{\sigma}$, i.e., $h(m_0) = 2h_0$. By uniqueness, it is easy to see that m_0 is equal to the product measure $\mu_0 \times \mu_0$. We shall need to consider the function $s \mapsto P(sR)$, $s \in \mathbb{R}$. This function is real analytic and has an analytic extension to a neighbourhood of the real line, furthermore, at $s = 0$, it takes the value $2h_0$,

$$\frac{d}{ds}P(sR)\Big|_{s=0} = \int R(x,y)dm_0(x,y) = \int r(x)d\mu_0(x) - \int r(y)d\mu_0(y) = 0 \quad (1.1)$$

and, since, by Lemma 1.3, R is not cohomologous to a constant,

$$\frac{d^2}{ds^2}P(sR)\Big|_{s=0} > 0. \tag{1.2}$$

In our subsequent analysis, we shall be particularly interested in $P(sr)$ and $P(sR)$ for small imaginary values of s. The following lemma relates $P(itR)$ and $P(\pm itr)$.

Lemma 1.4. *Suppose that $|t|$ is sufficiently small that $P(itR)$, $P(itr)$ and $P(-itr)$ are defined. Then $P(itR)$ is real valued and $e^{P(itR)} = e^{P(itr)+P(-itr)}$. Furthermore,*

$$\sum_{n,m=1}^{N} e^{nP(itr)}e^{mP(-itr)} = \frac{e^{(N+1)P(itR)}}{(e^{P(itr)}-1)(e^{P(-itr)}-1)}(1+O(\tau^N)) \tag{1.3}$$

for some $0 < \tau < 1$.

Proof. For the first part we observe by the variational principle (cf. [33]) that for $s \in \mathbb{R}$,

$$\begin{aligned} e^{P(sR)} &= \lim_{n\to+\infty}\left(\sum_{\overline{\sigma}^n(x,y)=(x,y)} e^{sR^n(x,y)}\right)^{1/n} \\ &= \lim_{n\to+\infty}\left(\sum_{\sigma^n x=x} e^{sr^n(x,y)}\right)^{1/n}\left(\sum_{\sigma^n y=y} e^{-sr^n(x,y)}\right)^{1/n} \\ &= e^{P(sr)+P(-sr)}. \end{aligned}$$

The identity then follows by the uniqueness of the analytic extension. Furthermore, since $R^n(x,y) = -R^n(y,x)$, we have that $\sum_{\overline{\sigma}^n(x,y)=(x,y)} e^{itR^n(x,y)}$ is real valued and thus so is $P(itR)$. For the second part, we can write

$$\begin{aligned} \sum_{n,m=1}^{N} e^{nP(itr)}e^{mP(-itr)} &= \left(\frac{e^{(N+1)P(itr)}-1}{e^{P(itr)}-1}\right)\left(\frac{e^{(N+1)P(-itr)}-1}{e^{P(-itr)}-1}\right) \\ &= \frac{e^{(N+1)[P(itr)+P(-itr)]}}{(e^{P(itr)}-1)(e^{P(-itr)}-1)}(1+O(\tau^N)) \\ &= \frac{e^{(N+1)P(itR)}}{(e^{P(itr)}-1)(e^{P(-itr)}-1)}(1+O(\tau^N)), \end{aligned}$$

for some $0 < \tau < 1$, as required $\square$

The next lemma describes the local behaviour of $P(itR)$.

Lemma 1.5. *The function $t \mapsto e^{P(itR)}$ has a Taylor expansion*

$$e^{P(itR)} = e^{2h_0}\left(1 - \frac{\sigma^2 t^2}{2} + O(|t|^3)\right),$$

where the order term is uniform on any bounded interval. Moreover, there exists a change of coordinates $v = v(t)$ such that for $t \in (-\epsilon, \epsilon)$, we can have $e^{P(itR)} = e^{2h_0}(1 - v^2)$.

Proof. It follows from equations (1.1) and (1.2) that the function $t \mapsto P(itR)$ has the following properties: $\frac{d}{dt}P(itR)\big|_{t=0} = 0$ and $\frac{d^2}{dt^2}P(itR)\Big|_{t=0} < 0$ [25]. This gives the required expansion in t, with $\sigma^2 = -\frac{d^2}{dt^2}P(itR)\Big|_{t=0}$. For the second statement we can use the Morse Lemma to make the suitable change of coordinates and the result follows (cf. [12]). (Note that $v'(0) = \sigma/\sqrt{2}$.) $\square$

2. A sum over periodic points

In this section, we shall analyse the sums

$$\sum_{n,m=1}^{N} \sum_{\substack{\sigma^n x = x \\ \sigma^m y = y}} \chi(r^n(x) - r^m(y)),$$

over pairs of periodic points, for appropriate test functions χ. We shall use Fourier analysis to relate these to a family of exponential sums of periodic points. We begin with two estimates.

Lemma 2.1. *Let $K \subset \mathbb{R}$ be a compact set. There exists $\epsilon > 0$, $0 < \theta < 1$ and $C_0 > 0$ such that:*

1. *for $t \in K - (-\epsilon, \epsilon)$, we can bound $\left|\sum_{\sigma^n x = x} e^{\pm itr^n(x)}\right| \leq C_0 e^{h_0 n}\theta^{n/2}$; and*
2. *for $t \in (-\epsilon, \epsilon)$ we can bound,*

$$\sum_{\sigma^n x = x} e^{\pm itr^n(x)} = e^{nP(\pm itr)} + O(e^{h_0 n}\theta^{n/2}).$$

Proof. These estimates can be derived from [20]. $\square$

Let $\chi : \mathbb{R} \to \mathbb{C}$ be an integrable function such that its Fourier transform $\widehat{\chi}$ is compactly supported and such that, for $|t| < \epsilon$, we have $\widehat{\chi}(t) = \widehat{\chi}(0) + O(|t|)$. In particular, we shall suppose that the support of $\widehat{\chi}$ is contained in $[-M, M]$, for some $M > 0$. Let us denote

$$\mathcal{S}_N(t) = \sum_{n,m=1}^{N} \sum_{\substack{\sigma^n x = x \\ \sigma^m y = y}} e^{it(r^n(x) - r^m(y))} \text{ and } \psi_N(\chi) = \sum_{n,m=1}^{N} \sum_{\substack{\sigma^n x = x \\ \sigma^m y = y}} \chi(r^n(x) - r^m(y)).$$

Then, by the Fourier inversion formula,

$$\begin{aligned}
\psi_N(\chi) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{S}_N(t)\widehat{\chi}(t)dt \\
&= \frac{1}{2\pi}\int_{|t|<\epsilon} \left(\sum_{n,m=1}^{N} e^{nP(itr)}e^{mP(-itr)}(1 + O(\theta^{n+m}))\right)\widehat{\chi}(t)dt \\
&\quad + \frac{1}{2\pi}\int_{\epsilon \leq |t| \leq M} \mathcal{S}_N(t)\widehat{\chi}(t)dt + O(e^{2h_0 N}\theta^N) \\
&= \frac{1}{2\pi}\int_{|t|<\epsilon} \left(\frac{e^{(N+1)P(itR)}}{(e^{P(itr)} - 1)(e^{P(-itr)} - 1)}\right)\widehat{\chi}(t)du + O(e^{2h_0 N}\max\{\theta^N, \tau^N\}),
\end{aligned} \tag{2.1}$$

where we have used Lemma 2.1 (i) and (1.3). Using Lemma 1.5 we can estimate

$$\begin{aligned}\psi_N(\chi) = \frac{\widehat{\chi}(0)\sqrt{2}}{2\pi\sigma}\frac{e^{2h_0(N+1)}}{(e^{h_0}-1)^2}\int_{-\epsilon}^{\epsilon}\left(1-v^2\right)^{N+1}(1+R(v))dv \\ + O(e^{2h_0N}\max\{\theta^N,\tau^N\}),\end{aligned} \tag{2.2}$$

where $R(v)$ is a smooth function with $R(0)=0$ and for the final line we have also used

$$\frac{1}{(e^{P(itr)}-1)(e^{P(-itr)}-1)} = \frac{1}{(e^{h_0}-1)^2} + O(|t|).$$

Moreover, the leading term in (2.2) may be estimated using

$$\begin{aligned}\int_{-\epsilon}^{\epsilon}(1-v^2)^{N+1}dv &= 2\int_0^{\epsilon}(1-v^2)^{N+1}dv \\ &= \int_0^{\epsilon^2}(1-w)^{N+1}w^{-1/2}dw \\ &= \frac{\Gamma(N+2)\sqrt{\pi}}{\Gamma(N+2+1/2)} + O((1-\epsilon^2)^N).\end{aligned}$$

cf. [32, p.236]. Thus, since $\widehat{\chi}(0) = \int \chi(x)dx$, we have

$$\psi_N(\chi) \sim \frac{\int\chi(x)dx}{(2\pi)^{1/2}\sigma(e^{h_0}-1)^2}\frac{e^{2h_0(N+1)}}{\sqrt{N}}, \text{ as } N\to+\infty. \tag{2.3}$$

Finally, in the next lemma we remove the hypothesis that $\widehat{\chi}$ is compactly supported.

Lemma 2.2. *Let $\chi:\mathbb{R}\to\mathbb{R}$ be a continuous non-negative function with compact support. Then*

$$\psi_N(\chi) \sim \frac{e^{2h_0}}{(2\pi)^{1/2}\sigma(e^{h_0}-1)^2}\int\chi(x)dx\frac{e^{2h_0N}}{\sqrt{N}}, \textit{ as } N\to+\infty.$$

Proof. By equation (2.3), the required convergence holds whenever $\chi:\mathbb{R}\to\mathbb{C}$ is such that $\widehat{\chi}$ compactly supported and satisfies $\widehat{\chi}(t) = \widehat{\chi}(0) + O(|t|)$. In particular, (2.3) holds for all functions of the form $\chi(x) = e^{ivx}\chi_0(x)$, where $v\in\mathbb{R}$ and χ_0 is the strictly positive function

$$\chi_0(x) = \frac{\sin^2 x}{x^2} + \frac{\sin^2(\sqrt{2}x)}{2x^2}.$$

The lemma now follows by a standard result (see, for example, Theorem 10.7 of [6]). □

3. Proof of Theorem 1

Given a continuous compactly supported non-negative function $\chi:\mathbb{R}\to\mathbb{R}$, we can define

$$\rho_N(\chi) := \sum_{\substack{(\gamma,\gamma') \\ |\gamma|,|\gamma'|\le N}} \chi(\lambda(\gamma)-\lambda(\gamma')).$$

Theorem 1 follows from the following result.

Proposition 3.1. *We have that*

$$\rho_N(\chi) \sim \frac{e^{2h_0}}{(2\pi)^{1/2}\sigma(e^{h_0}-1)^2}\int \chi(x)dx \frac{e^{2h_0N}}{N^{5/2}}, \text{ as } N \to +\infty.$$

Proof. Suppose that γ is a prime closed geodesic of word length $|\gamma| = n$. Then γ corresponds to a prime periodic orbit $\{x, \sigma x, \ldots, \sigma^{n-1}x\}$ and hence to n terms in the sum $\sum_{\sigma^n x = x}$. Taking into account non-prime orbits, we have that

$$\begin{aligned}\rho_N(\chi) &= \sum_{n,m=1}^{N} \sum_{|\gamma|=n} \sum_{|\gamma'|=m} \chi(\lambda(\gamma) - \lambda(\gamma')) \\ &= \Xi_N(\chi) + O\left(||\chi||_\infty (\log N)^2 e^{3h_0N/2}\right),\end{aligned}$$

where

$$\Xi_N(\chi) = \sum_{n,m=1}^{N} \frac{1}{nm} \sum_{\substack{\sigma^n x = x \\ \sigma^m y = y}} \chi\left(r^n(x) - r^m(y)\right).$$

Thus we need only prove the asymptotics for $\Xi_N(\chi)$. We immediately have the asymptotic lower bound

$$\Xi_N(\chi) \geq \frac{1}{N^2}\psi_N(\chi) \sim \frac{e^{2h_0}}{(2\pi)^{1/2}\sigma(e^{h_0}-1)^2}\int \chi(x)dx \frac{e^{2h_0N}}{N^{5/2}}, \text{ as } N \to +\infty.$$

To get an asymptotic upper bound we fix $\frac{1}{2} < \alpha < 1$ and then

$$\begin{aligned}\Xi_N(\chi) &= \sum_{n,m=[\alpha N]+1}^{N} \frac{1}{nm} \sum_{\substack{\sigma^n x = x \\ \sigma^m y = y}} \chi\left(r^n(x) - r^m(y)\right) + O(||\chi||_\infty N^2 e^{(1+\alpha)h_0N}) \\ &\leq \frac{1}{(\alpha N)^2} \sum_{n,m=[\alpha N]+1}^{N} \sum_{\substack{\sigma^n x = x \\ \sigma^m y = y}} \chi\left(r^n(x) - r^m(y)\right) + O(||\chi||_\infty N^2 e^{(1+\alpha)h_0N}) \\ &\sim \frac{1}{\alpha^2}\frac{e^{2h_0}}{(2\pi)^{1/2}\sigma(e^{h_0}-1)^2}\int \chi(x)dx \frac{e^{2h_0N}}{N^{5/2}} + O(||\chi||_\infty N^2 e^{(1+\alpha)h_0N}).\end{aligned}$$

However, since we can choose α arbitrarily close to 1 the result follows. □

Proof of Theorem 1. This follows from Proposition 3.1 and a standard L^1 approximation of the characteristic function of $[a, b]$ from above and below by smooth functions. The interpretation in terms of closed geodesics comes from Lemma 1.1 □

It is easy to see that the same methods give a similar comparison for orbits of precisely the same word length. In particular, we can show the following.

Proposition 3.2. *We have that*

$$\#\left\{(\gamma, \gamma') \,:\, |\gamma| = |\gamma'| = n,\ a \leq \lambda(\gamma) - \lambda(\gamma') \leq b\right\} \sim \frac{(b-a)}{(2\pi)^{1/2}\sigma}\frac{e^{2h_0 n}}{n^{5/2}},$$

as $n \to +\infty$.

4. Some estimates

Before giving the proof of Theorem 2, we need to prove some preliminary asymptotic estimates. Their importance will become apparent in the next section. We first define, for some small $\epsilon > 0$,

$$A_1(N,z) = \left| \int_{-\epsilon\sigma\sqrt{N}}^{\epsilon\sigma\sqrt{N}} e^{izt/\sigma\sqrt{N}} \left\{ e^{-2h_0 N} \mathcal{S}_N\left(\frac{t}{\sigma\sqrt{N}}\right) \widehat{\chi}\left(\frac{1}{\kappa_N}\frac{t}{\sigma\sqrt{N}}\right) - \frac{1}{\sqrt{2\pi}} \frac{\left(\int \chi(x)dx\right) e^{2h_0}}{(e^{h_0}-1)^2} e^{-t^2/2} \right\} dt \right|,$$

where $\mathcal{S}_N$ is as defined in section 2.

Lemma 4.1. $\sup_{z\in\mathbb{R}} A_1(N,z) \to 0$ *as* $N \to +\infty$

Proof. Using the proof of Lemma 1.4.

$$A_1(N,z) = \left| \int_{-\epsilon\sigma\sqrt{N}}^{\epsilon\sigma\sqrt{N}} e^{izt/\sigma\sqrt{N}} \left\{ e^{-2h_0 N} \left(\frac{e^{(N+1)P(itR/\sigma\sqrt{N}}(1+O(\tau^N))}{(e^{P(itr/\sigma\sqrt{N})}-1)(e^{P(-itr/\sigma\sqrt{N})}-1)} \widehat{\chi}\left(\frac{1}{\kappa_N}\frac{t}{\sigma\sqrt{N}}\right)\right) - \frac{\left(\int \chi(x)dx\right) e^{2h_0}}{(e^{h_0}-1)^2} e^{-t^2/2} \right\} dt \right|.$$

On the domain of integration, we see that as $N \to +\infty$:

(i) $\widehat{\chi}\left(\frac{1}{\kappa_N}\frac{t}{\sigma\sqrt{N}}\right)$ converges to $\widehat{\chi}(0) = \int \chi(x)dx$;
(ii) $e^{P(itR/\sigma\sqrt{N})}(e^{P(itr/\sigma\sqrt{N})}-1)^{-1}(e^{P(-itr/\sigma\sqrt{N})}-1)^{-1}$ converges to $e^{2h_0}(e^{h_0}-1)^{-2}$; and
(iii) $e^{N(P(itR/\sigma\sqrt{N})-2h_0)}$ converges to $e^{-t^2/2}$,

Furthermore, we have the bounds $e^{N(P(itR/\sigma\sqrt{N})-2h_0)} \leq e^{-t^2/4}$ and

$$|e^{N(P(itR/\sigma\sqrt{N})-2h_0)} - e^{-t^2/2}| \leq 2e^{-t^2/4}.$$

Applying the Dominated Convergence Theorem, gives $\lim_{N\to+\infty} \sup_{z\in\mathbb{R}} A_1(N,z) = 0$, as required. □

Let us now define

$$A_2(N,z) = \left| \int_{|t|\geq\epsilon\sigma\sqrt{N}} e^{izt/\sigma\sqrt{N}} \left\{ e^{-2h_0 N} \left(\mathcal{S}_N\left(\frac{t}{\sigma\sqrt{N}}\right) \widehat{\chi}\left(\frac{1}{\kappa_N}\frac{t}{\sigma\sqrt{N}}\right)\right)\right\} dt \right|.$$

To bound $A_2(N,z)$ we shall need certain estimates on transfer operators. We begin by recalling the definitions. Given $\alpha > 0$, we let $C^\alpha(\Sigma)$ be the Banach space of Hölder continuous functions $f : \Sigma \to \mathbb{R}$ with norm $||f|| = |f|_\alpha + ||f||_\infty$, where

$$|f|_\alpha = \sup\left\{ \frac{|f(x)-f(y)|}{d(x,y)^\alpha} : x, y \in \Sigma \right\}$$

and $||f||_\infty$ is the supremum norm. Let $\mathcal{L}_{itr} : C^\alpha(\Sigma) \to C^\alpha(\Sigma)$ be the transfer operator defined by

$$\mathcal{L}_{itr} w(x) = \sum_{\sigma x' = x} e^{itr(x')} w(x').$$

We can similarly denote by $C^\alpha(\overline{\Sigma})$ the space of Hölder continuous functions on $\overline{\Sigma}$ and let $\mathcal{L}_{itR} : C^\alpha(\overline{\Sigma}) \to C^\alpha(\overline{\Sigma})$ be the transfer operator defined by

$$\mathcal{L}_{itR} w(x,y) = \sum_{\overline{\sigma}(x',y')=(x,y)} e^{itR(x',y')} w(x',y').$$

The following lemma gives the connection between the transfer operators and the sum over periodic orbits.

Lemma 4.2. *There exists* $0 < \theta < 1$ *such that for any* $x_0, y_0 \in \Sigma$

$$\sum_{\overline{\sigma}^n(x,y)=(x,y)} e^{itR^n(x,y)} = (\mathcal{L}_{itR}^n 1)(x_0,y_0)\left(1 + O(\max\{1,|t|\}n\theta^n)\right).$$

Proof. This result appears in [26]. $\square$

The next result is central to our analysis and is a reinterpretation of a result of Dolgopyat [8].

Lemma 4.3. *There exists* $C > 0$, $0 < \theta < 1$ *and* $\alpha > 0$ *such that for* $|t| > \epsilon$ *and* $n \geq 1$, *we can bound*

$$||\mathcal{L}_{itR}^n 1||_\infty \leq C e^{2h_0 n} \min\{\theta^n |t|^\alpha, 1\}.$$

Proof. Observe from the definitions that we can immediately write

$$\begin{aligned}\mathcal{L}_{itR}^n 1(x_0,y_0) &= \sum_{\substack{\sigma^n x = x_0 \\ \sigma^n y = y_0}} e^{itR^n(x,y)} = \sum_{\sigma^n x = x_0} e^{itr^n(x)} \sum_{\sigma^n y = y_0} e^{-itr^n(y)} \\ &= \mathcal{L}_{itr}^n 1(x_0) \mathcal{L}_{-itr}^n 1(y_0).\end{aligned}$$

However, Dolgopyat [8] showed that there exists $C > 0$ and $0 < \theta < 1$ such that for $|t| > \epsilon$ and $p[\log|t|] \leq n \leq (p+1)[\log|t|]$, where $p \geq 1$,

$$||\mathcal{L}_{itr}^n 1||_\infty, ||\mathcal{L}_{-itr}^n 1||_\infty \leq C^{1/2} e^{h_0 n} \theta^{p[\log|t|]/2}, \text{ for } n \geq 1.$$

In particular, there exists $C > 0$ and $0 < \theta < 1$ such that for $|t| > \epsilon$ and $p[\log|t|] \leq n \leq (p+1)[\log|t|]$, where $p \geq 1$, we can bound

$$||\mathcal{L}_{itR}^n 1||_\infty \leq C e^{2h_0 n} \theta^{p[\log|t|]} \leq C e^{2h_0 n} \theta^n \theta^{-\log|t|}.$$

In particular, the lemma follows with $\alpha = |\log \theta|$. $\square$

The following estimate is useful.

Lemma 4.4. *If* χ *is* C^k *and compactly supported then* $\widehat{\chi}(u) = O(|u|^{-k})$.

Proof. This standard result follows with integration by parts on the definition of the Fourier transform. $\square$

We are now in a position to bound $A_2(N,z)$:

Lemma 4.5. $\sup_{z\in\mathbb{R}} A_2(N,z) \to 0$ *as* $N \to +\infty$.

Proof. Using Lemma 4.2 and 4.3, there exists $C > 0$ such that we can bound

$$A_2(N,z) \le C\theta^N \int_{|t|\ge\epsilon\sigma\sqrt{N}} \left|\widehat{\chi}\left(\frac{1}{\kappa_N}\frac{t}{\sigma\sqrt{N}}\right)\right| \left(\frac{|t|}{\sigma\sqrt{N}}\right)^{\alpha} dt.$$

In particular, for $\beta > 0$ we can bound

$$A_2(N,z) \le \frac{C\theta^N ||\widehat{\chi}||_\infty}{(\sigma\sqrt{N})^\alpha} \int_{\epsilon\sigma\sqrt{N}}^{e^{\beta N}} |t|^\alpha dt + \int_{e^{\beta N}}^{\infty} \left|\widehat{\chi}\left(\frac{1}{\kappa_N}\frac{t}{\sigma\sqrt{N}}\right)\right| \left(\frac{|t|}{\sigma\sqrt{N}}\right)^{\alpha} dt. \quad (4.1)$$

The first term in (4.1) is of order $O(\theta^N e^{\beta N(\alpha+1)})$. This tends to zero (uniformly in z) as $N \to +\infty$ provided we choose $\beta > 0$ sufficiently small. For the second term we can use Lemma 4.4 to bound the integral

$$\begin{aligned}\int_{e^{\beta N}}^{\infty} \left|\widehat{\chi}\left(\frac{1}{\kappa_N}\frac{t}{\sigma\sqrt{N}}\right)\right| \left(\frac{|t|}{\sigma\sqrt{N}}\right)^{\alpha} dt &= O\left(\kappa_N^k N^{(k-\alpha)/2} \int_{e^{\beta N}}^{\infty} \frac{1}{t^{k-\alpha}} dt\right) \\ &= O\left(\frac{\kappa_N^k N^{(k-\alpha)/2}}{e^{(k-1-\alpha)\beta N}}\right)\end{aligned}$$

which tends to zero as $N \to +\infty$, provided we assume $k > 1 + \alpha$. □

Finally, let us define

$$A_3(N,z) = \left| \int_{|t|\ge\epsilon\sigma\sqrt{N}} e^{izt/\sigma\sqrt{N}} \left\{ \frac{\left(\int \chi(x)dx\right) e^{2h_0}}{(e^{h_0}-1)^2} e^{-t^2/2} \right\} dt \right|.$$

Lemma 4.6. $\sup_{z\in\mathbb{R}} A_3(N,z) \to 0$ *as* $N \to +\infty$.

Proof. This follows easily from the inequality

$$A_3(N,z) \le \frac{2\left(\int \chi(x)dx\right) e^{2h_0}}{(e^{h_0}-1)^2} \int_{\epsilon\sigma\sqrt{N}}^{\infty} e^{-t^2/2} dt,$$

which is independent of z and tends to zero as $N \to +\infty$. □

5. Proof of Theorem 2

In this section $\chi : \mathbb{R} \to \mathbb{R}$ will denote a smooth integrable non-negative function. (Ultimately, χ will be used to approximate the indicator function of the interval $[a,b]$.) In order to obtain results for the shrinking intervals $[z+\epsilon_N a, z+\epsilon_N b]$, we shall consider a sequence of rescaled functions $\chi_N^{(z)}$, defined by $\chi_N^{(z)}(x) = \chi(\epsilon_N^{-1}(x-z))$. We can write

$$\widehat{\chi}_N^{(z)}(u) = e^{izu} \frac{1}{\kappa_N} \widehat{\chi}\left(\frac{u}{\kappa_N}\right). \quad (5.1)$$

We need to consider

$$A(N,z) := \left| \frac{\sigma\sqrt{N}}{\epsilon_N e^{2h_0 N}} \psi_N(\chi_N^{(z)}) - \frac{e^{2h_0}\int \chi(x)dx}{(e^{h_0}-1)^2} e^{-z^2/2\sigma^2 N} \right|,$$

where $\psi_N(\chi_N^{(z)}) = \sum_{n,m=1}^N \sum_{\substack{\sigma^n x = x \\ \sigma^m y = y}} \chi_N^{(z)}(r^n(x) - r^m(y))$.

Proposition 5.1.

$$\lim_{N\to+\infty} \sup_{z\in\mathbb{R}} A(N,z) = 0.$$

We begin with the following observation.

Lemma 5.1. *We can write*

$$e^{-z^2/2\sigma^2 N} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iuz/\sigma\sqrt{N}} e^{-u^2/2} du.$$

Using Fourier inversion and (5.1) we can write

$$\begin{aligned}\frac{\sigma\sqrt{N}\kappa_n}{e^{2h_0 N}}\psi_N(\chi_N^{(z)}) &= \frac{1}{2\pi}\frac{\sigma\sqrt{N}\kappa_n}{e^{2h_0 N}}\int_{-\infty}^{\infty} \mathcal{S}_N(u)\widehat{\chi}_N^{(z)}(u)du \\ &= \frac{1}{2\pi}\frac{\sigma\sqrt{N}}{e^{2h_0 N}}\int_{-\infty}^{\infty} \mathcal{S}_N(u)e^{izu}\widehat{\chi}\left(\frac{u}{\kappa_N}\right) du\end{aligned} \tag{5.2}$$

We can substitute $t = u\sigma\sqrt{N}$ and then (5.2) becomes:

$$\begin{aligned}&\frac{\sigma\sqrt{N}\kappa_n}{e^{2h_0 N}}\psi_N(\chi_N^{(z)}) \\ &= \frac{e^{-2h_0 N}}{2\pi}\int_{-\infty}^{\infty}\left(\mathcal{S}_N\left(\frac{t}{\sigma\sqrt{N}}\right)e^{izt/\sigma\sqrt{N}}\widehat{\chi}\left(\frac{1}{\kappa_N}\frac{t}{\sigma\sqrt{N}}\right)\right) dt.\end{aligned} \tag{5.3}$$

We can write

$$\begin{aligned}2\pi A(z,N) = \Bigg|\int_{-\infty}^{\infty} e^{izt/\sigma\sqrt{N}}\Bigg\{&e^{-2h_0 N}\mathcal{S}_N\left(\frac{t}{\sigma\sqrt{N}}\right)\widehat{\chi}\left(\frac{1}{\kappa_N}\frac{t}{\sigma\sqrt{N}}\right) \\ &-\frac{\left(\int\chi(x)dx\right)e^{2h_0}}{(e^{h_0}-1)^2}e^{-t^2/2}\Bigg\} dt\Bigg|\end{aligned}$$

In particular, we can bound

$$2\pi A(z,N) \le A_1(N,z) + A_2(N,z) + A_3(N,z)$$

and thus we can complete the proof of Proposition 5.1 with the bounds in Lemma 4.1, 4.2 and 4.6. In order to prove Theorem 2, we first need to replace $\psi_N(\chi_N^{(z)})$ with a sum over (prime) closed geodesics. More precisely, we define

$$\rho_N(\chi_N^{(z)}) = \sum_{n,m=1}^{N}\sum_{|\gamma|=n}\sum_{|\gamma'|=m}\chi_N^{(z)}(\lambda(\gamma)-\lambda(\gamma')).$$

As in section 3, we have the estimate

$$\rho_N(\chi_N^{(z)}) = \Xi_N(\chi_N^{(z)}) + O\left(||\chi||_\infty(\log N)^2 e^{3h_0 N/2}\right),$$

where

$$\Xi_N(\chi_N^{(z)}) = \sum_{n,m=1}^{N} \frac{1}{nm} \sum_{\substack{\sigma^n x = x \\ \sigma^m y = y}} \chi_N^{(z)}(r^n(x) - r^m(y))$$

and the implied constant in the big-O term is independent of z. Clearly we have that

$$\frac{N^{5/2}\kappa_N}{e^{2h_0 N}} \Xi_N(\chi_N^{(z)}) \geq \frac{N^{1/2}\kappa_N}{e^{2h_0 N}} \psi_N(\chi_N^{(z)}).$$

On the other hand, for any $0 < \alpha < 1$,

$$\begin{aligned}\frac{N^{5/2}\kappa_N}{e^{2h_0 N}} \Xi_N(\chi_N^{(z)}) &= \frac{N^{5/2}\kappa_N}{e^{2h_0 N}} \sum_{n,m=[\alpha N]+1}^{N} \frac{1}{nm} \sum_{\substack{\sigma^n x = x \\ \sigma^m y = y}} \chi_N^{(z)}(r^n(x) - r^m(y)) \\ &\qquad + O\left(||\chi||_\infty (\log N)^2 N^{5/2} \kappa_N e^{(\alpha-1)h_0 N}\right) \\ &\leq \frac{N^{1/2}\kappa_N}{\alpha e^{2h_0 N}} \psi_N(\chi_N^{(z)}) + O\left(||\chi||_\infty (\log N)^2 N^{5/2} \kappa_N e^{(\alpha-1)h_0 N}\right).\end{aligned}$$

Also, by Proposition 5.1 we have that

$$\lim_{N\to+\infty} \sup_{z\in\mathbb{R}} \frac{N^{1/2}\kappa_N}{e^{2h_0 N}} \psi_N(\chi_N^{(z)}) = \frac{e^{2h_0} \int \chi(x)dx}{(2\pi)^{1/2}\sigma(e^{h_0}-1)^2}.$$

Thus, we see that

$$\begin{aligned}0 &\leq \limsup_{N\to+\infty} \sup_{z\in\mathbb{R}} \left(\frac{N^{5/2}\kappa_N}{e^{2h_0 n}} \Xi_N(\chi_N^{(z)}) - \frac{N^{1/2}\kappa_N}{e^{2h_0 n}} \psi_N(\chi_N^{(z)}) \right) \\ &\leq \left(\frac{1}{\alpha} - 1\right) \frac{e^{2h_0} \int \chi(x)dx}{(2\pi)^{1/2}\sigma(e^{h_0}-1)^2}.\end{aligned}$$

Since we may take α arbitrarily close to 1, the above limit exists and is equal to zero. We have shown the following.

Proposition 5.2.

$$\lim_{N\to+\infty} \sup_{z\in\mathbb{R}} \left| \frac{N^{5/2}}{\epsilon_N e^{2h_0 N}} \rho_N(\chi_N^{(z)}) - \frac{e^{2h_0} \int \chi dx}{(2\pi)^{1/2}\sigma(e^{h_0}-1)^2} e^{-z^2/2\sigma^2 N} \right| = 0.$$

The final step in the proof of Theorem 2 is to replace the smooth function χ by the indicator function $\chi_{[a,b]}$ of the interval $[a,b]$. Given $\epsilon > 0$ we can choose compactly supported smooth functions $\chi_- \leq \chi_{[a,b]} \leq \chi_+$ such that

$$\int \chi_{[a,b]}(x)dx - \epsilon \leq \int \chi_-(x)dx \leq \int \chi_+(x)dx \leq \int \chi_{[a,b]}(x)dx + \epsilon.$$

¿From this we can deduce that

$$\begin{aligned}-\frac{e^{2h_0}}{\sigma(e^{h_0}-1)^2}\epsilon &\leq \liminf_{N\to+\infty} \sup_{z\in\mathbb{R}} \left(\frac{\sigma N^{5/2}}{\epsilon_N e^{2h_0 N}} \pi(N, I_N(z)) - \frac{e^{2h_0} \int \chi_{[a,b]} dx}{(e^{h_0}-1)^2} e^{-\sigma^2 z^2/2N} \right) \\ &\leq \limsup_{N\to+\infty} \sup_{z\in\mathbb{R}} \left(\frac{\sigma N^{5/2}}{\epsilon_N e^{2h_0 N}} \pi(N, I_N(z)) - \frac{e^{2h_0} \int \chi_{[a,b]} dx}{(e^{h_0}-1)^2} e^{-\sigma^2 z^2/2N} \right) \\ &\leq \frac{e^{2h_0}}{\sigma(e^{h_0}-1)^2}\epsilon.\end{aligned}$$

Since $\epsilon > 0$ can be chosen arbitrarily small, Theorem 2 follows. We can similarly improve Proposition 3.2 to the following statement.

Proposition 5.3. *If we write*

$$\omega(n, I_n(z)) = \#\left\{(\gamma, \gamma') : |\gamma| = |\gamma'| = n,\ z + \epsilon_n a \le \lambda(\gamma) - \lambda(\gamma') \le z + \epsilon_n b\right\}$$

then

$$\lim_{n\to+\infty} \sup_{z\in\mathbb{R}} \left| \frac{n^{5/2}}{\epsilon_n e^{2h_0 n}} \omega(n, I_n(z)) - \frac{1}{(2\pi)^{1/2}\sigma}(b-a)e^{-z^2/2\sigma^2 n} \right| = 0.$$

A more careful consideration of the proof of Proposition 5.3 shows that it remains valid under a slightly weaker hypothesis than Theorem 2, namely that $\epsilon_n^{-1} = O(e^{\eta n})$, for sufficiently small $\eta > 0$. Unfortunately, there is no effective estimate on the value of $\eta > 0$.

Remark on multiplicities. It is known by a result of Randol (based on work of Horowitz) that any such surface of constant negative curvature must have unbounded multiplicities in its length spectra [22]. Moreover, Buser observed that for each such surface there exists $C > 0$ and lengths T_n for which the multiplicity is at least CT_n^{β}, where $\beta = \frac{\log 2}{\log 5}$ [7]. If we consider arithmetic surfaces then the set of traces $\{\mathrm{tr}(g) : g \in \Gamma\}$ takes values in an appropriate algebraic number field. Each g corresponds to a closed geodesic γ of length $\lambda(\gamma) = \cosh^{-1}(\mathrm{tr}(g))$. Unfortunately, determining the multiplicity of geodesics with a given length is a very difficult problem. For example, with $\Gamma = PSL(2, \mathbb{Z})$ these multiplicities are class numbers [27]. However, the distributions of such numbers are irregular, their average over intervals $[l, l + \Delta l]$ being at least $ce^{l/2}/l$ [16]. Moreover, Luo and Sarnak established the Bounded Cluster property, i.e., the number of lengths of closed geodesics between T and $T + e^{-T}$ is uniformly bounded [14]. It is an easy exercise using Theorem 2 to give (for any surface) an exponentially growing lower bound for the number of distinct pairs of closed orbits of maximal length T with lengths differing by a subexponentially shrinking sequence ϵ_n.

6. Orderings by lengths

It might seem natural to consider counting problems for pairs of closed geodesics ordered solely by their (geometric) lengths, rather than in terms of word length. However, as well shall see in this section, it only seems possible to obtain relatively weak asymptotic results. Let $\chi : \mathbb{R} \to \mathbb{R}$ be a smooth compactly supported non-negative function and let us denote

$$\xi(T) = \sum_{\lambda(\gamma)+\lambda(\gamma')\le T} \chi(\lambda(\gamma) - \lambda(\gamma')) \left(\lambda(\gamma)\lambda(\gamma')e^{-h(\lambda(\gamma)+\lambda(\gamma'))}\right).$$

The following weak asymptotic estimate for $\xi(T)$ implies the result (0.3) stated in the introduction.

Proposition 6.1. *There exists $A > 0$ such that $\xi(T) = AT$, as $T \to +\infty$.*

The proof is based on a sequence of lemmas. Let us first define

$$\xi^*(T) = \sum_{\lambda(\gamma)+\lambda(\gamma')\le T} \chi * k(\lambda(\gamma) - \lambda(\gamma')) \left(\lambda(\gamma)\lambda(\gamma')e^{-h(\lambda(\gamma)+\lambda(\gamma'))}\right),$$

where $k : \mathbb{R} \to \mathbb{R}$ is a smooth function whose Fourier transform $\widehat{k}$ is compactly supported in an interval $[-a, a]$. This has the technical advantage of making more tractable the associated Dirichlet series defined, for $Re(s)$ sufficiently large, by

$$\eta(s) = \sum_{\gamma,\gamma'} \lambda(\gamma)\lambda(\gamma')\chi * k(\lambda(\gamma) - \lambda(\gamma'))e^{-s(\lambda(\gamma)+\lambda(\gamma'))}. \tag{6.1}$$

It is also useful for us to also introduce the Dirichlet series $\kappa(s) = \sum_\gamma \lambda(\gamma)e^{-s\lambda(\gamma)}$. This is useful for formulating the following lemma.

Lemma 6.2. *$\kappa(s)$ converges to an analytic function for $Re(s) > h$. In a neighbourhood of $s = h$ we may write*

$$\kappa(s) = \frac{1}{s-h} + F(s),$$

where $F(s)$ is analytic in a neighbourhood of $Re(s) \geq h$.

Proof. The result follows easily from results on the domain of the zeta function for a weak-mixing hyperbolic flow, see for example [19]. □

The description of the domain of $\kappa(s)$, in the above lemma, can now be used in the proof of the following lemma, on the domain of $\eta(s)$.

Lemma 6.3.

(1) *The abscissa of convergence of $\eta(s)$ is the topological entropy $h > 0$;*
(2) *There exists $D > 0$, such that, for real values of s, $\eta(s) \sim D(s-h)^{-1}$ as $s \searrow h$.*

Proof. Using the Fourier inversion theorem we can write $\eta(s)$ as

$$\int_{-a}^{a} \kappa(s-iu)\kappa(s+iu)\widehat{\chi}(u)\widehat{k}(u)du = \int_{-a}^{a} \frac{\widehat{\chi}(u)\widehat{k}(u)}{(s-h)^2+u^2}du$$
$$+ \int_{-a}^{a} \frac{F(s+iu)\widehat{\chi}(u)\widehat{k}(u)}{(s-h)-iu}du + \int_{-a}^{a} \frac{F(s-iu)\widehat{\chi}(u)\widehat{k}(u)}{(s-h)+iu}du + G(s)$$

where $F(s)$ and $G(s)$ are analytic functions on the half plane $Re(s) > h$. For part (1), we observe that this expression is analytic on $Re(s) > h$. For part (2) a simple change of variables shows that there exists $D > 0$ such that, for real values of s,

$$\int_{-a}^{a} \frac{\widehat{\chi}(u)\widehat{k}(u)}{(s-h)^2+u^2}du \sim \frac{D}{s-h}, \text{ as } s \searrow h,$$

while the remaining terms make a contribution which is at worst $O(\log(s-h))$. □

The final ingredient in the proof of Proposition 6.1 is the following result, due to Freud, which is a variant on the famous Hardy-Littlewood Tauberian Theorem [21, p.30].

Lemma 6.4. *Let $\alpha(T)$ be a monotone increasing. Assume that, for real values of s, the function $f(s) = \int_0^\infty e^{-st} d\alpha(t)$ satisfies*

$$f(s) = \frac{A}{s}\left(1 + O\left(s^\epsilon\right)\right), \text{ as } s \searrow 0,$$

where $\epsilon > 0$. Then $\alpha(T) = Ax\left(1 + O\left((\log T)^{-1}\right)\right)$, as $T \to +\infty$.

We are now in a position to complete the proof of Proposition 6.1. Since $\eta(s + h) = \int_0^\infty e^{-st} d\xi^*(t)$, we can use Lemma 6.3 and Lemma 6.4 to deduce that

$$\xi^*(T) = Ax\left(1 + O\left(\frac{1}{\log T}\right)\right), \text{ as } T \to +\infty.$$

Finally, we use an approximation argument to remove the dependence on k and to give the asymptotic formula in Proposition 6.1.

Remark. At first, it might seem more natural to modify $\xi(T)$ by removing the exponential term and consider instead counting functions of the form

$$\sum_{\lambda(\gamma)+\lambda(\gamma')\le T} \chi(\lambda(\gamma) - \lambda(\gamma'))\lambda(\gamma)\lambda(\gamma'),$$

say. However, it does not appear to be possible to obtain an asymptotic expression for this function, as this would require $\eta(s)$ to have an extension to neighbourhoods of points $h + it$, $t \neq 0$, and this is not the case.

7. Related results

7.1 Higher dimensional correlation functions. Following Rudnick and Sarnak [23], we can consider more general k-correlation functions

$$\pi_k\left(n, \prod_{i=1}^{k-1}[a_i, b_i]\right) := \sum_{|\gamma_1|,\dots,|\gamma_k|\le n} \prod_{i=1}^{k-1} \chi_i\left(\lambda(\gamma_{i+1}) - \lambda(\gamma_1)\right),$$

where $\chi_1, \dots, \chi_{k-1} : \mathbb{R} \to \mathbb{R}$ are the indicator functions of $[a_1, b_1], \dots, [a_{k-1}, b_{k-1}]$, respectively. Here the symbolic dynamics should be that for $\overline{\Sigma}_k = \underbrace{\Sigma \times \cdots \times \Sigma}_{k}$ and the function $R : \overline{\Sigma}_k \to \mathbb{R}^{k-1}$ given by $R(x_1, \dots, x_k) = (r(x_2) - r(x_1), \dots, r(x_k) - r(x_1))$. Modifying the proof of Theorem 1 shows that there exists $\sigma_k > 0$ such that

$$\pi_k\left(n, \prod_{i=1}^{k-1}[a_i, b_i]\right) \sim \frac{1}{(2\pi)^{(k-1)/2}\sigma_k^{k-1}} \prod_{i=1}^{k-1}(b_i - a_i) \frac{e^{kh_0}}{(e^{h_0} - 1)^k} \frac{e^{kh_0 n}}{n^{k+\frac{1}{2}(k-1)}},$$

as $n \to +\infty$. Furthermore, Theorem 2 has a particular interesting generalization. For each $i = 1, \dots k-1$, we let $\epsilon_n^{(i)} > 0$, for $n \ge 1$, be sequences which tend to zero subexponentially. For $z = (z_1, \dots, z_{k-1}) \in \mathbb{R}^{k-1}$, we shall write $I_n^{(i)}(z) = [z_i + a_i\epsilon_n^{(i)}, z_i + b_i\epsilon_n^{(i)}]$, for $i = 1, \dots k-1$, and $\underline{I}_n(z) = \prod_{i=1}^{k-1} I_n^{(i)}(z)$. We shall also write $\underline{\epsilon}_n = \prod_{i=1}^{k-1} \epsilon_n^{(i)}$. The generalization of Theorem 2 is the following. (We write $\langle \cdot, \cdot \rangle$ for the usual Euclidean inner product on $\mathbb{R}^{k-1}$.)

Theorem 3. *Given sequences $\epsilon_n^{(i)} > 0$, $i = 1, \ldots, k-1$, which tend to zero at a subexponential rate, we have that*

$$\lim_{n\to+\infty} \sup_{z\in\mathbb{R}^{k-1}} \left| \frac{\sigma_k^{k-1} n^{k+(k-1)/2}}{\underline{\epsilon}_n e^{2h_0 N}} \pi(n, \underline{I}_n(z)) - C_k e^{-\langle z, A^{-1} z\rangle/2n} \right| = 0,$$

where the matrix $A = (a_{ij})$ is defined by $a_{ij} = \frac{\partial^2}{\partial s_i \partial s_j} P(\langle s, R\rangle)\Big|_{s=0}$, $\sigma_k = (\det A)^{\frac{1}{2k}}$ and

$$C_k = \frac{1}{(2\pi)^{(k-1)/2}} \prod_{i=1}^{k-1} (b_i - a_i) \frac{e^{kh_0}}{(e^{h_0} - 1)^k}.$$

In particular, this implies that, for any fixed $z \in \mathbb{R}^{k-1}$,

$$\pi(n, \underline{I}_n(z)) \sim \frac{C_k}{\sigma_k^{k-1}} \underline{\epsilon}_n \frac{e^{2h_0 n}}{n^{k+(k-1)/2}}, \text{ as } n \to +\infty.$$

The proof follows the same lines as that of Theorem 2.

7.2 Orbital counting functions. As we remarked in the introduction, closed geodesics on V correspond to conjugacy classes in the fundamental group $\pi_1(V)$. However, we may also consider elements of $\pi_1(V)$ itself. It is well-known that $\{g \in \pi_1(V) : |g| \le n\} \sim Ce^{h_0 n}$, for some $C > 0$. The fundamental group has a natural isometric action on the universal cover $\widetilde{V}$ and, if we fix a point $x_0 \in \widetilde{V}$, say, we may consider the set of displacements $\{d(x_0, gx_0) : g \in \pi_1(V)\}$. A natural analogue of the questions we have considered for closed geodesics is to study the asymptotics of the number of pairs $(g, g') \in \pi_1(V) \times \pi_1(V)$ satisfying $|g|, |g'| \le n$ such that $d(x_0, gx_0) - d(x_0, g'x_0)$ lies in a prescribed interval. For example, one can show that there exists $C' > 0$ such that, for $a < b$,

$$\{(g, g') : |g|, |g'| \le n,\ a < d(x_0, gx_0) - d(x_0, g'x_0) < b\} \sim C'(b-a)\frac{e^{2h_0 n}}{n^{1/2}},$$

as $n \to +\infty$. It is also possible to prove a stronger result analogous to Theorem 2.

References

1. R. Adler and L. Flatto, *Geodesic flows, interval maps, and symbolic dynamics*, Bull. Amer. Math. Soc. **25** (1991), 229-334.
2. R. Aurich, E. Bogomolny and F. Steiner, *Periodic orbits on the regular hyperbolic octagon*, Physica D **48** (1991), 91-101.
3. R. Aurich and F. Steiner, *On the periodic orbits of a strongly chaotic system*, Physica D **32** (1988), 451-460.
4. M. Berry, *Semiclassical theory of spectral rigidity*, Proc. Roy. Soc. London Ser. A **400** (1985), 229-251.
5. J. Birman and C. Series, *Dehn's algorithm revisited, with applications to simple curves on surfaces*, Combinatorial group theory and topology (Alta, Utah, 1984), Ann. of Math. Stud., 111, Princeton University Press, Princeton, NJ, 1987, pp. 451-478.
6. L. Brieman, *Probability*, Addison-Wesley, Reading, Mass., 1968.
7. P. Buser, *Geometry and spectra of compact Riemann surfaces*, Progress in Mathematics, 106, Birkhauser, Boston, 1992.
8. D. Dolgopyat, *On decay of correlations in Anosov flows*, Ann. of Math. **147** (1998), 357-390.

9. H. Huber, *Zur analytischen Theorie hyperbolischer Raumformen und Bewegungsgruppen. II*, Math. Ann. **142** (1960), 385-398.
10. A. Katok, *Four applications of conformal equivalence to geometry and dynamics*, Ergodic Theory Dynam. Systems **8*** (1988), 139-152.
11. A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems*, Encyclopedia of Mathematics and its Applications, 54, Cambridge University Press, Cambridge, 1995.
12. A. Katsuda and T. Sunada, *Closed orbits in homology classes*, Inst. Hautes Études Sci. Publ. Math. **71** (1990), 5-32.
13. J. Keating, *Periodic orbits, spectral statistics, and the Riemann zeros*, Supersymmetry and Trace Formulae: Chaos and Disorder, NATO ASI Series, Series B: Physics Vol. 370, Kluwer, New York, 1999, pp. 1-17.
14. W.Luo and P. Sarnak, *Number variance for arithmetic hyperbolic surfaces*, Comm. Math. Phys. **161** (1994), 419-432.
15. G. Margulis, *Certain applications of ergodic theory to the investigation of manifolds of negative curvature*, Funkcional. Anal. i Priložen. **3** (1969), 89-90.
16. J. Marklof, *On multiplicities in length spectra of arithmetic hyperbolic three-orbifolds*, Nonlinearity **9** (1996), 517-536.
17. J. Marklof, *Pair correlation densities of inhomogeneous quadratic forms*, Ann. of Math. **158** (2003), 419-471.
18. H. Montgomery, *The pair correlation of zeros of the zeta function*, Proc. Sympos. Pure Math. **24** (1973), 181-193.
19. W. Parry and M. Pollicott, *Zeta functions and the periodic orbit structure of hyperbolic dynamics*, Asterisque **187-188** (1990), 1-268.
20. M. Pollicott and Sharp, *Rates of recurrence for $\mathbb{Z}^q$ and $\mathbb{R}^q$ extensions of subshifts of finite type*, J. London Math. Soc. **49** (1994), 401-416.
21. A. Postnikov, *Introduction to analytic number theory*, Translations of Mathematical Monographs, Vol. 68, Amer. Math. Soc., Providence, R.I., 1988.
22. B. Randol, *The length spectrum of a Riemann surface is always of unbounded multiplicity*, Proc. Amer. Math. Soc. **78** (1980), 455-456.
23. Z. Rudnick and P. Sarnak, *The n-level correlations of zeros of the zeta function*, C. R. Acad. Sci. Paris Sér. I Math. **319** (1994), 1027-1032.
24. Z. Rudnick and P. Sarnak, *The pair correlation function of fractional parts of polynomials*, Comm. Math. Phys. **194** (1998), 61-70.
25. D. Ruelle, *Thermodynamic formalism*, Addison-Wesley, New York, 1978.
26. D. Ruelle, *An extension of the theory of Fredholm determinants*, Inst. Hautes Études Sci. Publ. Math. **72** (1990), 175-193.
27. P. Sarnak, *Class numbers of indefinite binary quadratic forms*, J. Number Theory **15** (1982), 229-247.
28. C. Series, *Symbolic dynamics for geodesic flows*, Acta Math. **146** (1981), 103-128.
29. C. Series, *The infinite word problem and limit sets in Fuchsian groups*, Ergodic Theory Dynam. Systems **1** (1981), 337-360.
30. C. Series, *Geometrical Markov coding of geodesics on surfaces of constant negative curvature*, Ergodic Theory Dynam. Systems **6** (1986), 601-625.
31. C. Series, *Geometrical methods of symbolic coding*, Ergodic theory, symbolic dynamics, and hyperbolic spaces (Trieste, 1989) (T. Bedford, M. Keane and C. Series, eds.), Oxford University Press, Oxford, 1991, pp. 125-151.
32. G. Stephenson, *Mathematical methods for science students*, Longman, London, 1961.
33. P. Walters, *An introduction to ergodic theory*, Graduate Texts in Mathematics, 79, Springer-Verlag, New York-Berlin, 1982.

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