LARGE DEVIATIONS AND THE DISTRIBUTION OF PRE-IMAGES OF RATIONAL MAPS

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Abstract. In this article we prove a large deviation result for the pre-images of a point in the Julia set of a rational mapping of the Riemann sphere. As a corollary, we deduce a convergence result for certain weighted averages of orbital measures, generalizing a result of Lyubich.

0. Introduction

Let $\hat{\mathbb{C}}$ denote the Riemann sphere and let $T : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map of degree $d \geq 2$, say. Every point has d pre-images (counted according to their multiplicities). There is a well-known result of Lyubich which shows that for a point $x \in \mathbb{J}$ in the Julia set an evenly distributed weight on the set of d^n pre-images

$$S_n(x) = \{y \in \hat{\mathbb{C}} : T^n y = x\}$$

converges (in the weak* topology) to a measure μ_0 as $n \to +\infty$ [4], [8]. The measure μ_0 is precisely the unique measure of maximal entropy for the map T [2], [5].

Since $T : \mathbb{J} \to \mathbb{J}$ is a continuous map on a compact metric space we can define the pressure of a continuous function $f : \mathbb{J} \to \mathbb{R}$ by

$$P(f) = \sup \left\{ h(\nu) + \int f d\nu : \nu \text{ is a } T\text{-invariant probability} \right\},$$

where $h(\nu)$ denotes the entropy of T with respect to ν. An equilibrium state for f is a T-invariant probability μ realising this supremum.

Let $\mathcal{M}$ denote the set of all probability measures on $\mathbb{J}$. We shall show the following stronger "large deviation" result on the pre-images of a point $x \in \mathbb{J}$.

Theorem 1. *Let $f : \mathbb{J} \to \mathbb{R}$ be a Hölder continuous function such that $P(f) > \sup f$ and let μ be the unique equilibrium state for f. Let $x \in \mathbb{J}$. Then for any weak* open neighbourhood $\mathcal{U} \subset \mathcal{M}$ of μ we have that the weighted proportion of the measures*

$$\mu_{y,n} = \frac{1}{n} \left(\delta_y + \delta_{Ty} + \ldots + \delta_{T^{n-1}y} \right) \notin \mathcal{U}$$

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Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX *tends to zero exponentially fast, in the sense that there exists* $C > 0$ *and* $0 < \eta < 1$ *such that*

$$\frac{\sum_{\substack{T^n y = x \\ \mu_{y,n} \notin \mathcal{U}}} e^{f^n(y)}}{\sum_{T^n y = x} e^{f^n(y)}} \leq C\eta^n, \qquad \text{for } n \geq 0,$$

where we denote $f^n(y) = f(y) + f(Ty) + \ldots + f(T^{n-1}y)$.

Remark. The condition $P(f) > \sup f$ was first introduced by Urbanski in [9]. In the present article, as well as being required as a hypothesis for Lemmas 2 and 3, it is also used to give the lower bound required in establishing Lemma 6.

In the special case that $f = 0$ then μ becomes the measure of maximal entropy μ_0 for $T : \mathbb{J} \to \mathbb{J}$. The theorem then reduces to the following.

Corollary 1. *Let* $x \in \mathbb{J}$. *For any weak* open neighborhood* $\mathcal{U} \subset \mathcal{M}$ *of* μ_0 *the proportion of the points* $y \in S_n(x)$ *such that*

$$\mu_{y,n} = \frac{1}{n}\left(\delta_y + \delta_{Ty} + \ldots + \delta_{T^{n-1}y}\right) \notin \mathcal{U}$$

tends to zero exponentially fast, i.e., there exists $C > 0$ *and* $0 < \eta < 1$ *such that*

$$\frac{1}{d^n}\#\{y \in S_n(x) \,:\, \mu_{y,n} \notin \mathcal{U}\} \leq C\eta^n, \qquad \text{for } n \geq 0.$$

A second corollary to the theorem is given by the following convergence result.

Corollary 2. *Let* $x \in \mathbb{J}$. *Let* $f : \mathbb{J} \to \mathbb{R}$ *be a Hölder continuous function such that* $P(f) > \sup f$ *and let* μ *be the unique equilibrium state for* f. *Then the averages*

$$\frac{\sum_{T^n y = x} e^{f^n(y)} \mu_{y,n}}{\sum_{T^n y = x} e^{f^n(y)}}$$

converges to μ *in the weak* topology as* $n \to +\infty$.

These two corollaries provide two different generalizations of the following well-known result of Lyubich.

Lyubich's Theorem ([4], [8], [2]). *Let* $x \in \mathbb{J}$. *Then the averages*

$$\frac{1}{d^n}\sum_{T^n y = x} \mu_{y,n}$$

converge to μ_0 *in the weak* topology as* $n \to +\infty$.

Remark. Strictly speaking, Lyubich established that the averages $\frac{1}{d^n}\sum_{T^n y = x} \mu_{y,n}$ converge to a non-atomic T-invariant probability measure supported on $\mathbb{J}$. This measure was subsequently shown to maximise the entropy in [2] and it was shown that there is a unique measure of maximal entropy in [5]. Whereas Lyubich's Theorem can be established using normality of sequences of functions and the Montel-Carathéodory theorem [8], these stronger results (Corollaries 1 and 2) seem to require a different argument.

Previous applications of large deviation ideas to rational maps include the result of Lopes dealing with almost everywhere convergence of Birkhoff averages [3]. (Lopes restricted himself to the case of hyperbolic Julia sets but, as Przytycki observed [7] this is unnecessary provided we assume $P(f) > \sup f$.)

1. Some properties of rational maps

In this section we shall recall some of the basic properties of rational maps which we shall need later. Let $T : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map of degree $d \geq 2$. i.e.

$$T(z) = \frac{a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0}{b_m z^m + b_{m-1} z^{m-1} + \ldots + b_1 z + b_0}$$

where $a_n, \ldots, a_0, b_m, \ldots, b_0 \in \mathbb{C}$ (with $a_n, b_m \neq 0$) and $d = max\{n, m\} \geq 2$.

Counted according to multiplicity, every point $x \in \hat{\mathbb{C}}$ will have d-pre-images. If we consider the n-th iterate $T^n : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ then the set of pre-images $S_n(x) = \{y : T^n y = x\}$ of a point $x \in \hat{\mathbb{C}}$ will have cardinality d^n.

Definition. The *Julia set* $\mathbb{J} \subset \mathbb{C}$ is defined to be the closure of the set of all periodic points $T^n x = x$ for which $|(T^n)'(x)| > 1$.

Clearly $\mathbb{J}$ is closed T-invariant set and we shall be interested in the restriction $T : \mathbb{J} \to \mathbb{J}$ of the map T. We shall write $C^0(\mathbb{J})$ for the space of real valued continuous functions on $\mathbb{J}$. We denote by $\mathcal{M}_T$ the space of all T-invariant probability measures on $\mathbb{J}$.

Definition. For any continuous function $f : \mathbb{J} \to \mathbb{R}$ we can define the *pressure* by

$$P(f) = \sup\{h(\nu) + \int f d\nu : \nu \in \mathcal{M}_T\},$$

where $h(\nu)$ denotes the entropy of T with respect to the measure ν.

We let $h(T) = P(0)$ denote the *topological entropy* of $T : \mathbb{J} \to \mathbb{J}$.

The following results about entropy will be useful to us.

Lemma 1 [2], [4]**.**
(i) $h(T) = \log d$.
(ii) *The map* $\nu \to h(\nu)$ *is upper semi-continuous in the weak star topology.*
(iii) *There is a unique measure of maximal entropy* μ_0 *for* $T : \mathbb{J} \to \mathbb{J}$.

Statement (iii) of Lemma 1 has the following generalization.

Lemma 2 (Denker and Urbanski [1]). *If $f : \mathbb{J} \to \mathbb{R}$ is a Hölder continuous function such that $P(f) > \sup f$ them f has a unique equilibrium state μ.*

The following lemma gives us some information about the relationship between pressure and the pre-images of a point.

Lemma 3 (Przytycki [6]). *Let $x \in \mathbb{J}$. Let f be a Hölder continuous fuction on the Julia set $\mathbb{J}$ such that $P(f) > \sup f$ and let g be a continuous function on $\mathbb{J}$. Then*

(i)

$$\lim_{n \to +\infty} \frac{1}{n} \log \sum_{T^n y = x} e^{f^n(y)} = P(f);$$

(ii)

$$\limsup_{n \to +\infty} \frac{1}{n} \log \sum_{T^n y = x} e^{f^n(y) + g^n(y)} \leq P(f+g).$$

Finally, the following lemma gives us an alternative characterisation of the entropy.

Lemma 4.

(i) *If $\nu \in \mathcal{M}_T$ then*

$$h(\nu) = \inf\{P(g) - \int g d\nu \,:\, g \in C^0(\mathbb{J})\}.$$

(ii) *If $\nu \in \mathcal{M} - \mathcal{M}_T$ then*

$$0 \geq \inf\{P(g) - \int g d\nu \,:\, g \in C^0(\mathbb{J})\}.$$

Proof. In fact, these two results hold for any continuous mapping T of a compact metric space for which $h(T) < +\infty$ and, in the case of (i), the map $\nu \to h(\nu)$ is upper semi-continuous (cf. [10,pp. 221-222]).

2. Proof of Theorem 1

In this section we will give the proof of Theorem 1, using the results from the previous section. As before $f : \mathbb{J} \to \mathbb{R}$ is a Hölder continuous function satisfying $P(f) > \sup f$. We can define a map $Q : C^0(\mathbb{J}) \to \mathbb{R}$ by $Q(g) = P(f+g) - P(f)$. For $\nu \in \mathcal{M}$, we then denote the *Legendre transform* of $Q(g)$ by

$$I(\nu) = \sup_{g \in C^0(\mathbb{J})} (\int g d\nu - Q(g)).$$

Given any weak* closed (and hence compact) subset $\mathcal{K} \subset \mathcal{M}$ we define $\rho = \rho_{\mathcal{K}} := \inf_{\nu \in \mathcal{K}} I(\nu)$.

Our proof will be based upon the following estimate.

Lemma 5.

$$\limsup_{n\to+\infty}\frac{1}{n}\log\left(\frac{\sum_{\substack{T^n y=x\\ \mu_{y,n}\in\mathcal{K}}} e^{f^n(y)}}{\sum_{T^n y=x} e^{f^n(y)}}\right)\le -\rho \tag{1}$$

Proof. Fix a choice of $\epsilon>0$. From the definition of ρ, for every $\nu\in\mathcal{K}$, there exists $g\in C^0(\mathbb{J})$ such that

$$\int g d\nu - Q(g) > \rho-\epsilon.$$

Thus we have that

$$\mathcal{K}\subset\bigcup_{g\in C^0(\mathbb{J})}\left\{\nu\in\mathcal{M}:\int g d\nu - Q(g)>\rho-\epsilon\right\}$$

and by weak* compactness we can choose a finite subcover

$$\mathcal{K}\subset\bigcup_{i=1}^{k}\left\{\nu\in\mathcal{M}:\ \int g_i d\nu - Q(g_i)>\rho-\epsilon\right\}.$$

Therefore we have the inequality

$$\begin{aligned}\sum_{\substack{T^n y=x\\ \mu_{y,n}\in\mathcal{K}}} e^{f^n(y)} &\le \sum_{i=1}^{k}\left(\sum_{\substack{T^n y=x\\ \frac{1}{n}g_i^n(y)-Q(g_i)>\rho-\epsilon}} e^{f^n(y)}\right)\\ &\le \sum_{i=1}^{k} e^{-n(Q(g_i)+(\rho-\epsilon))}\left(\sum_{T^n y=x} e^{f^n(y)+g_i^n(y)}\right)\end{aligned}$$

Taking limits we get that

$$\begin{aligned}&\limsup_{n\to+\infty}\frac{1}{n}\log\left(\frac{\sum_{\substack{T^n y=x\\ \mu_{y,n}\in\mathcal{K}}} e^{f^n(y)}}{\sum_{T^n y=x} e^{f^n(y)}}\right)\\ &\le \sup_{1\le i\le k}\left\{-Q(g_i)-\rho+\epsilon+\limsup_{n\to+\infty}\frac{1}{n}\log\left(\sum_{T^n y=x} e^{f^n(y)+g_i^n(y)}\right)-\liminf_{n\to+\infty}\frac{1}{n}\log\left(\sum_{T^n y=x} e^{f^n(y)}\right)\right\}\\ &\le \sup_{1\le i\le k}\{-Q(g_i)-\rho+\epsilon+P(f+g_i)-P(f)\}\\ &=-\rho+\epsilon,\end{aligned}$$

where the second inequality uses Lemma 3. Since $\epsilon>0$ can be chosen arbitrarily small this completes the proof of the lemma.

We next want to show that if $\mathcal{K}$ does not contain μ then $\rho>0$. This will follow from the next lemma.

Lemma 6.

(i) *If $\nu \neq \mu$ then $I(\nu) > 0$.*

(ii) *The map $\nu \to I(\nu)$ is lower semi-continuous on $\mathcal{M}_T$ and I is bounded away from 0 on $\mathcal{M} - \mathcal{M}_T$.*

Proof. For part (i) we have that

$$\begin{aligned} I(\nu) &= \sup_{g \in C^0(\mathbb{J})} \left(\int g d\nu - P(f+g) + P(f) \right) \\ &= \sup_{g \in C^0(\mathbb{J})} \left(\int (g - f) d\nu - P(g) + P(f) \right) \\ &= \sup_{g \in C^0(\mathbb{J})} \left(\int g d\nu - P(g) \right) + P(f) - \int f d\nu \\ &= - \inf_{g \in C^0(\mathbb{J})} \left(P(g) - \int g d\nu \right) + P(f) - \int f d\nu. \end{aligned}$$

If $\nu \in \mathcal{M}_T$ then, by part (i) of Lemma 4, this is equal to $-h(\nu) + P(f) - \int f d\nu$, and, by the uniqueness of the equilibrium state μ, $-h(\nu) + P(f) - \int f d\nu > 0$. On the other hand, if $\nu \in \mathcal{M} - \mathcal{M}_T$, then

$$\inf_{g \in C^0(\mathbb{J})} (P(g) - \int g d\nu) < 0$$

by part (ii) of Lemma 4 and so

$$\begin{aligned} I(\nu) &> 0 + P(f) - \int f d\nu \\ &\geq P(f) - \sup f > 0. \end{aligned}$$

For the proof of (ii) we first notice that $I(\nu) = -h(\nu) + P(f) - \int f d\nu$. We then complete the proof with the the lower bound in the proof of (i) above. This completes the proof of the lemma.

Since $\mathcal{K}$ is compact, we can conclude that if $\mu \notin \mathcal{K}$ then $\rho > 0$. Theorem 1 now follows by setting $\mathcal{K} = \mathcal{M} - \mathcal{U}$.

Proof of Corollary 2. We shall show that for any $g \in C^0(\mathbb{J})$ we have that

$$\frac{1}{\Sigma(f,n)} \sum_{T^n y = x} e^{f^n(y)} \frac{g^n(y)}{n} \quad \to \quad \int g d\mu, \text{ as } n \to +\infty,$$

where $\Sigma(f,n) = \sum_{T^n y = x} e^{f^n(y)}$.

Given $\epsilon > 0$, define an open neighbourhood $\mathcal{U}$ of μ by

$$\mathcal{U} = \{\nu \in \mathcal{M} : |\int g d\nu - \int g d\mu| < \epsilon\}.$$

Then we may write

$$\begin{aligned}
&\frac{1}{\Sigma(f,n)} \sum_{T^n y = x} e^{f^n(y)} \frac{g^n(y)}{n} \\
&= \frac{1}{\Sigma(f,n)} \sum_{\substack{T^n y = x \\ \mu_{y,n} \in \mathcal{U}}} e^{f^n(y)} \frac{g^n(y)}{n} + \frac{1}{\Sigma(f,n)} \sum_{\substack{T^n y = x \\ \mu_{y,n} \notin \mathcal{U}}} e^{f^n(y)} \frac{g^n(y)}{n} \\
&= \frac{1}{\Sigma(f,n)} \sum_{\substack{T^n y = x \\ \mu_{y,n} \in \mathcal{U}}} e^{f^n(y)} \left\{ \int g d\mu + E_n(y) \right\} + O(\eta^n),
\end{aligned}$$

where $|E_n(y)| < \epsilon$ and $0 < \eta < 1$ is given by Theorem 1.

Thus we conclude (by adding appropriate constants to g if necessary) that

$$\limsup_{n \to +\infty} \frac{1}{\Sigma(f,n)} \sum_{T^n y = x} e^{f^n(y)} \frac{g^n(y)}{n} \leq \int g d\mu + \epsilon$$

and

$$\liminf_{n \to +\infty} \frac{1}{\Sigma(f,n)} \sum_{T^n y = x} e^{f^n(y)} \frac{g^n(y)}{n} \geq \int g d\mu - \epsilon.$$

Since $\epsilon > 0$ is arbitrary, the result is proved.

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