POINCARÉ SERIES AND ZETA FUNCTIONS FOR SURFACE GROUP ACTIONS ON ℝ-TREES

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0. Introduction

For a discrete group of isometries Γ acting freely on n-dimensional hyperbolic space $\mathbb{H}^n$ we can define a Poincaré series for each point $x \in \mathbb{H}^n$ by

$$\eta(s) = \sum_{g \in \Gamma} e^{-sd(x,gx)}$$

(where s is a complex variable, and the series converges to an analytic function providing $Re(s)$ is sufficiently large). If $[g]$ denotes the conjugacy class of $g \in \Gamma$ then we define a zeta function by

$$\zeta(s) = \prod_{[g]} \left(1 - e^{-sl(g)}\right)^{-1}$$

where the product is over all conjugacy classes of primitive elements in Γ (i.e., elements which are not a positive power of another element). Here

$$l(g) = \inf\{d(x, gx) : x \in \mathbb{H}^n\}$$

and it is easy to see that this is constant on conjugacy classes.

In this setting, it can be shown that $\eta(s)$ and $\zeta(s)$ have extensions as meromorphic functions to the entire complex plane. The proof relies on non-commutative harmonic analysis and the functions are studied via the spectral properties of the Laplace-Beltrami operator.

In this note we shall consider an analogous situation where we replace $\mathbb{H}^n$ by an $\mathbb{R}$-tree. $\mathbb{R}$-trees are a class of metric spaces which generalize the more familiar simplicial trees. In recent years there has been much interesting work on group actions on $\mathbb{R}$-trees (for a good survey see [10]) which can, in part, be viewed as a generalization of the now classical Bass-Serre theory of group actions on trees [16]. In particular, Morgan and Shalen have shown that the fundamental groups Γ of compact surfaces M with Euler characteristic strictly less than -1 act freely on $\mathbb{R}$-trees. More precisely, they show that given a hyperbolic structure on M, there

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exists an $\mathbb{R}$-tree $\mathcal{T}$ and a free action of Γ by isometries on $\mathcal{T}$ [13]. For any such action, pick a point $x \in \mathcal{T}$ and define the associated Poincaré series by

$$\eta(s) = \sum_{g \in \Gamma} e^{-sd(x,gx)},$$

where s is a complex variable. It is not difficult to show that the series converges absolutely if $Re(s)$ is sufficiently large and so defines an analytic function in a half-plane. Our first main result is the following.

Theorem 1. *Let Γ be the fundamental group of a compact surface with Euler characteristic strictly less than -1 and suppose that $\Gamma \times \mathcal{T} \to \mathcal{T}$ is a free, isometric action of Γ on $\mathcal{T}$, as constructed by Morgan and Shalen. Then $\eta(s)$ has an extension as a meromorphic function to the entire complex plane.*

We can define a zeta function for an action $\Gamma \times \mathcal{T} \to \mathcal{T}$ as for the case of actions on $\mathbb{H}^n$ by

$$\zeta(s) = \prod_{[g]} \left(1 - e^{-sl(g)}\right)^{-1}$$

where the product is over all conjugacy classes of primitive elements in Γ and $l(g) = \inf\{d(gx, x) : x \in \mathcal{T}\}$. Our second main result is the following.

Theorem 2. *Let Γ be the fundamental group of a compact surface with Euler characteristic strictly less than-1 and suppose that $\Gamma \times \mathcal{T} \to \mathcal{T}$ is a free, isometric action of Γ on $\mathcal{T}$, as constructed by Morgan and Shalen. Then $\zeta(s)$ has an extension as a meromorphic function to the entire complex plane.*

Our definition of a zeta function has strong parallels with the Ihara zeta function associated to torsion-free subgroups of $SL_2(\mathbb{Q}_p)$ or, more generally, finite graphs [1],[9] and [17]. Let G be a finite graph with fundamental group $\Gamma = \pi_1(M)$ (so that Γ is a free group). Given a unitary representation $R_\chi : \Gamma \to U(d)$ we can define the Ihara zeta function by

$$L(z, R_\chi) = \prod_{[g]} det\left(1 - R_\chi(g) z^{|g|}\right)^{-1}$$

where $|g|$ denotes the number of edges in the loop corresponding to g. (This covers the case of finitely generated torsion-free subgroups of $SL_2(\mathbb{Q}_p)$, since such groups act freely on an associated tree [16] and we can take G to be the quotient graph.) By the work of Ihara [9] it is known that $L(z, R_\chi)$ is a rational function. In [1], Bass proved an analogous result for the case where $\Gamma < SL_2(\mathbb{Q}_p)$ is not torsion-free. Such zeta functions have interesting connections with Ramanujan graphs and curves over finite fields [1, p.721].

We shall now outline the contents of the paper. In section 1 we recall the definition and some basic properties of $\mathbb{R}$-trees. In section 2 we recall the definition of strongly Markov groups. In section 3, we show the strong Markov structure and the special properties of $\mathbb{R}$-trees may be used to write $\eta(s)$ in terms of a family of matrices and hence show that it can be extended as a meromorphic function to the entire complex plane $\mathbb{C}$. In section 4 we carry out the parallel analysis for the zeta function.

1. Some basic properties of $\mathbb{R}$-trees.

We begin by recalling the definition of $\mathbb{R}$-trees. An $\mathbb{R}$-tree is a metric space $(\mathcal{T}, d)$ such that

(i) for any two points $x, y \in \mathcal{T}$ there is a unique map $\phi : [0, d(x,y)] \to \mathcal{T}$ which is isometric onto its image and has $\phi(0) = x$ and $\phi(d(x,y)) = y$;

(ii) no subset of $\mathcal{T}$ is homeomorphic to S^1.

The $\mathbb{R}$-trees defined above generalize the familiar notion of a simplicial tree, i.e., a (non-empty) connected, simply connected 1-complex with the natural metric structure. They were first introduced by Tits [18] (with the extra assumption of completeness) and were given the name $\mathbb{R}$-trees by Morgan and Shalen in [11]. Earlier, Chiswell, in his study of real valued length functions on groups, had constructed a space which was subsequently proved to be an $\mathbb{R}$-tree [6]. As an example of a space which is an $\mathbb{R}$-tree but not a simplicial tree, consider $\mathbb{R}^2$ with the metric

$$d((x,y),(x',y')) = \begin{cases} |y| + |x - x'| + |y'| & \text{if } x \neq x' \\ |y - y'| & \text{if } x = x'. \end{cases}$$

This can still be thought of as a collection of vertices and edges but here the vertices are dense in $\{(x,y) : y = 0\}$.

A very important property of $\mathbb{R}$-trees is that they are hyperbolic spaces in the sense of Gromov, in fact they are 0-hyperbolic, i.e., one side of a (geodesic) triangle lies in the union of the other two sides [8]. (From another point of view, $\mathbb{R}$-trees are spaces with "curvature" $-\infty$.)

It is well known that that only groups that act freely on simplicial trees are free groups [16, Theorem 4, p.27] and it is natural to ask which groups act freely on $\mathbb{R}$-trees. A major breakthrough in this direction was provided by the work of Morgan and Shalen. In particular, they proved the following result.

Proposition 1 ([13]). *Let Γ be the fundamental group of a compact surface M with Euler characteristic less than -1. Given a hyperbolic structure on M there exists an $\mathbb{R}$-tree $\mathcal{T}$ and a free isometric action of Γ on $\mathcal{T}$.*

It is important for what follows to recall how these $\mathbb{R}$-trees are constructed. Given a hyperbolic structure on M (i.e., a Riemannian metric of constant curvature -1) Morgan and Shalen show that there exists a measured geodesic lamination $(\mathcal{L}, \mu)$ on M such that both the leaves and the complementary regions are simply connected. (For a definition and detailed discussion of measured laminations, see [12].) They then lift $(\mathcal{L}, \mu)$ to a measured lamination $(\tilde{\mathcal{L}}, \tilde{\mu})$ of the hyperbolic plane $\mathbb{H}^2$. Denote the support of $\tilde{\mathcal{L}}$ by $Y \subset \mathbb{H}^2$ and let C denote the set of components of $\mathbb{H}^2 - Y$. Define a metric on C in the following way. For components $c_0, c_1 \in C$, let ω be a path from c_0 to c_1 which is transverse to $\tilde{\mathcal{L}}$ and crosses each leaf of $\tilde{\mathcal{L}}$ at most once. Define $d(c_0, c_1)$ to be the geometric intersection number of ω with $(\tilde{\mathcal{L}}, \tilde{\mu})$. The $\mathbb{R}$-tree $\mathcal{T}$ is constructed by joining components c_0 and c_1 with segements of length $d(c_0, c_1)$. Of course, one must check that the axioms for an $\mathbb{R}$-tree are satisfied. The action of Γ on $\mathcal{T}$ is induced by its action on C. The fact that the complementary regions of $\mathcal{L}$ are simply connected guarantees that the action is free.

Remark. More recently, Rips has announced that the only finitely generated groups which act freely on $\mathbb{R}$-trees are free products of surface groups of the above type and free abelian groups.

We shall require some estimates relating quantities defined on $\mathcal{T}$ and on Γ. These will follow once we have established that $\mathcal{T}$ and Γ (with the word metric) are quasi-isometric metric spaces (via the map $f : \Gamma \to \mathcal{T}$ given by $f(g) = gx$). A result of this kind is proved for a discrete isometric action of a group Γ on a metric space X in Proposition 19 of [8, Chapitre 3] under the assumption that X/Γ is compact and that closed balls in X are compact. This fails for $\mathcal{T}$ and Γ; however a careful examination of the proof shows that, in fact, one only requires the following two statements:

(1) the metric space $\mathcal{T}/\Gamma$ has finite diameter, i.e.,

$$R = \sup\{d_{\mathcal{T}/\Gamma}(x, y) : x, y \in \mathcal{T}/\Gamma\} < +\infty;$$

(2) let B be a ball of radius R in $\mathcal{T}$, then $\{g \in \Gamma : gB \cap B \neq \emptyset\}$ is finite.

Both of these follow from the construction of $\mathcal{T}$ and the action of Γ adumbrated above. To prove (1), first note that it is sufficient to bound the distances $d_{\mathcal{T}/\Gamma}(x, y)$, where x, y are the images of branch points in $\mathcal{T}$ (those points which are constructed from components of $\mathbb{H}^2 - Y$). However, there are only finitely many such points in $\mathcal{T}/\Gamma$ ([10, Theorem 22]), the distances are bounded. To see (2), it suffices to show that for any $x \in \mathcal{T}$ and $R' > 0$, $d(x, gx) \leq R'$ for only finitely many $g \in \Gamma$. We may suppose that $x = x_c$ is a branch point of $\mathcal{T}$, constructed from the component $c \in C$. Then $d(x_c, gx_c) = d(c, gc)$ is equal to the geometric intersection number of ω with $(\tilde{\mathcal{L}}, \tilde{\mu})$, where ω is a path from c to gc crossing each leaf of $\tilde{\mathcal{L}}$ at most once. Since $\tilde{\mathcal{L}}$ is the lift of the measured lamination $\mathcal{L}$ on $M = \mathbb{H}^2/\Gamma$ and the complementary regions of $\mathcal{L}$ are simply connected, this number tends to infinity as $d_{\mathbb{H}^2}(p, gp)$ ($p \in \mathbb{H}^2$) tends to infinity, so, in particular, $d(c, gc) \leq R'$ for only a finite number of $g \in \Gamma$.

We can conclude from this that for any $x \in \mathcal{T}$ and any finite generating set S for Γ there exist constants $C_1, C_2 > 0$ such that

$$C_1|g| \leq d(x, gx) \leq C_2|g|$$

for all $g \in \Gamma$, where $|g|$ denotes the word-length of g, i.e., the minimal length of g written as a word in S. Furthermore, if we define the Gromov products $(y, w)_x := 1/2(d(x, y) + d(x, w) - d(y, w))$ and $(g, g') := 1/2(|g| + |g'| - |g^{-1}g'|)$ then there exist constants $A > 0$, $K > 0$ such that

$$(g, g') \leq A(gx, g'x)_x + K$$

for all $g, g' \in \Gamma$.

The following result will prove useful later.

Lemma 1. *Let $\mathcal{T}$ be an $\mathbb{R}$-tree and let x, y, w, z be four points in $\mathcal{T}$ such that $d(x, z) \leq l$. If $(y, w)_x = 1/2(d(y, x) + d(w, x) - d(y, w)) > l$ then $d(x, w) + d(y, z) - d(x, y) - d(w, z) = 0$.*

Proof. Suppose first that the four points are pairwise distinct. From the definition of $\mathcal{T}$ precisely one of the two pictures in the figure must occur.

FIGURE

If we are in case (a), then it is clear that the required quantity is zero. To complete the proof, we aim to show that if $(y,w)_x$ is sufficiently large then case (b) cannot occur.

We recall the elementary fact that for $\mathbb{R}$-trees, $(y,w)_x = d(x,[yw])$, where $[yw]$ denotes the (unique) segment with endpoints y and w [8]. However, observe that in case (b) we have that $d(x,[yw]) \leq l$, but by hypothesis we have that $(y,w)_x > l$ so this case cannot occur and we are done.

It remains to consider the cases where two or more of x, y, w, z are identical. These can easily be dealt with by a case by case analysis.

2. Strongly Markov groups

In this section we shall describe how to associate to the group Γ (and a symmetric set of generators S) a finite directed graph which can be thought of as encoding the group. This is a key object proving the rationality of the growth series and will, likewise, prove fundamental in constructing the meromorphic extension of the Poincaré series.

We start by recalling the definition of strongly Markov groups.

Definition. We say that a finitely generated group is strongly Markov if for every finite symmetric generating set S we can find:

- (i) a finite directed graph G, with edges E and vertices V, and a distinguished vertex $*$ with no edge terminating at $*$; and
- (ii) an edge labelling $\lambda : E \to S$,

such that for each $n \geq 1$ there is a bijection between:

- (a) elements $g \in \Gamma - \{e\}$ with $|g| = n$; and
- (b) paths of length n in G which start at $*$ and pass through n consecutive edges $e_1, \ldots, e_n$,

given by $g = \lambda(e_1) \ldots \lambda(e_n)$.

The following result tells us that the fundamental groups of surfaces that we are considering are always strongly Markov.

Proposition 2 (Cannon [4]). *The fundamental group of a compact manifold admitting a hyperbolic structure is strongly Markov.*

This theorem was originally proved in 1984 by Cannon [4] and a particularly nice account can be found in [8].

When Γ is the fundamental group of an orientable compact surface M with Euler characteristic less than -1 we have the following standard one relator presentation

$$\Gamma = < a_1, \ldots, a_g, b_1, \ldots, b_g : \prod_{i=1}^{g} [a_i, b_i] = 1 > .$$

For generators in the standard presentation, Series [15] anticipated parts of this theory. It is her work that will prove more suitable for the study of zeta functions and the proof of Theorem 2. In particular, we shall need the following result.

Proposition 3 (Series [15]). *Let Γ be the fundamental group of a compact surface with negative Euler characteristic with the standard presentation. There is a correspondence between conjugacy classes $[g]$ (with $g \in \Gamma - \{e\}$) and closed loops $e_1, \dots, e_n$, where $[g] = [\lambda(e_1) \dots \lambda(e_n)]$, with at most finitely many exceptions.*

Remark. In Series's formulation a (finite type) set of rules is specified for admissible sequences of generators in $S = \{a_1, \dots, a_g, b_1, \dots, b_g\}$. (This is easily expressed in terms of a labelled directed graph). Any element in Γ is shown to have a unique shortest *admissible* representation as a product of generators in S ([15], Theorem 3.5). Moreover, there is a map π mapping each admissible infinite sequence of generators $(e_n)_{n\geq 0}$ to a point on the boundary $\pi((e_n)_{n\geq 0}) \in \partial\mathbb{H}^2$ of the hyperbolic plane (using the equivalent Poincaré disc model) which is equivariant with respect to the shift map on sequences with a natural action on the the boundary (i.e. $\pi((e_{n+1})_{n\geq 0}) = e_0\pi((e_n)_{n\geq 0})$). Every conjugacy class $[g]$ corresponds to a fixed point $gx = x \in \partial\mathbb{H}^2$ (this can be seen, for example, by considering lifts of closed geodesics) and thus to a periodic sequence of admissible generators. Moreover, this correspondence is one-to-one, except at for at most finitely many (prime) periodic sequences (cf. [15, Proposition 4.6]).

We shall need to work with the incidence matrix of the graph G; however, it is convenient to introduce an extra vertex 0 as described below. (This idea has been used by Bourdon [2] and by the authors [14]. In this context, its usefulness will be seen in allowing us to obtain a matrix representation of the Poincaré series, in particular in deriving equation (3.1).) Let $k = |V| + 1$ then we associate to G a $k \times k$ matrix A with rows and columns indexed by $V \cup \{0\}$ and entries given by

(i) $A(i, j) = 1$ for $i, j \in V$ connected by an edge in E;
(ii) $A(i, 0) = 1$ for $i \in V$
(iii) $A(0, 0) = 1$
(iv) $A(i, j) = 0$ for all other entries.

This procedure gives us a new edge set E' and we extend the labelling map $\lambda : E \to S$ to a map $\lambda : E' \to S \cup \{e\}$ by defining $\lambda(i, 0) = e$ for $i \in V \cup \{0\}$, where e is the identity element in Γ. By a simple recoding argument, we may assume without loss of generality that there is at most one edge joining any ordered pair of vertices.

3. The proof of Theorem 1

Given a free isometric action of Γ on an $\mathbb{R}$-tree $\mathcal{T}$ and a point $x \in \mathcal{T}$, it is natural to consider the Poincaré series defined by

$$\eta(s) = \sum_{g\in\Gamma} e^{-sd(g)},$$

whenever the series converges. (Here, for convenience, we have written $d(g) = d(x, gx)$.)

Fix a finite symmetric generating set S for Γ. Notice that (in the domain of convergence) we may write $\eta(s)$ in the following way:

$$\eta(s) = 1 + \sum_{n=1}^{\infty} \sum_{\substack{g\in\Gamma \\ |g|=n}} e^{-sd(g)}.$$

It is not hard to see that $\eta(s)$ converges to an analytic function providing the real part $Re(s)$ is sufficiently large. To see this, recall that there exist constants $C_1, C_2 > 0$ such that $C_1|g| \leq d(g) \leq C_2|g|$ for all $g \in \Gamma$ and so, for $s \in \mathbb{R}$, we have the inequalities $\xi(e^{-C_2 s}) \leq \eta(s) \leq \xi(e^{-C_1 s})$, where $\xi(z)$ denotes the growth series

$$\xi(z) = 1 + \sum_{n=1}^{\infty} Card\{g \in \Gamma : |g| = n\} z^n,$$

associated to (Γ, S). Since Γ contains the free group on two genertors as a subgroup, it is an elementary observation that the coefficients in $\xi(z)$ have exponential growth. We can therefore conclude that there exists $h > 0$ such that $\eta(s)$ converges absolutely for $Re(s) > h$ (and diverges for $Re(s) < h$).

The following proposition shows that because of the special nature of $\mathbb{R}$-trees the displacement function $d : \Gamma \to \mathbb{R}^+$ must satisfy the rather strong condition described below.

Proposition 4. *There exists an integer $N \geq 2$ such that for all sequences of vertices $i_0, i_1, \ldots, i_m$, $m \geq N$ with $A(i_r, i_{r+1}) = 1$, for $r = 0, \ldots, m-1$, we have that*

$$\begin{aligned} &d(\lambda(i_0, i_1)\lambda(i_1, i_2) \ldots \lambda(i_{m-1}, i_m)) - d(\lambda(i_1, i_2) \ldots \lambda(i_{m-1}, i_m)) \\ &= d(\lambda(i_0, i_1)\lambda(i_1, i_2) \ldots \lambda(i_{N-1}, i_N)) - d(\lambda(i_1, i_2) \ldots \lambda(i_{N-1}, i_N)). \end{aligned}$$

Proof. Write $g_i = \lambda(i_{r-1}, i_r) \in S$, $r = 1, \ldots, m$. Then

$$\begin{aligned} &(g_1 \ldots g_N, g_1 \ldots g_N g_{N+1} \ldots g_m) \\ &= 1/2(|g_1 \ldots g_N| + |g_1 \ldots g_N \ldots g_m| - |g_{N+1} \ldots g_m|) \\ &= 1/2(N + m - (m - N)) \\ &= N. \end{aligned}$$

Write $g = g_1 \ldots g_N$ and $g' = g_1 \ldots g_N g_{N+1} \ldots g_m$. Then there exist $A > 0$ and $K > 0$ such that $N = (g, g') \leq A(gx, g'x)_x + K$, for all $g, g' \in \Gamma$. We need to show that

$$d(x, g'x) - d(x, g_1^{-1} g'x) = d(x, gx) - d(x, g_1^{-1} gx).$$

Note that

$$\begin{aligned} &d(x, gx) - d(x, g'x) - d(x, g_1^{-1} gx) + d(x, g_1^{-1} g'x) \\ &= d(x, gx) - d(x, g'x) - d(g_1 x, gx) + d(g_1 x, g'x) \\ &= Q, \text{ say.} \end{aligned}$$

Apply Lemma 1 with $x = x, z = g_1 x, w = gx, y = g'x$ and note that $d(x, z) \leq l = max\{d(x, ax) : a \in S\}$. Provided $(gx, g'x)_x > l$, we have that $Q = 0$. To ensure this, we need only require that $N > Al + K$.

We shall now show that the function $\eta(s)$ can be extended as a meromorphic function to the whole of $\mathbb{C}$. Indeed, we shall give a closed form for the extension in terms of a family of matrices which we now define.

We need first to consider a matrix $\hat{A}$ whose columns and rows are indexed by the set of admissible words $\hat{\imath} = (i_0, \dots, i_{N-1})$ of length $|\hat{\imath}| = N$ (that is we require that $A(i_r, i_{r+1}) = 1$ for $0 \leq r \leq N-1$). We define

$$\hat{A}(\hat{\imath}, \hat{\jmath}) = \begin{cases} 1, & \text{if } i_0 \neq 0 \text{ and } i_r = j_{r-1}, r = 1, \dots, N-1 \\ 0, & \text{otherwise.} \end{cases}$$

Note that by this definition, $\hat{A}((0,0,\dots,0),(0,0,\dots,0)) = 0$.

Given a pair of admissible words $\hat{\imath}, \hat{\jmath}$ such that $\hat{A}(\hat{\imath}, \hat{\jmath}) = 1$, we shall write

$$r(\hat{\imath}, \hat{\jmath}) = d(\lambda(i_0, i_1)\lambda(i_1, i_2)\dots\lambda(i_{N-1}, j_{N-1})) - d(\lambda(i_1, i_2)\dots\lambda(i_{N-1}, j_{N-1})).$$

For each complex value $s \in \mathbb{C}$ we can then define a matrix P^s again indexed with the admissible words described above and with entries $P^s(\hat{\imath}, \hat{\jmath}) = \hat{A}(\hat{\imath}, \hat{\jmath})e^{-sr(\hat{\imath}, \hat{\jmath})}$.

Using Proposition 4, we can write

$$\sum_{\substack{g \in \Gamma \\ |g| = n}} e^{-sd(g)} = \sum_{\hat{\imath} \in \mathcal{I}, \hat{\jmath} \in \mathcal{J}} (P^s)^{(n)}(\hat{\imath}, \hat{\jmath}) \tag{3.1}$$

where

(a) $\mathcal{I} = \{\hat{\imath} : i_0 = *\}$

(b) $\mathcal{J} = \{\hat{\jmath} : j_0 \neq 0 \text{ and } j_r = 0, r = 1, \dots, N-1\}$.

To see this, suppose that $g = \lambda(i_0, i_1)\lambda(i_1, i_2)\dots\lambda(i_{n-1}, i_n)$ and that $n \geq N$, where N is given by Proposition 4. (If $n < N$ then the proof is easier.) Write $\hat{\imath} = \hat{\imath}^{(0)} = (i_0, i_1, \dots, i_{N-1})$ and define $\hat{\imath}^{(m)} = (\hat{\imath}_0^{(m)}, \hat{\imath}_1^{(m)}, \dots, \hat{\imath}_{N-1}^{(m)})$, for $m = 1, \dots, n$, by

$$\hat{\imath}_k^{(m)} = \begin{cases} i_{k+m} & \text{if } k + m \leq n \\ 0 & \text{if } k + m > n. \end{cases}$$

Thus, $\hat{\jmath} = \hat{\imath}^{(n)} = (i_n, 0, \dots, 0)$. We claim that

$$R(i_0, i_1, \dots, i_n) := r(\hat{\imath}^{(0)}, \hat{\imath}^{(1)}) + r(\hat{\imath}^{(1)}, \hat{\imath}^{(2)}) + \dots + r(\hat{\imath}^{(n-1)}, \hat{\imath}^{(n)}) = d(g).$$

From the defintion of r we have that

$$\begin{aligned}
&R(i_0, i_1, \dots, i_n) \\
&\quad = (d(\lambda(i_0, i_1)\lambda(i_1, i_2)\dots\lambda(i_{N-1}, i_N)) - d(\lambda(i_1, i_2)\dots\lambda(i_{N-1}, i_N))) \\
&\qquad + (d(\lambda(i_1, i_2)\lambda(i_2, i_3)\dots\lambda(i_N, i_{N+1})) - d(\lambda(i_2, i_3)\dots\lambda(i_N, i_{N+1}))) \\
&\qquad + \dots \\
&\qquad + \begin{pmatrix} d(\lambda(i_{n-N}, i_{n-N+1})\lambda(i_{n-N+1}, i_{n-N+2})\dots\lambda(i_{n-1}, i_n)) \\ -d(\lambda(i_{n-N+1}, i_{n-N+2})\dots\lambda(i_{n-1}, i_n)) \end{pmatrix} \\
&\qquad + \begin{pmatrix} d(\lambda(i_{n-N+1}, i_{n-N+2})\lambda(i_{n-N+2}, i_{n-N+3})\dots\lambda(i_n, 0)) \\ -d(\lambda(i_{n-N+2}, i_{n-N+3})\dots\lambda(i_n, 0)) \end{pmatrix} \\
&\qquad + \begin{pmatrix} d(\lambda(i_{n-N+2}, i_{n-N+3})\lambda(i_{n-N+3}, i_{n-N+4})\dots\lambda(0, 0)) \\ -d(\lambda(i_{n-N+3}, i_{n-N+4})\dots\lambda(0, 0)) \end{pmatrix} \\
&\qquad + \dots \\
&\qquad + (d(\lambda(i_{n-1}, i_n)\lambda(i_n, 0)\dots\lambda(0, 0)) - d(\lambda(i_n, 0)\dots\lambda(0, 0)))
\end{aligned}$$

Since all of the labels of the form $\lambda(i_n, 0)$ and $\lambda(0,0)$ are equal to the identity this becomes

$$
\begin{aligned}
&R(i_0, i_1, \dots, i_n) \\
&\quad = (d(\lambda(i_0,i_1)\lambda(i_1,i_2)\dots\lambda(i_{N-1},i_N)) - d(\lambda(i_1,i_2)\dots\lambda(i_{N-1},i_N))) \\
&\quad\quad + (d(\lambda(i_1,i_2)\lambda(i_2,i_3)\dots\lambda(i_N,i_{N+1})) - d(\lambda(i_2,i_3)\dots\lambda(i_N,i_{N+1}))) \\
&\quad\quad + \dots \\
&\quad\quad + \begin{pmatrix} d(\lambda(i_{n-N},i_{n-N+1})\lambda(i_{n-N+1},i_{n-N+2})\dots\lambda(i_{n-1},i_n)) \\ -d(\lambda(i_{n-N+1},i_{n-N+2})\dots\lambda(i_{n-1},i_n)) \end{pmatrix} \\
&\quad\quad + \begin{pmatrix} d(\lambda(i_{n-N+1},i_{n-N+2})\lambda(i_{n-N+2},i_{n-N+3})\dots\lambda(i_{n-1},i_n)) \\ -d(\lambda(i_{n-N+2},i_{n-N+3})\dots\lambda(i_{n-1},i_n)) \end{pmatrix} \\
&\quad\quad + \begin{pmatrix} d(\lambda(i_{n-N+2},i_{n-N+3})\lambda(i_{n-N+3},i_{n-N+4})\dots\lambda(i_{n-1},i_n)) \\ -d(\lambda(i_{n-N+3},i_{n-N+4})\dots\lambda(i_{n-1},i_n)) \end{pmatrix} \\
&\quad\quad + \dots \\
&\quad\quad + d(\lambda(i_{n-1},i_n))
\end{aligned}
$$

Applying Proposition 4 to successive pairs of terms, we obtain that

$$
\begin{aligned}
R(i_0,i_1,\dots,i_n) =& d(\lambda(i_0,i_1)\lambda(i_1,i_2)\dots\lambda(i_{n-1},i_n)) - d(\lambda(i_1,i_2)\dots\lambda(i_{n-1},i_n)) \\
&+ d(\lambda(i_1,i_2)\lambda(i_2,i_3)\dots\lambda(i_{n-1},i_n)) - d(\lambda(i_2,i_3)\dots\lambda(i_{n-1},i_n)) \\
&+ \dots \\
&+ d(\lambda(i_{n-N},i_{n-N+1})\lambda(i_{n-N+1},i_{n-N+2})\dots\lambda(i_{n-1},i_n)) \\
=& d(\lambda(i_0,i_1)\lambda(i_1,i_2)\dots\lambda(i_{n-1},i_n)) \\
=& d(g),
\end{aligned}
$$

as required.

To complete the proof of (3.1), note that

$$\sum_{\hat{i}\in\mathcal{I},\hat{j}\in\mathcal{J}} (P^s)^{(n)}(\hat{i},\hat{j}) = \sum_{i_0=*,i_1,\dots,i_n} \hat{A}(\hat{i}^{(0)},\hat{i}^{(1)})\dots\hat{A}(\hat{i}^{(n-1)},\hat{i}^{(n)})e^{-sR(i_0,i_1,\dots,i_n)}$$

and that there is a one-to-one correspondence between terms in the summation and elements $g \in \Gamma$ with $|g| = n$.

Substituting identity (3.1) into the definition of $\eta(s)$ we can write

$$
\begin{aligned}
\eta(s) &= 1 + \sum_{n=1}^{\infty} \sum_{\substack{g\in\Gamma \\ |g|=n}} e^{-sd(g)} \\
&= \sum_{n=0}^{\infty} \sum_{\hat{i}\in\mathcal{I},\hat{j}\in\mathcal{J}} (P^s)^{(n)}(\hat{i},\hat{j}) \\
&= \sum_{\hat{i}\in\mathcal{I},\hat{j}\in\mathcal{J}} \left(\sum_{n=0}^{\infty} (P^s)^{(n)}(\hat{i},\hat{j}) \right) \qquad (3.2) \\
&= \sum_{\hat{i}\in\mathcal{I},\hat{j}\in\mathcal{J}} (I - P^s)^{-1}(\hat{i},\hat{j}) \\
&= \sum_{\hat{i}\in\mathcal{I},\hat{j}\in\mathcal{J}} \left(\frac{\mathrm{Minor}(I-P^s)(\hat{i},\hat{j})}{\mathrm{Det}(I-P^s)} \right).
\end{aligned}
$$

This not only shows that $\eta(s)$ has a meromorphic extension to the entire space $\mathbb{C}$ but also gives an explicit closed form for the extension. This completes the proof of Theorem 1.

We want to formulate the following conjecture.

Conjecture. *The value at $s = 0$ is given by $\eta(0) = 1/\chi(\Gamma)$, where $\chi(\Gamma)$ represents the Euler characteristic of the group Γ (or equivalently, the Euler characteristic of the associated surface).*

This conjecture has some support in the work of Cannon and Wagreich [3,5] and Floyd and Plotnick [7] concerning the growth series $\xi(z)$ associated to (Γ, S), defined earlier. By the work of Cannon [4], this is a rational function and thus it is possible to ask about the value at $z = 1$. In the articles [3] (see also [5]) and [7] it is shown that for many choices of generators (including those giving the standard presentation) we have the identity $\xi(1) = 1/\chi(\Gamma)$.

4. The proof of Theorem 2

In this section, so that we can take advantage of Proposition 3, we must work with the generators $S = \{a_1, \dots, a_g, b_1, \dots, b_g\}$ giving the standard one-relator presentation. To study the zeta function $\zeta(s)$ we need an identity on closed loops for the graph G. We call a closed loop *prime* if it is not an iterate of a shorter closed loop. To a prime closed loop τ corresponding to a string $i_0, \dots, i_{n-1}, i_n = i_0, \dots$ ($i_k \neq 0$ $\forall k$) we associate the weighting $L(\tau) = \sum_{k=0}^{n-1} r\left((i_k, \dots, i_{k+N-1}), (i_{k+1}, \dots, i_{k+N})\right)$, where N is given by Proposition 4.

Lemma 2. *If $g = \lambda(i_0, i_1) \dots \lambda(i_{n-1}, i_0)$ then $L(\tau) = l(g) := \inf\{d(x, gx) : x \in \mathcal{T}\}$. (In fact, this quantity only depends on the conjugacy class $[g]$ of g.)*

Proof. We begin by showing that

$$L(\tau) = \lim_{m \to +\infty} \frac{1}{m} d(x, g^m x).$$

Choose $m \geq N$ (so that, in particular, $mn \geq N$). We have that $d(x, g^m x) = d(\lambda(i_0, i_1)\lambda(i_1, i_2) \dots \lambda(i_{mn-1}, i_{mn})$. Applying Propostion 4 $(mn - N + 1)$ times, we obtain

$$\begin{aligned} d(x, g^m x) = &\sum_{k=0}^{mn-N} \left\{ \begin{array}{c} d(\lambda(i_k, i_{k+1})\lambda(i_{k+1}, i_{k+2}), \dots, \lambda(i_{k+N-1}, i_{k+N})) \\ -d(\lambda(i_{k+1}, i_{k+2}), \dots, \lambda(i_{k+N-1}, i_{k+N})) \end{array} \right\} \\ &+ d(\lambda(i_{mn-N+1}, i_{mn-N+2}), \dots, \lambda(i_{mn-1}, i_{mn})) \\ &= (m - N)L(\tau) \\ &+ \sum_{k=(m-N)n+1}^{mn-N} \left\{ \begin{array}{c} d(\lambda(i_k, i_{k+1})\lambda(i_{k+1}, i_{k+2}), \dots, \lambda(i_{k+N-1}, i_{k+N})) \\ -d(\lambda(i_{k+1}, i_{k+2}), \dots, \lambda(i_{k+N-1}, i_{k+N})) \end{array} \right\} \\ &+ d(\lambda(i_{mn-N+1}, i_{mn-N+2}), \dots, \lambda(i_{mn-1}, i_{mn})). \end{aligned}$$

Thus, using the inequality $d(g) \leq C_2|g|$, we have the estimate

$$|d(x, g^m x) - (m - N)L(\tau)| \leq (2N(n-1) + 1)C_2 N,$$

yielding the required limit upon division by m.

To prove the lemma, we shall show that this limit is equal to $l(g)$. Noting that, since g is an isometry, $d(g^m x, g^m y) = d(x, y)$, for any $x, y \in \mathcal{T}$, we can apply the triangle inequality to conclude that

$$d(x, g^m x) - 2d(x, y) \leq d(y, g^m y) \leq d(x, g^m x) + 2d(x, y).$$

Thus we see that $D(g) = \lim_{m \to +\infty} \frac{1}{m} d(x, g^m x)$ is independent of the choice of $x \in \mathcal{T}$.

To show that $D(g) = l(g)$ we can choose x to lie on the axis $\mathrm{Axis}(g)$ of g i.e. the isometric copy of $\mathbb{R}$ in $\mathcal{T}$ upon which g acts as a translation by $l(g)$. With such a choice of $x \in \mathrm{Axis}(g)$ we see that $D(g) = D_x(g) = l(g)$, as required [10].

Let $\tau = \{i_0, \ldots, i_{n-1}\}$ denote the loop passing through the vertices $i_0, \ldots, i_{n-1}$. Let $(i_0, \ldots, i_{n-1})$ denote the same loop but with a distinguished starting point. If we define $H(s) = \prod_{\substack{\tau=\{i_0,\ldots,i_{n-1}\} \\ \text{prime}}} \left(1 - e^{-sL(\tau)}\right)^{-1}$ (for $Re(s) > 0$ sufficiently large) then we have that

$$\begin{aligned}
H(s) &= exp\left(-\sum_{n=1}^{\infty} \sum_{\substack{\{i_0,\ldots,i_{n-1}\} \\ \text{prime}}} \log\left(1 - e^{-sL(\tau)}\right)\right) \\
&= exp\left(\sum_{n=1}^{\infty} \sum_{\substack{\{i_0,\ldots,i_{n-1}\} \\ \text{prime}}} \left(\sum_{k=1}^{\infty} \frac{e^{-skL(\tau)}}{k} \right)\right) \\
&= exp\left(\sum_{n=1}^{\infty} \sum_{\substack{(i_0,\ldots,i_{n-1}) \\ \text{prime}}} \left(\sum_{k=1}^{\infty} \frac{e^{-skL(\tau)}}{nk} \right)\right) \\
&= exp\left(\sum_{m=1}^{\infty} \sum_{(i_0,\ldots,i_{m-1})} \frac{e^{-s\sum_{j=0}^{m-1} r((i_j,\ldots,i_{j+N-1}),(i_{j+1},\ldots,i_{j+N}))}}{m} \right)
\end{aligned}$$

(where $m = kn$ and $(i_0, \ldots, i_{m-1})$ is n concatenations of the prime orbit $(i_0, \ldots, i_{n-1})$, say)

$$\begin{aligned}
&= exp\left(\sum_{m=1}^{\infty} \frac{\mathrm{Trace}(P^s)^{(m)}}{m} \right) \\
&= \frac{1}{\mathrm{Det}(I - P^s)}. \qquad (4.1)
\end{aligned}$$

(The penultimate identity makes use of the fact that $\hat{A}((0,0,\ldots,0),(0,0,\ldots,0)) = 0$, so the loop involving 0 does not appear.) Using Proposition 3 we can make the formal identification $\zeta(s) = H(s)T(s)$ (for $Re(s)$ sufficiently large) where $T(s)$ is the ratio of two trigonometric polynomials. Thus by (4.1), $\zeta(s)$ has the meromorphic extension $\frac{T(s)}{\mathrm{Det}(I-P^s)}$ to $\mathbb{C}$.

Remark. In analogy with the Ihara zeta function, it is possible to introduce a unitary representation R_χ of Γ into the definition of the zeta function to obtain a function $L(s, R_\chi)$. The methods we employ in this paper could be adapted to provide a meromorphic extension in this more general setting.

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