DISTRIBUTION OF ERGODIC SUMS FOR HYPERBOLIC MAPS

Mark Pollicott and Richard Sharp

University of Manchester

Dedicated to Anatoly Vershik on the occasion of his seventieth birthday.

Abstract. In this paper we study statistical properties of hyperbolic maps. In particular, we estimate how sums of functions along orbits are distributed relative to intervals which shrink in size.

0. Introduction

In this article we shall study statistical properties for the orbits of dynamical systems. Given any measurable map $T : X \to X$ and ergodic probabilty measure m we can consider an integrable function $f : X \to \mathbb{R}$ such that $\int f dm = 0$. Let $f^n(x) := f(x) + f(Tx) + \cdots + f(T^{n-1}x)$ denote the sum along the first n points in the orbit of $x \in X$. The Birkhoff Ergodic Theorem implies that $f^n(x)/n \to 0$, as $n \to +\infty$, for almost every $x \in X$ with respect to m. An important problem in ergodic theory is to obtain a more detailed understanding of such ergodic sums $f^n(x)$ and, in particular, the fluctuations from their mean behaviour.

To get interesting results, we need to consider a more restricted class of systems. In particular, we shall study the important class of (mixing) hyperbolic diffeomorphisms and expanding maps $T : X \to X$, where X is a compact subset of a Riemannian manifold M. Let m be a Gibbs state (for a Hölder continuous function $g : X \to \mathbb{R}$) and let $f : X \to \mathbb{R}$ be a Hölder continuous function for which the variance

$$\sigma^2(f) := \lim_{n\to+\infty} \frac{1}{n} \int (f^n)^2 dm$$

is non-zero. In this case, the sums $f^n(x)$ satisfy the stronger Central Limit Theorem [10],[1], i.e., for any real numbers $a < b$, we have that

$$\lim_{n\to+\infty} m\left\{x \in X : a \le \frac{f^n(x)}{\sqrt{n}} \le b\right\} = \frac{1}{\sqrt{2\pi}\sigma}\int_a^b e^{-t^2/2\sigma^2} dt.$$

Moreover, the sums also satisfy the weak invariance principle and law of the iterated logarithm, both of which are consequences of a more general almost sure invariance principle in [2].

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For non-lattice functions, Lalley showed the following Local Limit Theorem: for any real numbers $a < b$, we have that

$$m\{x \in X : a \leq f^n(x) \leq b\} \sim \frac{(b-a)}{\sqrt{2\pi}\sigma} \frac{1}{\sqrt{n}}, \tag{0.1}$$

as $n \to +\infty$ [5]. (Here we have used the notation $A(n) \sim B(n)$, as $n \to +\infty$, if $\lim_{n\to+\infty} A(n)/B(n) = 1$). Related results were proved by Guivarc'h and Hardy.

In this paper we strengthen (0.1) to give an asymptotic formula when the interval $[a, b]$ is allowed to shrink, at a suitably slow rate, as n increases. Towards this end, we need to impose modest additional restrictions on the function $f : X \to \mathbb{R}$.

Definition. We say that the function $f : X \to \mathbb{R}$ is *diophantine* if we can find periodic orbits $T^{n_1}(x_1) = x_1$, $T^{n_2}(x_2) = x_2$ and $T^{n_3}(x_3) = x_3$ such that

$$\alpha = \frac{f^{n_2}(x_2) - f^{n_1}(x_1)}{f^{n_3}(x_3) - f^{n_1}(x_1)}$$

is a Diophantine number (i.e., there exists $C > 0$ and $\gamma > 2$ such that $|\alpha - p/q| \geq C/q^\gamma$, for all $p, q \in \mathbb{N}$).

Theorem 1. *Let $T : X \to X$ be a C^1 expanding map and let m be the Gibbs state for a Hölder continuous function. Suppose that $f : X \to \mathbb{R}$ is a Hölder continuous function satisfying the Diophantine condition and such that $\int f dm = 0$. Then there exists $\delta > 0$ such that for any $z \in \mathbb{R}$, $a < b$ and sequence $\epsilon_n > 0$ which tends to zero and satisfies $\epsilon_n^{-1} = O(n^\delta)$, we have that*

$$m\{x \in X : \ z + \epsilon_n a \leq f^n(x) \leq z + \epsilon_n b\} \sim \frac{(b-a)}{\sqrt{2\pi}\sigma} \frac{\epsilon_n}{\sqrt{n}},$$

as $n \to +\infty$.

Remark. It is easy to see that typical functions satisfy the Diophantine condition. Indeed, the diophantine condition holds for generic functions in various topologies [7], [4].

Theorem 1 also holds if expanding maps are replaced by one-sided subshifts of finite type. This will be apparent from the proof. We shall also show analogous results for Axiom A diffeomorphisms restricted to a basic set (Theorem 2) and periodic orbits (Theorem 3).

In section 1 we shall recall some preliminary results. In section 2, we present the proof of Theorem 1. In section 3 we extend these results to Axiom A diffeomorphisms. Finally, in section 4 we will present an analogous result for periodic points.

1. Preliminaries

Let M be a compact connected smooth Riemannian manifold and suppose that $X \subset U \subset M$ with X compact and U open. Let $T : U \to M$ be a C^1 map. Suppose that there exists $\lambda > 1$ such that $||DT_x(v)|| \geq \lambda ||v||$ for all $x \in U$ and all $v \in T_xM$ and that $X = \bigcap_{n\geq 0} T^{-n}U$. We shall then refer to $T : X \to X$ as an *expanding map.* In the special case where $X = U = M$, we shall call T an expanding endomorphism of M; in this case T is topologically conjugate to an expanding endomorphism of an infranilmanifold. In addition, we shall suppose that $T : X \to X$ is topologically mixing.

We recall that two continuous functions $f, f' : X \to \mathbb{R}$ are cohomologous if $f - f' = h \circ T - h$, for some continuous function $h : X \to \mathbb{R}$. By Livsic's Theorem, this is equivalent to the statement that $f^n(x) = (f')^n(x)$ whenever $T^n x = x$ is a periodic point. In particular, the assumption that $\sigma^2 > 0$ is equivalent to the statement that f is not cohomologous to a constant. We say that a function f has integer periods if $\{f^n(x) : T^n x = x, n \geq 1\} \subset \mathbb{Z}$. We recall that f is called a *non-lattice* function if one has the stronger condition that f is not cohomologous to a function of the form $a + b\psi$, where $a, b \in \mathbb{R}$ and ψ is a function with integer periods. If f satisfies the diophantine condition then f is a non-lattice function and, in particular, f is not cohomologous to a constant.

We shall write $\mathcal{M}$ for the space of T-invariant probability measures on X. For $\nu \in \mathcal{M}$, we write $h(\nu)$ for the entropy of T with repect to ν. Given a continuous function $g : X \to \mathbb{R}$ we define its pressure $P(g)$ by

$$P(g) = \sup\left\{h(\nu) + \int g d\nu \,:\, \nu \in \mathcal{M}\right\}.$$

If g is Hölder continuous then the above supremum is attained for a unique measure called the equilibrium state for g. If $g - g'$ is cohomologous to a constant then g and g' have the same equilibrium state.

Given a $k \times k$ matrix A with entries 0 or 1, we define a space

$$\Sigma^+ = \{x = (x_n)_{n=0}^{\infty} \,:\, A(x_n, x_{n+1}) = 1 \ \forall n \in \mathbb{Z}^+\}$$

and a shift map $\sigma : \Sigma^+ \to \Sigma^+$ given by $(\sigma x)_n = x_{n+1}$. The pair (Σ^+, σ) is called a (one-sided) shift of finite type. There is a metric on Σ^+ given by $d(x, y) = 2^{-N}$, where $N = \sup\{n : x_i = y_i, i \leq n\}$. The map $\sigma : \Sigma^+ \to \Sigma^+$ is mixing if the matrix A is aperiodic, i.e., there exists $N \geq 1$ such that $A^N(i, j) \geq 1$ for any $1 \leq i, j \leq k$.

An important feature of expanding maps is that they may be modelled by shifts of finite type.

Proposition 1.1. *Let $T : X \to X$ be a (mixing) expanding map. Then there exists a mixing subshift of finite type $\sigma : \Sigma^+ \to \Sigma^+$ and a Hölder continuous map $\pi : \Sigma^+ \to X$ such that*

(i) $T \circ \pi = \pi \circ \sigma$
(ii) *π is surjective, bounded-to-one and one-to-one almost everywhere with respect to any ergodic measure on Σ^+.*

Given $\alpha > 0$, we let $C^\alpha(\Sigma^+)$ be the Banach space of Hölder continuous functions $f : \Sigma^+ \to \mathbb{R}$ with norm $||f|| = |f|_\alpha + |f|_\infty$, where

$$|f|_\alpha = \sup\left\{\frac{|f(x) - f(y)|}{d(x, y)^\alpha} \,:\, x, y \in \Sigma^+\right\}$$

and $|f|_\infty$ is the supremum norm. Let $\mathcal{L}_{g+iuf} : C^\alpha(\Sigma^+) \to C^\alpha(\Sigma^+)$ be the transfer operator defined by

$$\mathcal{L}_{g+iuf} w(x) = \sum_{\sigma y = x} e^{g(y) + iuf(y)} w(y).$$

We will say that g is normalized if $\mathcal{L}_g 1 = 1$; by adding a coboundary and a constant it is always possible to arrange that g is normalized. The following result is standard (see [7] for parts (1) and (2) and [8] for part (3)).

Lemma 1.2.

(1) *When* $u = 0$ *the operator* $\mathcal{L}_{g+iuf}$ *has a maximal eigenvalue* $e^{P(g)}$ *and the rest of the spectrum is contained in a disc of strictly smaller radius. In particular, if* g *is normalized then* $P(g) = 0$ *and* $\mathcal{L}_g^* \mu = \mu$*, where* μ *is the equilibrium state for* g*.*

(2) *There exists* $a > 0$ *such that, for* $|u| < a$*,* $\mathcal{L}_{g+iuf}$ *has a simple maximal eigenvalue* $e^{P(g+iuf)}$*, satisfying* $|e^{P(g+iuf)}| \leq e^{P(g)}$*, and the rest of the spectrum is contained in* $\{z : |z| \leq \theta e^{P(g)}\}$*, for some* $0 < \theta < 1$*. In particular,* $u \mapsto e^{P(g+iuf)}$ *is analytic for* $|u| < a$*. Furthermore, we write* $\left.\frac{d^2 P(g+iuf)}{du^2}\right|_{u=0} = -\sigma^2$*.*

(3) *There exists a change of coordinates* $v = v(u)$ *such that for* $|u| < a$*, we can expand* $e^{P(g+iuf)} = e^{P(g)}(1 - v^2 + iQ(v))$*, where* $Q(v)$ *is real valued and satisfies* $Q(v) = O(|v|^3)$*. In particular,* $v'(0) = \sigma/\sqrt{2}$*.*

The following identity will be important in subsequent calculations.

Lemma 1.3. *Let* μ *denote the equilibrium state for* g*. If* g *is normalized then*

$$\int e^{iuf^n(x)} d\mu(x) = \int \mathcal{L}_{g+iuf}^n 1(x) d\mu(x).$$

Proof. This follows from the identity $\mathcal{L}_g^* \mu = \mu$ by a simple calculation. $\square$

A key point in our proof will be a bound which involves an estimate on iterates of $\mathcal{L}_{g+iuf}$; estimates of the kind we require were developed in [3] and [9].

Lemma 1.4. *Assume that* f *satisfies the Diophantine condition and that* g *is normalized. Then there exists* $\gamma > 0$*,* $D > 0$ *and* $C, c > 0$ *such that, for* $|u| \geq a$*, we have that*

$$||\mathcal{L}_{g+iuf}^{2Nm} 1||_\infty \leq C \left(1 - \frac{c}{|u|^\gamma}\right)^m, \text{ for } n \geq 1, \tag{1.1}$$

where $N = [D \log |u|]$*.*

Proof. Since we are assuming the Diophantine condition, the hypotheses of Proposition 2 in [9] hold. This gives the inequality (1.1). $\square$

2. Proof of Theorem 1

In the section we will present a proof of the Theorem 1 using properties of the transfer operator from section 2. Let m be the equilibrium state for a Hölder continuous function $g : X \to \mathbb{R}$ and choose $g_0 : \Sigma^+ \to \mathbb{R}$ be a normalized Hölder continuous function on Σ^+ which is cohomologous to $g \circ \pi$. Let μ denote the equilibrium state for g_0. Given a Hölder continuous function $f : X \to \mathbb{R}$ (with $\int f dm = 0$), we define $f_0 : \Sigma^+ \to \mathbb{R}$ by $f_0 = f \circ \pi$. Since π is Hölder, $f_0 \in C^\alpha(\Sigma^+)$, for some $\alpha > 0$. Then

$$\begin{aligned} &m \{x \in X : \ z + \epsilon_n a \leq f^n(x) \leq z + \epsilon_n b\} \\ &= \mu \left\{x \in \Sigma^+ : \ z + \epsilon_n a \leq f_0^n(x) \leq z + \epsilon_n b\right\}. \end{aligned}$$

Thus, to prove Theorem 1, it suffices to prove the corresponding asymptotic formula for $\mu \{x \in \Sigma^+ : \ z + \epsilon_n a \leq f_0^n(x) \leq z + \epsilon_n b\}$. For the remainder of this section, we shall abuse notation and write f and g for f_0 and g_0.

We shall first prove a modified result, where the interval $[z+\epsilon_n a, z+\epsilon_n b]$ is replaced by a sequence of smooth test functions. Let $\chi:\mathbb{R}\to\mathbb{R}$ be a compactly supported C^k function (where k will be chosen later). We shall write $\chi_n(x) = \chi(\epsilon_n^{-1}(x-z))$ and we note that the Fourier transform satisfies $\widehat{\chi}_n(u) = e^{izu}\epsilon_n\widehat{\chi}(\epsilon_n u)$. Let us define

$$\rho(n) := \int \chi_n(f^n(x))d\mu.$$

Proposition 2.1. *Let γ be as in Lemma 1.4. Then, provided that $\epsilon_n^{-1} = O(n^\delta)$, for some $\delta < 1/\gamma$, we have that*

$$\rho(n) \sim \frac{\int\chi(x)dx}{\sqrt{2\pi}\sigma}\frac{\epsilon_n}{\sqrt{n}}, \quad \text{as } n\to+\infty.$$

To prove Proposition 2.1 we first use the inverse Fourier transform and Fubini's Theorem to write

$$\begin{aligned}\rho(n) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int e^{iuf^n(x)}d\mu(x)\right)\widehat{\chi}_n(u)du\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int \mathcal{L}^n_{g+iuf}1(x)d\mu(x)\right)\widehat{\chi}_n(u)du,\end{aligned}\tag{2.1}$$

using Lemma 1.3 for the last equality.

Choose $a>0$ sufficiently small. Using part (3) of Lemma 1.2, we can change coordinates on $(-a,a)$ to $v=v(u)$ and write $e^{P(g+iuf)} = (1-v^2+iQ(v))$, for $|v|<a$, say. If $\mathcal{P}_{g+iuf}: C^\alpha(\Sigma^+)\to C^\alpha(\Sigma^+)$ is the associated one dimensional eigenprojection, then by perturbation theory $\mathcal{P}_{g+iuf}(1) = 1+O(|v|)$. Using the formula $\mathcal{L}^n_{g+iuf}1 = e^{nP(g+iuf)}(1+O(|v|))+O(\theta^n)$, we may write

$$\begin{aligned}&\int_{-a}^{a}\left(\int \mathcal{L}^n_{g+iuf}1(x)d\mu(x)\right)\widehat{\chi}_n(u)du\\ &= \int_{-a}^{a}(1-v^2+iQ(v))^n\,(1+O(|v|))\,\widehat{\chi}_n(u(v))\frac{du}{dv}dv + O(\theta^n)\\ &= \frac{\epsilon_n\widehat{\chi}(0)\sqrt{2}}{\sigma}\int_{-a}^{a}(1-v^2+iQ(v))^n\,(1+O(|v|))\,dv + O\left(\frac{\epsilon_n}{n}\right)+O(\theta^n),\end{aligned}\tag{2.2}$$

where the $O(\epsilon_n n^{-1})$ estimate follows from a simple calculation in [8, p.409]. Using another easy calculation in [8, pp.408-409], we see that the principle term in the last line of (2.2) is asymptotic to $\int_{-a}^{a}(1-v^2)^n dv$; by making the substitution $w=v^2$, we may estimate this as

$$\begin{aligned}\frac{\epsilon_n\widehat{\chi}(0)\sqrt{2}}{\sigma}\int_{-a}^{a}(1-v^2)^n dv &= 2\frac{\epsilon_n\widehat{\chi}(0)\sqrt{2}}{\sigma}\int_{0}^{a}(1-v^2)^n dv\\ &= \frac{\epsilon_n\widehat{\chi}(0)\sqrt{2}}{\sigma}\int_{0}^{a^2}\frac{(1-w)^n}{w^{1/2}}dw\\ &= \frac{\epsilon_n\widehat{\chi}(0)\sqrt{2}}{\sigma}\int_{0}^{1}\frac{(1-w)^n}{w^{1/2}}dw + O((1-a^2)^n)\\ &\sim \sqrt{2\pi}\frac{\widehat{\chi}(0)}{\sigma}\frac{\epsilon_n}{\sqrt{n}},\end{aligned}\tag{2.3}$$

as $n \to +\infty$ (cf. [15, p.236]). Moreover, the term rising from the $O(|v|)$ term in the integrand is of order

$$\int_{-a}^{a} (1-v^2)^n |v| dv = \int_0^{a^2} (1-w)^n dw = O\left(\frac{1}{n}\right).$$

It remains to estimate the integral in (2.1) over $|u| \geq a$. To do this we shall use the bound on the transfer operators $\mathcal{L}_{g+iuf}$ contained in Lemma 1.4. We shall also use the following simple lemma.

Lemma 2.2. *If $\chi : \mathbb{R} \to \mathbb{R}$ is C^k and compactly supported then the Fourier transform $\widehat{\chi}(u)$ satisfies $\widehat{\chi}(u) = O(|u|^{-k})$, as $|u| \to \infty$.*

Proof. This is a standard application of integration by parts. □

To complete the proof of Proposition 2.1, we can bound

$$\begin{aligned} &\int_{|u|\geq a} \left(\int \mathcal{L}^n_{g+iuf} 1(x) d\mu(x) \right) \widehat{\chi}_n(u) du \\ &= \epsilon_n \int_{|u|\geq a} e^{izu} \left(\int \mathcal{L}^n_{g+iuf} 1(x) d\mu(x) \right) \widehat{\chi}(\epsilon_n u) du \qquad (2.4) \\ &= O\left(\frac{1}{\epsilon_n^{k-1}} \int_a^{\infty} \left(1 - \frac{c}{u^\gamma}\right)^{n/2[D \log |u|]} u^{-k} du \right). \end{aligned}$$

We need to show that this quantity tends to zero more quickly than $\epsilon_n n^{-1/2}$. To see this we shall split the integral in (2.4) into two parts:

$$\begin{aligned} &\int_a^{\infty} \left(1 - \frac{c}{u^\gamma}\right)^{n/2[D \log |u|]} u^{-k} du \\ &= \int_a^{n^{\delta'}} \left(1 - \frac{c}{u^\gamma}\right)^{n/2[D \log |u|]} u^{-k} du + \int_{n^{\delta'}}^{\infty} \left(1 - \frac{c}{u^\gamma}\right)^{n/2[D \log |u|]} u^{-k} du, \end{aligned}$$

where we choose $\delta < \delta' < 1/\gamma$. The first integral may be bounded by

$$\int_a^{n^{\delta'}} \left(1 - \frac{c}{u^\gamma}\right)^{n/2[D \log |u|]} u^{-k} du = O\left(n^{\delta'} \left(1 - \frac{c}{n^{\delta'\gamma}}\right)^{n/2D\delta' \log n} \right)$$

and, since $\delta'\gamma < 1$, this tends to zero faster than the reciprocal of any polynomial. The second integral may be bounded by

$$\int_{n^{\delta'}}^{\infty} \left(1 - \frac{c}{u^\gamma}\right)^n u^{-k} du = O(n^{(1-k)\delta'}).$$

Combining these estimates we see that

$$\int_{|u|\geq a} \left(\int \mathcal{L}^n_{g+iuf} 1(x) d\mu(x) \right) \widehat{\chi}_n(u) du = O(\epsilon_n^{-(k-1)} n^{(1-k)\delta'}) = O(n^{(k-1)(\delta-\delta')}).$$

We obtain the required bound by choosing k sufficiently large that $(k-1)(\delta-\delta') < -\delta - 1/2$.

Finally, Theorem 1 may be deduced from Proposition 2.1 by a simple approximation argument. More precisely, given $\eta > 0$, we can choose smooth functions $\chi^- \leq \chi_{[a,b]} \leq \chi^+$, where $\chi_{[a,b]}$ denotes the indicator function of the interval $[a, b]$, such that $b - a - \eta \leq \int \chi^-(x)dx \leq \int \chi^+(x)dx \leq b - a + \eta$. Then

$$\limsup_{n\to+\infty} \frac{n^{1/2}}{\epsilon_n} \mu \left\{ x \in \Sigma^+ : \ z + a\epsilon_n \leq f^n(x) \leq z + \epsilon_n b \right\}$$
$$\leq \limsup_{n\to+\infty} \frac{n^{1/2}}{\epsilon_n} \int \chi_n^+(f^n(x))d\mu \leq \frac{b-a+\eta}{\sqrt{2\pi}\sigma}$$

and

$$\liminf_{n\to+\infty} \frac{n^{1/2}}{\epsilon_n} \mu \left\{ x \in \Sigma^+ : \ z + a\epsilon_n \leq f^n(x) \leq z + \epsilon_n b \right\}$$
$$\geq \liminf_{n\to+\infty} \frac{n^{1/2}}{\epsilon_n} \int \chi_n^-(f^n(x))d\mu \geq \frac{b-a-\eta}{\sqrt{2\pi}\sigma}.$$

Since $\eta > 0$ is arbitrary, this gives the result.

3. Axiom A Diffeomorphisms

In this section we shall show how the results of Theorem 1 can be extended to invertible systems. This requires some technical details which we shall describe in this section.

Let $T : M \to M$ be a C^1 diffeomorphism. We call an T-invariant set X a basic set if:

(i) we have a DT-invariant splitting $T_X M = E^s \oplus E^u$ such that $\exists C > 0, 0 < \lambda < 1$, such that $||DT^n|E^s|| \leq C\lambda^n$ and $||DT^{-n}|E^u|| \leq C\lambda^n$;
(ii) $\exists$ open set $U \supset X$ such that $X = \cap_{n=-\infty}^{\infty} T^{-n}U$;
(iii) $T : X \to X$ is transitive; and
(iv) the periodic orbits for $T|X$ are dense in X

We say that T satisfies Axiom A if the the non-wandering set Ω is hyperbolic. In particular, Ω is a finite union of hyperbolic fixed points and basic sets.

The analogue of Theorem 1 for Axiom A diffeomorphisms is the following.

Theorem 2. *Let $T : X \to X$ be an Axiom A diffeomorphism restricted to a non-trivial basic set. Suppose that $T : X \to X$ is mixing and let m be the Gibbs state for a Hölder continuous function. Suppose that $f : X \to \mathbb{R}$ is a Hölder continuous function satisfying the Diophantine condition and such that $\int f dm = 0$. Then there exists $\delta > 0$ such that for any $z \in \mathbb{R}$, $a < b$ and sequence $\epsilon_n > 0$ which tends to zero and satisfies $\epsilon_n^{-1} = O(n^\delta)$, we have that*

$$m \left\{ x \in \Lambda : \ z + \epsilon_n a \leq f^n(x) \leq z + \epsilon_n b \right\} \sim \frac{(b-a)}{\sqrt{2\pi}\sigma} \frac{\epsilon_n}{\sqrt{n}},$$

as $n \to +\infty$.

We begin by introducing the two sided version of subshifts of finite type.

3.1 Symbolic dynamics. As in section 1, given a $k \times k$ matrix A with entries 0 or 1, we define a space

$$\Sigma = \{x = (x_n)_{n=-\infty}^{\infty} : A(x_n, x_{n+1}) = 1 \ \forall n \in \mathbb{Z}\}$$

and a shift map $\sigma : \Sigma \to \Sigma$ given by $(\sigma x)_n = x_{n+1}$. The pair (Σ, σ) is called a (two-sided) shift of finite type. There is a metric on Σ given by $d(x, y) = 2^{-k}$, where $k = \sup\{n : x_i = y_i, |i| \le n\}$.

The following result reduces this to the study of the subshift $\sigma : \Sigma \to \Sigma$.

Proposition 3.1. *Let $T : X \to X$ be an Axiom A diffeomorphism restricted to a non-trivial basic set and suppose that T is mixing. Then there exists a mixing subshift of finite type $\sigma : \Sigma \to \Sigma$ and a Hölder continuous map $\pi : \Sigma \to X$ such that*

(i) $T \circ \pi = \pi \circ \sigma$

(ii) *π is surjective, bounded-to-one and one-to-one almost everywhere with respect to any ergodic measure on Σ.*

Given $\alpha > 0$, we let $C^\alpha(\Sigma)$ be the Banach space of Hölder continuous functions $f : \Sigma \to \mathbb{R}$ with norm $||f|| = |f|_\alpha + ||f||_\infty$, where

$$|f|_\alpha = \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)^\alpha} : x, y \in \Sigma, x \neq y \right\}$$

and $||f||_\infty$ is the supremum norm. Observe that for any $0 < \beta < \alpha$ we have that $|f|_\beta \le |f|_\alpha$.

Suppose that m is the Gibbs state for the Hölder continuous function $g : X \to \mathbb{R}$ and let μ be the Gibbs state for $g \circ \pi : \Sigma \to \mathbb{R}$. As in section 2, we have

$$\begin{aligned} &m\{x \in X : \ z + \epsilon_n a \le f^n(x) \le z + \epsilon_n b\} \\ &= \mu\{x \in \Sigma : \ z + \epsilon_n a \le (f \circ \pi)(x) \le z + \epsilon_n b\}, \end{aligned}$$

so it suffices to consider functions on Σ. We shall suppose that α is chosen so that $f \circ \pi, g \circ \pi \in C^\alpha(\Sigma)$. Once again, we shall abuse notation and write f and g instead of $f \circ \pi$ and $g \circ \pi$.

In order to obtain the asymptotics in Theorem 2, we need to relate f to functions defined on the corresponding one-sided shift Σ^+; then we can apply the analysis of section 2. It is well-known that it is possible to find a Hölder continuous function defined on Σ^+ which is cohomologous to f, however, $\mu\{x : z + \epsilon_n a \le f(x) \le z + \epsilon_n b\}$ is not invariant under this change. It is therefore necessary to employ a slightly more sophisticated approach involving approximations.

3.2 Introducing functions $\widetilde{f}_k$ on Σ^+. The first step in the proof is to approximate $f \in C^\alpha(\Sigma)$ by functions that only go finitely far into the "past" (i.e., we choose $f_k : \Sigma \to \mathbb{R}$ depending only on the co-ordinates $x_{-k}, x_{-k+1}, x_{-k+2}, \ldots$) sufficiently close to f, in a suitable sense. In particular, we want to let $k = k(n) = \eta \log n$, where $\eta = (1 + \gamma)(\alpha \log 2)^{-1} > 0$, where γ will be specified later. We then choose $f_k(x) = \inf\{f(y) : y_i = x_i, \ i \ge -k\}$.

The following relates f_k to f, and reduces Theorem 2 to proving the corresponding result for f_k, for some sufficiently small $\delta > 0$ (independent of k).

Lemma 3.1. *Assume that $\gamma > 1 > \delta$ and let $\epsilon_n^{-1} = o(n^\delta)$. Then*

$$\begin{aligned} &\mu\left\{x \in \Sigma : z + \epsilon_n a + n^{-\delta} \le f_k^n(x) \le z + \epsilon_n b - n^{-\delta}\right\} \\ &\le \mu\{x \in \Sigma : z + \epsilon_n a \le f^n(x) \le z + \epsilon_n b\} \qquad (3.1) \\ &\le \mu\left\{x \in \Sigma : z + \epsilon_n a - n^{-\delta} \le f_k^n(x) \le z + \epsilon_n b + n^{-\delta}\right\}, \end{aligned}$$

for all sufficiently large n.

Proof. This is similar to the approach in [5]. Observe that $2^{-\alpha k} = n^{-(1+\gamma)} = o(n^{-1}\epsilon_n)$, then $||f_k^n - f^n||_\infty \leq n|f|_\alpha 2^{-\alpha k} = o(n^{-\delta})$, provided that $\delta < \gamma$. In particular, we can compare:

$$\begin{aligned}&\mu\left\{x \in \Sigma : \ z + \epsilon_n a - n|f|_\alpha 2^{-\alpha k} \leq f_k^n(x) \leq z + \epsilon_n b + n|f|_\alpha 2^{-\alpha k}\right\} \\ &\leq \mu\left\{x \in \Sigma : \ z + \epsilon_n a \leq f^n(x) \leq z + \epsilon_n b\right\} \\ &\leq \mu\left\{x \in \Sigma : \ z + \epsilon_n a - n|f|_\alpha 2^{-\alpha k} \leq f_k^n(x) \leq z + \epsilon_n b + n|f|_\alpha 2^{-\alpha k}\right\}.\end{aligned} \tag{3.2}$$

This implies the required result. □

Thus we see that establishing asymptotic results for f^n suffices to show the required results for f_k^n, where $k = k(n)$ is as defined before. The basic idea is to show that γ sufficiently large gives rise to a suitable δ for the first and last terms in (3.2) are asymptotic (with an expression involving constants in terms of f, and independent of n).

In order to introduce transfer operators, we first want to shift each truncated function f_k into a function depending on the co-ordinates $x_0, x_1, x_2, \ldots$. More precisely, we shall write $\widetilde{f}_k := f_k \circ \sigma^k \in C^\alpha(\Sigma^+)$, for $k \geq 0$. Since μ is σ-invariant we can write

$$\begin{aligned}&\mu\left\{x \in \Sigma : \ z + \epsilon_n a \leq f_k^n(x) \leq z + \epsilon_n b\right\} \\ &= \mu\left\{x \in \Sigma^+ : \ z + \epsilon_n a \leq \widetilde{f}_k^n(x) \leq z + \epsilon_n b\right\},\end{aligned}$$

for each $k \geq 0$.

As in section 2, we need to introduce a sufficiently differentiable test function $\chi : \mathbb{R} \to \mathbb{R}$. By replacing f by $\widetilde{f}_k$ in (2.1) we see that we want to estimate

$$\rho_k(n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int \mathcal{L}^n_{g+iu\widetilde{f}_k} 1(x) d\mu(x) \right) \widehat{\chi}_n(u) du. \tag{3.3}$$

For future reference, we shall write $w_k := f_k + f_k \circ \sigma + \ldots + f_k \circ \sigma^{k-1}$. In particular, we have the trivial relation $\widetilde{f}_k = f_k + w_k \circ \sigma - w_k \in C^\alpha(\Sigma^+) \subset C^\beta(\Sigma^+)$, provided $\beta < \alpha$.

3.3 Introducing functions $\overline{f}_k$ on Σ^+. Although the shifted functions $\widetilde{f}_k$ give the above expression for $\rho_k(n)$, we have little a priori control over how the corresponding transfer operators $\mathcal{L}_{g+iu\widetilde{f}_k}$ behave as $k \to +\infty$. To address this problem, it is convenient to introduce a second sequence of better behaved functions $\overline{f}_k$, such that $\overline{f}_k$ is cohomologous to f_k, for each $k \geq 1$.

The properties of these new functions are described in the following simple lemma.

Lemma 3.2. *Let $\beta < \alpha$. There exists $\overline{f}, \overline{f}_k \in C^{\alpha/2}(\Sigma^+)$, for $k \geq 1$, and $u_{f_k} \in C^{\alpha/2}(\Sigma)$ and $C, C_0 > 0$ such that $\widetilde{f}_k = \overline{f}_k + u_{f_k} \circ \sigma - u_{f_k}$, where*

1. *$||\overline{f} - \overline{f}_k||_\infty \leq C|f|_\alpha 2^{-\alpha k}$, for $k \geq 1$;*
2. *$|\overline{f} - \overline{f}_k|_\beta \leq C|f|_\beta 2^{-(\alpha-\beta)k}$, for $k \geq 1$; and*
3. *$|u_f|_{\beta/2} \leq C_0|f|_\alpha$.*

Proof. Following [14] and [6], we may define a linear operator $\tau : C^\alpha(\Sigma) \to C^{\alpha/2}(\Sigma^+)$ such that $\tau : f \mapsto \overline{f} = f + u_f \circ \sigma - u_f$. More precisely, one can write $f(x) = \sum_{n=0}^\infty \mathfrak{f}_n(x)$ where $\mathfrak{f}_n(x) = \mathfrak{f}_n(x_{-n}, \dots, x_0, \dots, x_n)$ and $||\mathfrak{f}_n||_\infty \leq |f|_\alpha 2^{-\alpha n}$. Let

$$u_f(x) = \sum_{n=0}^\infty \sum_{m=0}^{n-1} \mathfrak{f}_n \circ \sigma^m$$

then one sees this is well-defined in $C^0(\Sigma^+)$ and

$$||u_f||_\infty \leq ||f||_\infty \sum_{n=0}^\infty n 2^{-\alpha n} \leq \frac{||f||_\infty}{(1-2^{-\alpha})^2} < +\infty.$$

To check the coboundary identity we see that

$$u_f \circ \sigma - u_f = \sum_{n=0}^\infty \mathfrak{f}_n \circ \sigma^n - \sum_{n=0}^\infty \mathfrak{f}_n = \overline{f} - f.$$

Parts (1) and (2) then come from the corresponding properties of f and $\mathfrak{f}_k$:

(i) $||f - \mathfrak{f}_k||_\infty \leq |f|_\alpha 2^{-\alpha k}$, for $k \geq 1$; and
(ii) $|f - \mathfrak{f}_k|_\beta \leq |f|_\beta 2^{-(\alpha-\beta)k}$, for $k \geq 1$.

These results can be found in [12].

To bound the norm of $u_f \in C^{\alpha/2}(\Sigma)$, assume that $x_i = y_i$, $i = -N, \dots, N$, then

$$\begin{aligned}
|u_f(x) - u_f(y)| &= \sum_{n=0}^\infty \sum_{m=0}^{n-1} |\mathfrak{f}_n(x_{-n+m}, \dots, x_{n+m}) - \mathfrak{f}_n(y_{-n+m}, \dots, y_{n+m})| \\
&\leq 2|f|_\alpha \sum_{n=[(N-1)/2]}^\infty \sum_{m=N-n}^{n} 2^{-\alpha(N-m)} \\
&\leq 2|f|_\alpha \left(\sum_{n=[(N-1)/2]}^\infty \frac{2^{-\alpha n/2}}{1-2^{-\alpha/2}} \right) \\
&\leq \left(\frac{2|f|_\alpha}{\left(1-2^{-\alpha/2}\right)^2} \right) 2^{-\alpha N/2}.
\end{aligned}$$

Thus we deduce that $|u_f|_{\alpha/2} \leq C_0 |f|_\alpha$, where $C_0 = 2(1-2^{-\alpha/2})^{-2}$. Since $|u_f|_{\beta/2} \leq |u_f|_{\alpha/2}$, this shows (3). □

Later, we shall want to choose a conveniently small value of $\beta > 0$.

3.4 Comparing transfer operators for $\widetilde{f}_k$ and $\overline{f}_k$. We now have two sequences of functions $\overline{f}_k, \widetilde{f}_k \in C^{\beta/2}(\Sigma^+)$, $k \geq 1$, each cohomologous to f_k, and we need to understand the coboundary that relates them. This is the purpose of the next lemma.

Lemma 3.3. *Let us denote $v_k := w_k + u_{f_k}$. Then $\widetilde{f}_k - \overline{f}_k = v_k \circ \sigma - v_k$ and $v_k \in C^{\beta/2}(\Sigma^+)$.*

Proof. Fix $k \geq 1$ then, a priori, we only know that $v_k \in C^{\beta/2}(\Sigma)$. However, since $\overline{f}_k - \widetilde{f}_k \in C^{\beta/2}(\Sigma^+)$ we can apply Livsic's theorem for periodic orbits [7, p.45] to

$\sigma : \Sigma^+ \to \Sigma^+$ to deduce that $v_k \circ \sigma - v_k = g_k \circ \sigma - g_k$, for some $g_k \in C^{\beta/2}(\Sigma^+)$. Thus $(v_k - g_k) \circ \sigma = (v_k - g_k)$, and by ergodicity (with respect to any Gibbs measure, say) we see that $v_k = g_k + K \in C^{\beta/2}(\Sigma^+)$, for some constant K. This completes the proof. □

We have now constructed two sequences of pairwise cohomologous functions $\widetilde{f}_k$ and $\overline{f}_k$, $k \geq 1$. The former are best suited for studying $\rho_k(n)$ in (3.3), but the latter are best suited for taking limits. The following is useful to relating the spectrum of the associated transfer operators.

Lemma 3.4. *The transfer operators* $\mathcal{L}_{g+iu\widetilde{f}_k}$ *and* $\mathcal{L}_{g+iu\overline{f}_k}$, $k \geq 1$, *are conjugate. In particular,*

$$\mathcal{L}_{g+iu\widetilde{f}_k} = \Delta(e^{iuv_k})\mathcal{L}_{g+iu\overline{f}_k}\Delta(e^{-iuv_k}),$$

where $\Delta(e^{iuv_k}) : C^{\beta/2}(\Sigma^+) \to C^{\beta/2}(\Sigma^+)$ *denotes multiplication by the function* $e^{iuv_k} \in C^{\beta/2}(\Sigma^+)$.

Proof. This follows immediately from the definition of the transfer operator and the identity $\widetilde{f}_k - \overline{f}_k = v_k \circ \sigma - v_k$ in Lemma 3.3. □

Thus, in order to compare bounds on the (common) spectra of the two transfer operators we need to understand better the conjugating operator. This is achieved using the following lemma.

Lemma 3.5. *There exists* $C_1 > 0$ *such that, for any* $h \in C^{\beta/2}(\Sigma^+)$, *we have the bounds* $||\Delta(e^{iuv_k})h||_\infty \leq ||h||_\infty$ *and* $|\Delta(e^{iuv_k})h|_{\beta/2} \leq C_1 2^{k\beta/2}|u||f|_\alpha|h|_\infty + |h|_{\beta/2}$.

Proof. The first inequality is obvious.

Using that $|h_1h_2|_{\beta/2} \leq |h_1|_{\beta/2}|h_2|_\infty + |h_2|_{\beta/2}|h_1|_\infty$ we can bound

$$\begin{aligned} |e^{iuv_k}h|_{\beta/2} &\leq |e^{iuv_k}|_{\beta/2}|h|_\infty + |e^{iuv_k}|_\infty|h|_{\beta/2} \\ &= |e^{iuv_k}|_{\beta/2}|h|_\infty + |h|_{\beta/2}. \end{aligned} \tag{3.4}$$

We therefore need to bound

$$|e^{iuv_k}|_{\beta/2} \leq |e^{iuu_{f_k}}|_{\beta/2} + |e^{iuw_k}|_{\beta/2} \leq |u|\left(|u_{f_k}|_{\beta/2} + |w_k|_{\beta/2}\right), \tag{3.5}$$

say. Observe here that whereas we can interpret $v_k \in C^{\beta/2}(\Sigma^+)$ (by Lemma 3.3), we still have to view $u_{f_k}, w_k \in C^{\beta/2}(\Sigma)$. However, this does not effect the bound on v_k.

Firstly, we can bound $|u_{f_k}|_{\beta/2} \leq C_0|f|_\alpha$ using Lemma 3.2. Secondly, from the definition of $|\cdot|_{\beta/2}$ we see that $|f_k \circ \sigma^i|_{\beta/2} = 2^{i\beta/2}|f_k|_{\beta/2}$, for $k \geq 1$. Thus, by the triangle inequality, we can also bound for each $k \geq 1$

$$\begin{aligned} |w_k|_{\beta/2} &= |f_k + f_k \circ \sigma + \ldots + f_k \circ \sigma^{k-1}|_{\beta/2} \\ &\leq |f_k|_{\beta/2} + |f_k \circ \sigma|_{\beta/2} + \ldots + |f_k \circ \sigma^{k-1}|_{\beta/2} \\ &\leq |f_k|_{\beta/2}\left(1 + \sum_{j=0}^{k-1} 2^{j\beta/2}\right) \\ &\leq \frac{2^{k\beta/2}}{2^{\beta/2}-1}|f|_{\beta/2} \end{aligned} \tag{3.6}$$

and thus by comparing (3.5) and (3.6) we get

$$|e^{iuv_k}|_{\beta/2} \leq \left(C_0 + \frac{2^{k\beta/2}}{2^{\beta/2}-1}\right)|u||f|_{\beta/2}$$

and since $|f|_{\beta/2} \leq |f|_\alpha$ the result follows from substituting into (3.4). □

3.5 Perturbation theory for u small. We return to the problem of relating the spectra of the two families of operators. This is easiest for the maximal eigenvalue. Since we now know that the functions $\overline{f}_k$ and $\widetilde{f}_k$ differ by a coboundary, it is immediate that the operators $\mathcal{L}_{g+iu\overline{f}_k}, \mathcal{L}_{g+iu\widetilde{f}_k} : C^{\beta/2}(\Sigma^+) \to C^{\beta/2}(\Sigma^+)$ have exactly the same maximal eigenvalues $e^{P(g+iu\overline{f}_k)} = e^{P(g+iu\widetilde{f}_k)}$ (provided $|u| < a$, where a is sufficiently small to make the "complex presssure" well-defined [7]).

By standard perturbation theory, both the maximal eigenvalue

$$C^{\beta/2}(\Sigma^+) \ni g_1 \mapsto e^{P(g_1)} \in \mathbb{R}$$

and the associated eigenprojection

$$C^{\beta/2}(\Sigma^+) \ni g_1 \mapsto \mathcal{P}_{g_1} \in B\left(C^{\beta/2}(\Sigma^+), C^{\beta/2}(\Sigma^+)\right),$$

(taking values in the space of bounded linear operators) are analytic [7]. Moreover, we have the following.

Lemma 3.6. *There exist $\epsilon > 0$ and $C > 0$ such that, for $||g_1||, ||g_2|| \leq \epsilon$, we have that*

(i) $||e^{P(g_1)} - e^{P(g_2)}|| \leq C||g_1 - g_2||$, *and*
(ii) $||\mathcal{P}_{g_1} - \mathcal{P}_{g_2}|| \leq C||g_1 - g_2||$,

where $||\cdot|| = |\cdot|_\infty + ||\cdot||_\infty$.

For $|u| < a$, can apply the bound in Lemma 3.6 for the maximal eigenvalue to bound

$$\begin{aligned} |e^{nP(g+iu\overline{f}_k)} - e^{nP(g+iu\overline{f})}| &= O\left(n||\overline{f} - \overline{f}_k||\right) \\ &= O\left(n|\overline{f}|_\beta \left(\frac{1}{2}\right)^{(\alpha-\beta)k}\right) \\ &= O\left(n^{1-(1-\frac{\beta}{\alpha})(1+\gamma)}|\overline{f}|_\beta\right), \end{aligned}$$

using Lemma 3.2 and the definition of $k = k(n)$. Thus, since the pressure in unchanged by adding coboundaries, we can estimate

$$\begin{aligned} e^{nP(g+iu\widetilde{f}_k)} &= e^{nP(g+iu\overline{f})} + \left(e^{nP(g+iu\overline{f}_k)} - e^{nP(g+iu\overline{f})}\right) \\ &= e^{nP(g+iu\overline{f})} + O\left(|\overline{f}|_\beta n^{1-(1-\frac{\beta}{\alpha})(1+\gamma)}\right) \\ &= e^{nP(g+iu\overline{f})} + O\left(\frac{1}{n}\right), \end{aligned} \tag{3.7}$$

provided $\gamma > 1$ is chosen sufficiently large and β is chosen sufficiently small. (For definiteness, if we assume $\beta < \alpha/2$ then it would suffice that $\gamma > 3$.) As in section

2, we can use the Morse lemma to change coordinates in a neigbourhood of the origin to $v = v(u)$, and write $e^{P(g+iu\overline{f})} = (1 - v^2 + iQ(v))$. In addition, using Lemma 3.6 (ii), we can estimate

$$\begin{aligned} \mathcal{P}_{g+iu\widetilde{f}_k} 1 &= 1 + O\left(\min\left\{1, |u||\widetilde{f}_k|_{\beta/2}\right\}\right) \\ &= 1 + O\left(\min\left\{1, |u|n^{\frac{\beta}{2\alpha}(1+\gamma)}\right\}\right), \end{aligned} \tag{3.8}$$

where we have used $|\widetilde{f}_k|_{\beta/2} \leq 2^{\beta k/2}|f_k|_{\beta/2} = O(2^{\beta k/2}) = O(n^{\frac{\beta}{2\alpha}(1+\gamma)})$, by definition of k.

We recall with the following standard result on the spectral gap for the transfer operators $\mathcal{L}_{g+iu\overline{f}_k}$.

Lemma 3.7. *We can choose* $0 < \theta < 1$ *and* $C > 0$ *such that*

$$||\mathcal{L}^n_{g+iu\overline{f}_k} - e^{nP(g+iu\overline{f}_k)}\mathcal{P}^n_{g+iu\overline{f}_k}|| \leq C\theta^n, \text{ for } n \geq 1,$$

whenever $|u| \leq a$ *and* $k \geq 1$.

We can use Lemmas 3.5 and 3.7 to get the slightly weaker bounds for $\mathcal{L}^n_{g+iu\widetilde{f}_k}$:

$$\begin{aligned} &||(\mathcal{L}^n_{g+iu\widetilde{f}_k} - e^{nP(g+iu\overline{f}_k)}\mathcal{P}_{g+iu\widetilde{f}_k})1||_\infty \\ &= ||\Delta(e^{iuv_k})\left(\mathcal{L}^n_{g+iu\overline{f}_k} - e^{nP(g+iu\overline{f}_k)}\mathcal{P}_{g+iu\overline{f}_k}\right)(e^{-iuv_k})||_\infty \\ &= ||\Delta(e^{iuv_k})||_\infty||\mathcal{L}^n_{g+iu\overline{f}_k} - e^{nP(g+iu\overline{f}_k)}\mathcal{P}_{g+iu\overline{f}_k}||||\Delta(e^{-iuv_k})|| \\ &= O\left(\theta^n 2^{\beta/2k}\right) = O\left(\theta^n n^{\frac{\beta}{2\alpha}(1+\gamma)}\right) \end{aligned}$$

where $||\Delta(e^{-iuv_k})|| = O\left(2^{\beta k/2}\right)$, by Lemma 3.5. Thus we can write

$$\mathcal{L}^n_{g+iu\widetilde{f}_k} 1 = \left(e^{nP(g+iu\overline{f})} + O\left(\frac{1}{n}\right)\right)\left(1 + O\left(|v|n^{\frac{\beta}{\alpha}(1+\gamma)}\right)\right) + O\left(\theta^n n^{\frac{\beta}{\alpha}(1+\gamma)}\right). \tag{3.9}$$

3.6 The integral for $|u|$ small. Using (3.9) we can now estimate the part of the integral (3.3) with $|u| < a$ by

$$\begin{aligned} &\frac{1}{2\pi}\int_{-a}^{a}\left(\int \mathcal{L}^n_{g+iu\widetilde{f}_k} 1(x)d\mu(x)\right)\widehat{\chi}_n(u)du \\ &= \int_{-a}^{a}(1 - v^2 + iQ(v))^n\left(1 + O\left(|v|n^{\frac{\beta}{2\alpha}(1+\gamma)}\right)\right)\widehat{\chi}_n(u(v))\frac{du}{dv}dv \\ &\qquad + O\left(\theta^n n^{\frac{\beta}{2\alpha}(1+\gamma)}\right) \\ &= \frac{\epsilon_n\widehat{\chi}(0)\sqrt{2}}{\sigma}\int_{-a}^{a}(1 - v^2 + iQ(v))^n\left(1 + O\left(|v|n^{\frac{\beta}{2\alpha}(1+\gamma)}\right)\right)dv \\ &\qquad + O\left(\frac{\epsilon_n}{n}\right) + O\left(\theta^n n^{\frac{\beta}{2\alpha}(1+\gamma)}\right). \end{aligned} \tag{3.10}$$

We can bound the error term coming from the integral by

$$n^{\frac{\beta}{\alpha}(1+\gamma)}\int_0^a v(1 - v^2)^n dv = O\left(n^{\frac{\beta}{\alpha}(1+\gamma)-1}\right).$$

In particular, can make all of the error terms of order $O\left(n^{-(\delta+\frac{1}{2})}\right)$ by choosing $\beta = \beta(\gamma) > 0$ sufficiently small. As in section 2, we see that the remaining principle asymptotic is $\widehat{\chi}(0)(\sqrt{2\pi}\sigma)^{-1}\epsilon_n n^{-1/2}$.

3.7 The integral for $|u|$ large. Finally, we want to bound

$$\int_{|u|\geq a}\left(\int \mathcal{L}_{g+iu\widetilde{f}_k}1d\mu\right)\widehat{\chi}_n(u)du \leq \int_{|u|\geq a}||\mathcal{L}_{g+iu\widetilde{f}_k}1||_\infty|\widehat{\chi}_n(u)|du. \tag{3.11}$$

Observe that for $|u|\geq a$ we can relate the operators associated to $\widetilde{f}_k$ and $\overline{f}_k$ by

$$\begin{aligned}||\mathcal{L}^n_{g+iu\widetilde{f}_k}1||_\infty &\leq ||\Delta(e^{iuv_k})||_\infty||\mathcal{L}^n_{g+iu\overline{f}_k}||||\Delta(e^{iuv_k})|| \\ &\leq \left(||\mathcal{L}^n_{g+iu\overline{f}}-\mathcal{L}^n_{g+iu\overline{f}_k}||+||\mathcal{L}^n_{g+iu\overline{f}}||\right)||\Delta(e^{iuv_k})||.\end{aligned} \tag{3.12}$$

Assuming χ is C^κ, say, the results in section 2 applied to $\overline{f}$, which also satisfies the Diophantine Condition, show that we can bound

$$\begin{aligned}\int_{|u|\geq a}||\mathcal{L}_{g+iu\overline{f}}1|||\widehat{\chi}_n(u)|du &= O\left(n^{-(1-\kappa)(\delta'-\delta)}2^{\beta k/2}\right) \\ &= O\left(n^{-(1-\kappa)(\delta'-\delta)}n^{(1+\gamma)\beta/(2\alpha)}\right),\end{aligned} \tag{3.13}$$

where $\delta<\delta'<\frac{1}{\gamma}$, say. This contribution is of order $O(n^{-(\delta+\frac{1}{2})})$ if $\kappa>0$ is sufficiently large and then $\beta=\beta(\kappa,\gamma)>0$ is chosen sufficiently small.

To bound the difference between (3.11) and (3.13) we shall use the following trivial identity

$$\mathcal{L}^n_{g+iu\overline{f}}-\mathcal{L}^n_{g+iu\overline{f}_k}=\sum_{j=1}^{n-1}\mathcal{L}^j_{g+iu\overline{f}}\left(\mathcal{L}_{g+iu\overline{f}}-\mathcal{L}_{g+iu\overline{f}_k}\right)\mathcal{L}^{(n-j)}_{g+iu\overline{f}_k}. \tag{3.14}$$

By Lemma 3.6, we can bound

$$||\mathcal{L}_{g+iu\overline{f}}-\mathcal{L}_{g+iu\overline{f}_k}||\leq C|u|||\overline{f}-\overline{f}_k||, \tag{3.15}$$

provided $|u|$ is sufficiently small. The following bound will also be quite useful.

Lemma 3.8. *There exists $C>0$ such that*

$$||\mathcal{L}^m_{g+iu\overline{f}}||\leq C|u| \text{ and } ||\mathcal{L}^m_{g+iu\overline{f}_k}||\leq C|u|, \text{ for } m\geq 1.$$

for all $k\geq 1$.

Let $\rho>0$ be a valus to be specified later. Using (3.14) we can bound

$$\begin{aligned}&\int_a^{n^\rho}||\mathcal{L}^n_{g+iu\overline{f}}-\mathcal{L}^n_{g+iu\overline{f}_k}||||\widehat{\chi}_n(u)||_\infty du \\ &\leq \int_a^{n^\rho}\left(\sum_{j=1}^{n-1}||\mathcal{L}^j_{g+iu\overline{f}}||_\infty||\mathcal{L}_{g+iu\overline{f}}-\mathcal{L}_{g+iu\overline{f}_k}||||\mathcal{L}^{(n-j)}_{g+iu\overline{f}_k}||||e^{iuv_k}||\right)||\widehat{\chi}_n(u)||_\infty du.\end{aligned} \tag{3.16}$$

Furthermore, we can use Lemmas 3.2, 3.5 and 3.8 to obtain a bound for (3.16) of order

$$\begin{aligned} &O\left(\frac{2^{k\beta/2}}{\epsilon_n^{\kappa-1}}(n-1)\int_a^{n^\rho} |u|^2||\overline{f}-\overline{f}_k||u^{-\kappa}du\right) \\ &= O\left(2^{k\beta/2}n^{(\kappa-1)\delta}n^{1+\rho}\left(\frac{1}{2}\right)^{k(\alpha-\beta/2)}\right) \\ &= O\left(n^{(1+\gamma)\beta/(2\alpha)}n^{(\kappa-1)\delta}n^{1+\rho}n^{-(1+\gamma)(1-\beta/(2\alpha))}\right). \end{aligned} \tag{3.17}$$

The remaining contribution to the integral can be bounded as

$$\begin{aligned} \int_{n^\rho}^\infty ||\mathcal{L}^n_{g+iu\overline{f}}-\mathcal{L}^n_{g+iu\overline{f}_k}||\ |\widehat{\chi}_n(u)|du &\leq \int_{n^\rho}^\infty 2C|u|\ |\widehat{\chi}_n(u)|du \\ &= O\left(\frac{1}{\epsilon_n^{\kappa-1}}\int_{n^\rho}^\infty \frac{1}{u^{\kappa-1}}du\right) \\ &= O\left(\frac{n^{\delta(\kappa-1)}}{n^{\rho(\kappa-2)}}\right) \end{aligned} \tag{3.18}$$

We can first choose $\rho = \rho(\kappa)$ sufficiently large that the contribution of (3.18) is $O(n^{-(\delta+\frac{1}{2})})$. If we then assume that $\gamma = \gamma(\kappa,\rho)$ is sufficiently large the contribution from (3.17) is of the same size. (If $\beta < \alpha/2$ then γ can be chosen independently of β. This is necessary, since earlier in the proof we need to choose $\beta = \beta(\gamma)$ small.) This finally completes the proof of Theorem 2.

4. Periodic orbits

In this final section, we shall sketch the proof of a version of Theorems 1 and 2 for sums over periodic points. In this case the result holds for hyperbolic diffeomorphisms, as well as expanding maps. We shall suppose that $T : X \to X$ is either the restriction of an Axiom A diffeomorphism to a non-trivial basic set or an expanding map. As above, we additionally assume that T is topologically mixing.

Let f be a non-lattice function. Suppose that there exists a Gibbs state m such that $\int f dm = 0$. Without loss of generality, we may choose the measure m to be the Gibbs state for a function $g = \xi f$, for some unique $\xi \in \mathbb{R}$ [8, Lemma 5]. With this choice, the supremum

$$\beta = \sup\left\{h(\nu) : \nu \in \mathcal{M} \text{ such that } \int f d\nu = 0\right\}$$

is attained at $\nu = m$, i.e., $\beta = h(m)$.

We recall the following asymptotic formula proved in [8].

Proposition 4.1. *Let f be a non-lattice function. Suppose that there exists a Gibbs state m such that $\int f dm = 0$. Then, for any real numbers $a < b$, we have that*

$$\#\{x \in \mathrm{Fix}(T^n) :\ a \leq f^n(x) \leq b\} \sim \frac{1}{\sqrt{2\pi}\sigma}\left(\int_a^b e^{-\xi t}dt\right)\frac{e^{\beta n}}{\sqrt{n}}, \tag{4.1}$$

as $n \to +\infty$, where ξ is defined as above.

We shall strengthen (4.1) to give an asymptotic formula when the interval $[a, b]$ is allowed to shrink at a subexponential rate.

Theorem 3. *Suppose that $f : X \to \mathbb{R}$ is a Hölder continuous function satisfying the Diophantine condition and that $\int f dm = 0$, where m is the Gibbs state for ξf. Then there exists $\delta > 0$ such that for any $z \in \mathbb{R}$, $a < b$ and sequence $\epsilon_n > 0$ which tends to zero and satisfies $\epsilon_n^{-1} = O(n^\delta)$, we have that*

$$\#\left\{x \in \mathrm{Fix}(T^n) \ : \ z + a\epsilon_n \leq f^n(x) \leq z + \epsilon_n b\right\} \sim \frac{1}{\sqrt{2\pi}\sigma}\left(\int_a^b e^{-\xi t} dt\right)\frac{\epsilon_n e^{\beta n}}{\sqrt{n}},$$

as $n \to +\infty$.

The proof follows the same lines as that of Theorem 1. The diffeomorphism $T : X \to X$ is modelled by a two-sided subshift of finite type $\sigma : \Sigma \to \Sigma$, where

$$\Sigma = \{x = (x_n)_{n=-\infty}^{\infty} \ : \ A(x_n, x_{n+1}) = 1 \ \forall n \in \mathbb{Z}\}.$$

The correspondence between periodic points for $T : X \to X$ and $\sigma : \Sigma \to \Sigma$ is not one-to-one but this discrepancy does not effect the asymptotics.

In order to use the transfer operator analysis from the earlier sections, one may pass from $\sigma : \Sigma \to \Sigma$ to the one-sided subshift $\sigma : \Sigma^+ \to \Sigma^+$. To do this, we apply the following standard lemma [14] (cf. Lemma 3.2).

Lemma 4.1. *Let $f : \Sigma \to \mathbb{R}$ be a Hölder continuous function. Then there exists a Hölder continuous function $f' : \Sigma \to \mathbb{R}$ which is cohomologous to f and has the property that $f'(x) = f'(y)$ if $x_n = y_n$ for $n \geq 0$. In particular, we may regard f' as a function $f' : \Sigma^+ \to \mathbb{R}$.*

For periodic orbits, Lemma 1.3 is replaced by the following result, which can be easily deduced from results in [13].

Lemma 4.2. *There exists $0 < \theta < 1$ such that, for any $x_0 \in \Sigma^+$,*

$$\sum_{\sigma^n x = x} e^{(\xi+iu)f^n(x)} = (\mathcal{L}^n_{(\xi+iu)f}1)(x_0)(1 + O(\max\{1, |u|\}n\theta^n)).$$

To to proceed we can use the identity

$$\begin{aligned}\rho(n) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\sum_{\sigma^n x = x} e^{iuf^n(x)}\right)\widehat{\chi}_n(u)du \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left((\mathcal{L}^n_{(\xi+iu)f}1)(x_0)(1 + O(\max\{1, |u|\}n\theta^n))\right)\widehat{\chi}_n(u)du.\end{aligned} \tag{4.2}$$

The proof now follows the same lines as Theorem 1. Using the bounds on the transfer operators from section 2, the same arguments give the estimate

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left((\mathcal{L}^n_{(\xi+iu)f}1)(x_0)\right)\widehat{\chi}_n(u)du \sim \frac{\int \chi(x)dx}{\sqrt{2\pi}\sigma}\frac{\epsilon_n}{\sqrt{n}}, \text{ as } n \to +\infty.$$

The contribution of the second term on the Right Hand Side of (4.2) is dominated by this principal term because of the θ^n factor.

Finally, the proof of Theorem 3 is completed by an approximation argument, as at the end of section 2.

Remark. The method of proof of the results in this paper should also lend itself to proving uniform "local limit theorems" for shrinking intervals (cf. [11]).

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Mark Pollicott, Department of Mathematics, University of Manchester, Oxford Road, Manchester, M13 9PL, U.K.

Current address: (From 1/10/04) Mathematics Institute, University of Warwick, Coventry, CV4 7AL, U.K.

Richard Sharp, Department of Mathematics, University of Manchester, Oxford Road, Manchester, M13 9PL, U.K.