A WEIL-PETERSSON TYPE METRIC ON SPACES OF METRIC GRAPHS

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Abstract. In this note, we discuss an analogue of the Weil-Petersson metric for spaces of metric graphs and some of its properties.

0. Introduction

Given a compact topological surface V with negative Euler characteristic, the Teichmüller space $\mathrm{Teich}(V)$ describes the marked Riemann metrics (with constant curvature $\kappa = -1$) which it supports. The moduli space $\mathrm{Mod}(V)$ describes the unmarked Riemann metrics on V and is obtained by quotienting $\mathrm{Teich}(V)$ by the Mapping Class Group of V.

There are several different metrics which can naturally be defined on $\mathrm{Teich}(V)$, for example, the Teichmüller metric and the Weil-Petersson metric, both of which are invariant under the Mapping Class Group and descend to $\mathrm{Mod}(V)$. There is a particularly illuminating formulation of the Weil-Petersson metric, due to Thurston and Wolpert, in terms of the second derivative of the length of a typical geodesic on V [Wo]. This was extended from Fuchsian to quasi-Fuchsian groups by Bridgeman and Taylor [BT]. A more dynamical characterization of this was proposed by Curt McMullen, who thereby extended the notion of the Weil-Petersson metric to a variety of settings (e.g., Fuchsian and quasi-Fuchsian groups, Blaschke products) in a unified way [Mc]. In this note we will introduce an analogue of the Weil-Petersson metric for families of metric graphs, and explore its properties through some simple examples.

To formulate an analogous definition for families of metric graphs we can replace the surface V by a finite (undirected) graph $\mathcal{G}$ with edge set $\mathcal{E}$. We can replace the Riemann metrics by edge weightings $l : \mathcal{E} \to \mathbb{R}^+$.

Definition. Let $\mathcal{M}_{\mathcal{G}}$ denote the space of all edge weightings $l : \mathcal{E} \to \mathbb{R}^+$ on $\mathcal{G}$.

Of course, the constant curvature -1 condition gives a natural normalization to metrics in $\mathrm{Teich}(V)$ or $\mathrm{Mod}(V)$ and it is natural to introduce a constraint on edge weightings on $\mathcal{G}$. One normalization (corresponding to curvature -1 metrics giving constant area $-2\pi\chi(V)$) would be that the sum of the edge lengths is equal to one, i.e., $\sum_{e\in\mathcal{E}} l(e) = 1$. However, the more dynamical approach described below is useful.

To motivate this approach, let us first consider a (not necessarily constant) negative curvature Riemannian metric g on on the surface V. Then, with respect to this metric, there are a countable infinity of closed geodesics γ with least period $l(\gamma)$. We can use these to define the *entropy* $h(g)$ of the metric (i.e. the topological entropy of the associated geodesic flow) by

$$h(g) = \lim_{t\to+\infty} \frac{1}{t} \log \mathrm{Card}\{\gamma : l(\gamma) \le t\}. \tag{0.1}$$

Of course, for Riemann surfaces of constant curvature $\kappa < 0$, we have that $h(g) = \sqrt{|\kappa|}$ and therefore, if we normalize the surfaces to have $\kappa = -1$, then we have that $h = 1$. By analogy with (0.1), we may

We would like to thank Lien-Yung Kao for pointing out some errors in the calculations in an earlier version.

normalize edge weightings on a graph $\mathcal{G}$ by their *entropy* characterized as, say, the rate of growth of closed loops, i.e.

$$h(l) = \lim_{t\to+\infty} \frac{1}{t} \log \mathrm{Card}\{\gamma : l(\gamma) \leq t\}, \tag{0.2}$$

where $\gamma = (e_0, e_1, \cdots, e_n = e_0)$ is a closed cycle of edges in $\mathcal{G}$ (without backtracking) and $l(\gamma) = \sum_{i=0}^{n-1} l(e_i)$.

Remark. Equivalently, $h(l)$ can be interpreted as the asymptotic volume growth of the graph, and thus depends only on edge lengths of the metric tree $\mathcal{T}$ which is the universal cover of the graph $\mathcal{G}$, i.e., if $\Gamma \cong \pi_1(\mathcal{G})$ is the covering group and $v \in \mathcal{T}$ is any vertex then

$$h(l) = \lim_{t\to+\infty} \frac{1}{t} \log \mathrm{Card}\{g \in \Gamma : d(gv, v) \leq t\}.$$

We shall only consider connected graphs which are non-trivial (i.e. which contain at least two distinct closed paths) and strictly positive length functions. In this case, we always have $h(l) > 0$.

The entropy can be used to define to the following dynamical normalization of edge weightings on graphs.

Definition. For fixed $h > 0$, let

$$\mathcal{M}_{\mathcal{G}}^h = \{l : \mathcal{E} \to \mathbb{R}^+ : h(l) = h\},$$

denote the space of all edge weightings with entropy $h(l) = h$. We shall concentrate attention on the space $\mathcal{M}_{\mathcal{G}}^1$.

Since we do not fix a marking on $\mathcal{G}$ (i.e. a homotopy equivalence to the graph with one vertex and $r = \mathrm{rank}(\pi_1(\mathcal{G}))$ edges), $\mathcal{M}_{\mathcal{G}}^1$ is an analogue of $\mathrm{Mod}(V)$, rather than $\mathrm{Teich}(V)$. (A more precise analogue of $\mathrm{Mod}(V)$ is the complex obtained by attaching spaces $\mathcal{M}_{\mathcal{G}}^1$ for graphs $\mathcal{G}$ with a given fundamental group.) The boundary $\mathcal{M}_{\mathcal{G}}^1$ corresponds to the situation where one or more of the edge lengths becomes equal to zero.

In §1 we recall the definition of Wolpert and McMullen of the Weil-Petersson metric. In §2 we give a definition of an analogue of the Weil-Petersson metric for graphs. In §3 we describe a number of the metric's properties, which help to illustrate the usefulness of the definition. In §4 and §5 we illustrate the definition for a variety of examples of graphs whose fundamental group is the free group on 2 generators. This makes an interesting connection with the Culler-Vogtmann space [CV, Vo], also known as the *outer space*, in rank 2.

1. The Weil-Petersson Metric on Moduli Space

We begin by reviewing some of the results of Wolpert, Bridgeman and Taylor, and McMullen for Riemann surfaces which are relevant to our analysis.

Let $\mathrm{Mod}(V)$ be the moduli space of Riemann metrics for a compact surface. We can consider a C^1 family of metrics $g_\lambda \in \mathrm{Mod}(V)$, $0 \leq \lambda \leq 1$. Let SV be the unit tangent bundle of V with respect to the metric g_{λ_0}, say. Let μ_{λ_0} be the corresponding Haar measure on SV. We denote by $\phi_t^{(\lambda_0)} : SV \to SV$ the geodesic flow. Since the geodesic flow for g_λ is volume preserving we have

$$\int \dot{g}_{\lambda_0}(v, v)\, d\mu_{\lambda_0}(v) = 0, \tag{1.1}$$

(cf. [Be]), where $\dot{g}_{\lambda_0}$ is defined via the expansion

$$g_\lambda = g_{\lambda_0} + \dot{g}_{\lambda_0}(\lambda - \lambda_0) + O((\lambda - \lambda_0)^2).$$

Definition. If we write $F(v) := \dot{g}_{\lambda_0}(v, v)$ we can define the *variance* by the following equivalent formulae

$$\sigma^2 = \sigma^2(F) := \lim_{t\to+\infty} \frac{1}{t} \int \left(\int_0^t F(\phi_u v)\, du \right)^2 d\mu(v) = \int_{-\infty}^{+\infty} \left(\int_{SV} F(\phi_t v) F(v)\, d\mu(v) \right) dt$$

(cf. [KS]).

We are now able to formulate the characterization of the Weil-Petersson metric by McMullen [Mc].

Proposition 1.1 (McMullen). *The Weil-Petersson metric is proportional to $\sigma^2(\cdot)$. More precisely,*

$$\sigma^2(F) = \frac{4}{3}\frac{\|g_\lambda\|^2_{WP}}{\mathrm{area}(V, g_\lambda)}.$$

Proposition 1.1 comes from a reinterpretation of Wolpert's formula for the Weil-Petersson metric in terms of the second derivative of lengths of generic geodesics. A simple interpretation for σ^2 is via the Central Limit Theorem, which we can express in terms of closed orbits. Let γ be a free homotopy class and let $l_g(\gamma)$ denote its length with respect to the metric g_λ.

Proposition 1.2 (Central Limit Theorem for surfaces). *For $a < b$ we have*

$$\lim_{t\to+\infty} te^{-t}\mathrm{Card}\left\{\gamma : l_{g_{\lambda_0}}(\gamma) \le t \text{ and } \frac{1}{\sqrt{t}}\left.\frac{\partial \log l_{g_\lambda}(\gamma)}{\partial\lambda}\right|_{\lambda=\lambda_0} \in [a,b]\right\}$$
$$= \frac{1}{\sqrt{2\pi}\sigma}\int_a^b e^{-y^2/(2\sigma^2)}\,dy.$$

Proof. This follows from standard results (cf. [La] and [Ra]) once we also observe that

$$\left.\frac{\partial \log l_{g_\lambda}(\gamma)}{\partial\lambda}\right|_{\lambda=\lambda_0} = \frac{1}{l_{g_{\lambda_0}}(\gamma)}\left.\frac{\partial l_{g_\lambda}(\gamma)}{\partial\lambda}\right|_{\lambda=\lambda_0} = \frac{1}{l_{g_{\lambda_0}}(\gamma)}\int_0^{l_{g_{\lambda_0}}(\gamma)} \dot{g}_{\lambda_0}(\phi_t^{(\lambda_0)}v_\gamma)\,dt,$$

where $v_\gamma \in T_\gamma V$. $\square$

Remark. There are several alternative equivalent definitions of σ^2. It can be interpreted as the value at zero of the spectral density. It can also be written in terms of an asymptotic quantity for lengths of weighted closed geodesics, and thus in terms of variants of zeta functions.

2. The Weil-Petersson Type Metric on Spaces of Metric Graphs.

We can associate to the graph $\mathcal{G}$ a subshift of finite type whose states are oriented edges of $\mathcal{G}$. This is constructed as follows. Each edge $e \in \mathcal{E}$ corresponds to two oriented edges which, abusing notation, we shall denote by e and $\overline{e}$. We shall write $\mathcal{E}^o$ for the set of oriented edges. We say that $e' \in \mathcal{E}^o$ follows $e \in \mathcal{E}^o$ if e' begins at the terminal endpoint of e. We then define a $|\mathcal{E}^o| \times |\mathcal{E}^o|$ matrix A, with rows and columns indexed by $\mathcal{E}^o$, by

$$A(e, e') = \begin{cases} 1 \text{ if } e' \text{ follows } e \text{ and } e' \neq \overline{e} \\ 0 \text{ otherwise.} \end{cases}$$

The shift space

$$\Sigma_A = \{\underline{e} = (e_n)_{n\in\mathbb{Z}} \in (\mathcal{E}^o)^{\mathbb{Z}} : A(e_n, e_{n+1}) = 1\ \forall n \in \mathbb{Z}\}$$

can be naturally identified with the space of all (two-sided) infinite paths (with a distinguished zeroth edge) in the graph $\mathcal{G}$. Then Σ_A is a compact zero dimensional space with respect to the Tychonoff product topology. We define the shift map $T : \Sigma_A \to \Sigma_A$ by $(T\underline{e})_n = e_{n+1}$, $n \in \mathbb{Z}$. Clearly, this is a homeomorphism.

Given any continuous function $f : \Sigma_A \to \mathbb{R}$ we can define the *pressure function* $P : C(\Sigma_A, \mathbb{R}) \to \mathbb{R}$ by

$$P(f) = \limsup_{n\to+\infty}\frac{1}{n}\log\left(\sum_{T^n x = x} e^{f(x)+f(Tx)+\cdots+f(T^{n-1}x)}\right).$$

We can associate to the oriented edges the weightings of the corresponding (unoriented) edges in the original graph. This leads to a locally constant function $l : \Sigma_A \to \Sigma_A$ defined by $l((e_n)_{n=-\infty}^{\infty}) = l(e_0)$. This function satisfies $l \circ i = l$ under the involution $i : \Sigma_A \to \Sigma_A$ given by $i((e_n)_{n=-\infty}^{\infty}) = (\overline{e_{-n}})_{n=-\infty}^{\infty}$, where $\overline{e}$ corresponds to e with the orientation reversed.

The following results on the function l are easily seen.

Lemma 2.1.

(1) *The pressure function is analytic on locally constant functions.*
(2) *The entropy h is characterized by $P(-h(l)l) = 0$.*
(3) *The entropy $\mathcal{M}_{\mathcal{G}} \ni l \mapsto h(l) \in \mathbb{R}^+$ varies analytically for $l > 0$.*

Proof. For part (1), the analyticity of the pressure is well known [PP].

To prove part (2), the entropy $h(l)$ can be interpreted as the entropy of the suspended flow over Σ_A with roof function l. Hence, using Abramov's theorem [Ab], it may be rewritten as

$$h(l) = \sup\left\{\frac{h(m)}{\int f\, dm} : m \text{ is a } T\text{-invariant probability measure}\right\},$$

where $h(m)$ denotes the entropy of $T : \Sigma_A \to \Sigma_A$ with respect to m (cf. [PP]). The first result now follows from the variational principle for pressure:

$$P(-tl) = \sup\left\{h(m) - t\int l\, dm : m \text{ is a } T\text{-invariant probability measure}\right\},$$

for $t \in \mathbb{R}$ [Wa].

Finally, for part (3) the result follows by the analyticity of the pressure function in part (1) and the implicit function theorem. □

We can define an analogue of the tangent space (at $l \in \mathcal{M}_{\mathcal{G}}$) to be a subspace of the locally constant functions (depending only on the zeroth coordinate e_0). By definition, the matrix $A_{h(l)l}$ given by $A_{h(l)l}(e,e') = A(e,e')e^{-h(l)l(e)}$ has spectral radius 1 and, by the Perron-Frobenius Theorem, has a strictly positive right eigenvector v such that $A_{h(l)l}v = v$. Then the matrix Q defined by $Q(e,e') = A_{h(l)l}(e,e')v_{e'}/v_e$ is stochastic. Let the probability vector p be a left eigenvector for Q and let μ be the associated Markov measure.

Definition. We define the *tangent space* to $\mathcal{M}_{\mathcal{G}}^h$ at $l \in \mathcal{M}_{\mathcal{G}}$ by

$$T_l\mathcal{M}_{\mathcal{G}}^h = \left\{f(\underline{e}) = f(e_0) : f(\overline{e}) = f(e) \text{ and } \sum_{e\in\mathcal{E}^o} f(e)p_e = 0\right\},$$

i.e., the signed edge weightings (depending only on the unoriented edges) whose appropriately weighted sum is zero. This is a finite dimensional space (indeed the dimension is precisely $|\mathcal{E}| - 1$).

Alternatively, we can associate to l the measure $\mu = \mu_l$ on Σ corresponding to the equilibrium state for $-h(l)l$, i.e., the unique T-invariant probability measure μ on Σ_A such that

$$P(-h(l)l) = h(\mu) - h(l)\int l\, d\mu.$$

Then we can write $T_l\mathcal{M}_{\mathcal{G}}^h = \{f : \Sigma_A \to \mathbb{R} : f(\underline{e}) = f(e_0), f\circ i = f, \text{ and } \mu(f) = 0\}$.

Definition. Given $f \in T_l\mathcal{M}_{\mathcal{G}}^h$ we define the *variance* by

$$\sigma^2(f) = \lim_{n\to+\infty}\frac{1}{n}\int (f^n(\underline{e}))^2\, d\mu(\underline{e}).$$

(This is strictly positive on non-zero functions since the condition $f\circ i = f$ forces any coboundary in $T_l\mathcal{M}_{\mathcal{G}}^h$ to be zero.)

There are several different ways to rewrite the variance. To begin with, it can be written in terms of the second derivative of the pressure (cf. [PP]).

Lemma 2.2. *Given $f \in T_l\mathcal{M}_{\mathcal{G}}^h$, we can write*

$$\sigma^2(f) = \frac{d^2}{dt^2} P(-(l+tf))\bigg|_{t=0}.$$

Since we are dealing with functions that depend only on one coordinate the formula for the variance is particularly simple.

Lemma 2.3. *Given $f \in T_l\mathcal{M}_{\mathcal{G}}^h$ we can write*

$$\sigma^2(f) = \sum_{e\in\mathcal{E}^o} p_e f(e)^2.$$

Proof. We can write

$$\begin{aligned}
&\frac{1}{n}\int\left(\sum_{i=0}^{n-1} f(e_i)\right)^2 d\mu(\underline{e}) = \frac{1}{n}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\int f(e_i)f(e_j)\,d\mu(\underline{e})\\
&= \frac{1}{n}\sum_{i=0}^{n-1}(n-i)\int f(e_0)f(e_i)\,d\mu(\underline{e})\\
&= \int |f(e_0)|^2\,d\mu(\underline{e}) + \frac{1}{n}\sum_{k=1}^{n}(n-k)\sum_{e_0}\sum_{i_1\cdots i_{k-1}}\sum_{e_k}\mu[e_0,i_1,\cdots,i_{k-1},e_k]f(e_0)f(e_k)\\
&= \int |f(e_0)|^2\,d\mu(\underline{e}) + \frac{1}{n}\sum_{e_0} p_{e_0}f(e_0)\sum_{k=1}^{n}(n-k)\sum_{i_1\cdots i_{k-1}}\sum_{e_k}P(e_0,i_1)\cdots P(i_{k-1},e_k)f(e_k)\\
&= \int |f(e_0)|^2\,d\mu(\underline{e}) + \frac{1}{n}\sum_{e_0} p_{e_0}f(e_0)\sum_{k=1}^{n}(n-k)\sum_{e_k}P^n(e_0,e_k)f(e_k)\\
&= \int |f(e_0)|^2\,d\mu(\underline{e}) + \frac{1}{n}\left(\sum_{e_0} p_{e_0}f(e_0)\right)\sum_{k=1}^{n}(n-k)\left(\sum_{e_k\in\mathcal{E}^o} p_{e_k}f(e_k) + O(\theta^n)\right),
\end{aligned}$$

for some $0 < \theta < 1$, since $\sum_{e\in\mathcal{E}^o} p(e)f(e) = 0$. Taking the limit as $n \to +\infty$ gives the required formula. □

Finally, we are in a position to define the analogue of the Weil-Petersson metric in the context of weighted graphs.

Definition (Weil-Petersson metric for graphs). We can define a norm on the tangent space $T_l\mathcal{M}_{\mathcal{G}}^h$ by

$$\|f\|_{WP}^2 = \sigma^2(f),$$

where $f \in T_1\mathcal{M}_{\mathcal{G}}^h$. We can then define the length of any continuously differentiable curve $\gamma : [0,1] \to \mathcal{M}_{\mathcal{G}}^h$ by

$$L(\gamma) = \int_0^1 \|\dot\gamma\|_{WP}\,dt$$

and thus define a path space metric on $\mathcal{M}_{\mathcal{G}}^h$ by $d(l_1,l_2) = \inf_\gamma\{L(\gamma)\}$, where the infimum is taken over all continuously differentiable curves with $\gamma(0) = l_1$ and $\gamma(1) = l_2$.

Remark. We can also express the Weil-Petersson metric for graphs in terms of the zeta function for weighted graphs. More precisely, we can associate to the graph $\mathcal{G}$ and a family of length functions $l_\lambda : \mathcal{G} \to \mathbb{R}^+$, $\lambda_0 - \epsilon < \lambda < \lambda_0 + \epsilon$, the zeta functions

$$\zeta_\lambda(s) = \prod_\gamma \left(1 - e^{-s l_{\lambda_0,\lambda}(\gamma)}\right)^{-1} \ (s \in \mathbb{C}),$$

where: the product is over prime closed non-backtracking loops γ on the graph $\mathcal{G}$; $l_{\lambda_0,\lambda} = l_{\lambda_0} + (\lambda - \lambda_0)\dot{l}_{\lambda_0}$ is a linearized length function for l_λ; and $l_{\lambda_0,\lambda}(\gamma)$ is the weight on γ. This can be viewed as an extension of the Ihara zeta function for graphs. The zeta function has a simple pole at $h(l_{\lambda_0,\lambda})$ and is analytic for $\mathrm{Re}(s) > h(l_{\lambda_0,\lambda})$. Then the Weil-Petersson metric is given by the residue at $s = h = h(l_{\lambda_0})$ of the second logarithmic derivative of the zeta function, i.e.

$$\left\| \frac{\partial l_\lambda}{\partial \lambda}\bigg|_{\lambda=\lambda_0} \right\|_{WP}^2 = -\lim_{s \to h} (s-h) \left. \frac{\partial^2 \log \zeta_\lambda(s)}{\partial \lambda^2} \right|_{\lambda=\lambda_0}.$$

3. Properties of the Weil-Petersson Metric for Graphs

In this section we want to establish some of the basic properties of the Weil-Petersson metric for graphs. We begin with the following analyticity result.

Theorem 1 (Analyticity of metric). *The metric $\|\cdot\|_{WP}$ is real analytic on $\mathcal{M}_\mathcal{G}$.*

Proof. This follows from the definition of the metric in terms of $\sigma^2(\cdot)$, the interpretation of $\sigma^2(\cdot)$ in terms of the second derivative of pressure by Lemma 2.2, and the analyticity of the pressure function in Lemma 2.1 (1).

It is known that the Weil-Petersson metric on Teichmüller space is incomplete, but has strictly negative curvature. It is natural to ask if there are analogous results in the case of spaces of graphs. In this context we have the following:

Theorem 2 (Properties of the metric).
- (i) *There exist examples of graphs for which the metric $\|\cdot\|_{WP}$ is not complete.*
- (ii) *There exist examples of graphs for which the curvature of the metric $\|\cdot\|_{WP}$ takes both positive and negative values.*
- (iii) *There exist examples of graphs for which the metric is incomplete.*

We present explicit examples of the properties in (i) and (ii) in later sections.

By the metric being incomplete, we mean that we can find a geodesic path in $\mathcal{M}_\mathcal{G}^1$ which reaches the boundary $\partial\mathcal{M}_\mathcal{G}^1$ in a finite distance with respect to $\|\cdot\|_{WP}$. There is a simple principle for establishing Theorem 2 (i) in many examples. Consider a graph $\mathcal{G}$ and fix an edge e_0. Moreover, let us consider those $\mathcal{G}$ for which there is a subgraph $\mathcal{G}' = \mathcal{G} - \{e_0\} \subset \mathcal{G}$, after deleting an edge e_0, say, corresponding to a subshift of finite type with non-zero topological entropy. We can consider a family $l_\lambda \in \mathcal{M}_\mathcal{G}^1$, for $\lambda \in (0,1)$, which associates to e_0 (and $\overline{e_0}$) the length $l_\lambda(e_0) = \lambda$, and to all other edges a common length to ensure $h(l_\lambda) = 1$. Then the associated probability vector p_λ converges to a non-zero vector p_0 as λ tends to zero. In particular, using Lemma 2.3 we can see that along this path $\|\cdot\|_{WP}$ is equivalent to Euclidean distance and the path meets the boundary of $\partial\mathcal{M}_\mathcal{G}^1$ in a finite distance.[1]

A geodesic corresponds to a piecewise smooth curve, possibly with points, or even more generally segments, in the boundary. Let us consider a graph $\mathcal{G}$ for which deleting any edge still gives a subgraph corresponding to a subshift of finite type with non-zero topological entropy. We see using Lemma 2.3 that for any geodesic approaching the boundary $\|\cdot\|_{WP}$ is equivalent to Euclidean distance. Assume for a

[1]The same basic argument shows that for a positive measure set of points and directions, geodesics can reach the boundary in finite time. This confirms the fact that the geodesic flow cannot be ergodic.

contradiction that a geodesic γ contains a point in the boundary then γ would have to meet the boundary non-tangentially. However, it is then easy to see we could then cut the corner(s) to get a shorter curve, giving a contradiction. Finally, once we have established that γ is in the interior of the simplex (except possibly for endpoints on different points) geodesic convexity follows.

The following can be compared with Wolpert's Theorem [Wo] and gives, perhaps, the most intuitive definition of the Weil-Petersson metric in the present setting. Consider a family of closed paths $\{\gamma_n\}$ which become evenly distributed with respect to μ_{λ_0} (i.e. γ corresponds to $T^n \underline{e} = \underline{e}$ with $l_\lambda(\gamma) = \sum_{j=0}^{n-1} l_\lambda(e_j)$ and the measure $\frac{1}{n}\sum_{j=0}^{n-1} \delta_{T^j \underline{e}}$ converges in the weak* topology to μ_{λ_0} on Σ_A).

Theorem 3 (Random Geodesic Theorem). *Let $l_\lambda \in \mathcal{M}_\mathcal{G}$ be a family of length functions for $0 \leq \lambda \leq 1$. Then for $0 < \lambda_0 < 1$,*

$$\lim_{n \to +\infty} -\frac{d^2}{d\lambda^2} \log\left(l_\lambda(\gamma_n)\right)\Big|_{\lambda=\lambda_0} = \|v\|^2_{WP}$$

where $v = \dot{l}_{\lambda_0} \in T_{l_{\lambda_0}}\mathcal{M}_\mathcal{G}$ is the tangent vector at $\lambda = \lambda_0$.

Proof. At the symbolic level we can characterize $h(l)$ and μ in terms of the pressure $P(\cdot)$. In particular, differentiating the expression $P(-l_\lambda) = 0$ on $\mathcal{M}_\mathcal{G}$ we see that $\int \dot{l}_{\lambda_0}\, d\mu_{l_\lambda} = 0$. Differentiating again we see that $\mathrm{var}(\dot{l}_{\lambda_0}) + \int \ddot{l}_{\lambda_0}\, d\mu_{l_{\lambda_0}} = 0$. We can write that

$$\frac{d^2}{d\lambda^2} \log\left(l_\lambda(\gamma)\right)\Big|_{\lambda=\lambda_0} = \frac{d}{d\lambda}\left(\frac{\dot{l}_\lambda(\gamma)}{l_\lambda(\gamma)}\right)\Big|_{\lambda=\lambda_0} = \frac{\ddot{l}_{\lambda_0}(\gamma)}{l_{\lambda_0}(\gamma)} - \left(\frac{\dot{l}_{\lambda_0}(\gamma)}{l_{\lambda_0}(\gamma)}\right)^2.$$

For typical geodesics $\frac{\dot{l}_{\lambda_0}(\gamma_n)}{l_{\lambda_0}(\gamma_n)} \to 0$ and $\frac{\ddot{l}_{\lambda_0}(\gamma_n)}{l_{\lambda_0}(\gamma_n)} \to \int \ddot{l}_{\lambda_0}\, d\mu_{l_{\lambda_0}}$. □

4. Some Examples

In this section we will consider some simple examples. In particular, we consider the graphs whose fundamental group in F_2, the free group of rank 2. In subsequent sections we will use them to illustrate properties of the metric.

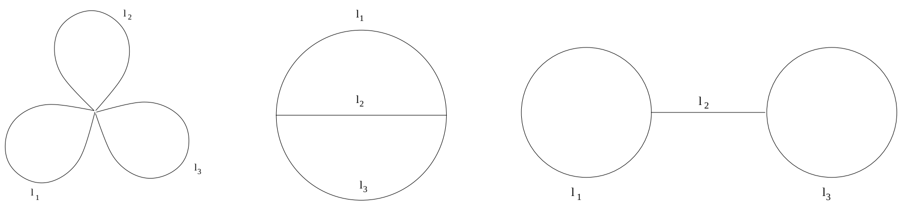

FIGURE 1. The three examples: (I) A rose (with $n = 2$); (II) A belt buckle; and (III) A dumbell

4.1. Example I (A rose).

A particularly simple example of an undirected graph is the rose with n-petals ($n \geq 2$), cf. Figure 1 (I).

Lemma 4.1. *The entropy h of the rose example is characterized by $\det(I - A_{h(l)l}) = 0$ where*

$$A_{-h(l)} = \left((2 - \delta_{ij})e^{-h(l)l_i}\right)_{ij=1}^n \quad \text{with } \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

Proof. The graph is regular with valency $2n$. Furthermore, since there is no doubling back allowed, a path arriving at any vertex has $(2n-1)$ allowed choices for the subsequent edges with all associated lengths appearing twice, except for that corresponding to the edge last traversed, which appears only once. The entropy $h(l)$ is then characterised as the value for which the associated matrix $A_{-h(l)}$ has 1 as a maximal eigenvalue (the remark after Lemma 2.1). $\square$

In particular, for the figure eight graph (corresponding to $n = 2$) with edge lengths $l_1, l_2 > 0$ the entropy $h = h(l)$ has to satisfy $1 = e^{-hl_1} + e^{-hl_2} + 3e^{-h(l_1+l_2)}$.

4.2. Example II (Belt buckles). We can consider a graph with two vertices, each connected by three edges, cf. Figure 1 (II). By shrinking any of the edges to a point we get the homotopy equivalent figure eight graph (i.e., a 2-rose, as above).

In the associated tree the valency is three and the three edges meeting at a vertex and have the three lengths $l_1, l_2, l_3 > 0$.

Lemma 4.2. *The entropy $h = h(l) > 0$ is characterised by*

$$e^{h(l_1+l_2+l_3)} = 2 + e^{hl_1} + e^{hl_2} + e^{hl_3}$$

In particular, when $h = 1$ we see that for $l = (l_1, l_2, l_3) \in \mathcal{M}_{\mathcal{G}}^1$ we require that $l_3 > 0$ satisfies

$$e^{-l_3} = \frac{e^{l_1+l_2} - 1}{2 + e^{l_1} + e^{l_2}}. \tag{4.1}$$

Proof. The graph is regular with valency 3. Since, as before, there is no doubling back allowed at vertices, the allowed paths have two choices of subsequent edges at each vertex. The associated matrix (whose rows and columns correspond to the three undirected edges) takes the form

$$A_{hl} = \begin{pmatrix} 0 & e^{-hl_1} & e^{-hl_1} \\ e^{-hl_2} & 0 & e^{-hl_2} \\ e^{-hl_3} & e^{-hl_3} & 0 \end{pmatrix}. \tag{4.2}$$

Then the entropy $h = h(l) > 0$ is characterised as the value for which the maximal eigenvalue λ_l of M_{hl} is equal to 1, as in the proof of Lemma 4.1. By solving for

$$0 = \det(I - A_{hl}) = -1 + e^{-h(l_1+l_2)} + e^{-h(l_2+l_3)} + e^{-h(l_3+l_1)} + 2e^{-h(l_1+l_2+l+3)}$$

and multiplying through by $e^{h(l_1+l_2+l_3)}$ the identity in the statement follows. $\square$

4.3. Example III (Dumbbells). We can consider a graph with two vertices, each being the start and end of an edge, and joined by a third edge, cf. Figure 1 (III).

Lemma 4.3. *The entropy $h = h(l)$ of the dumbbell example is characterised by*

$$4 = e^{2hl_2}(e^{hl_1} - 1)(e^{hl_3} - 1)$$

In particular, when $h = 1$ we see that for $l = (l_1, l_2, l_3) \in \mathcal{M}_{\mathcal{G}}^1$ we require that $l_3 > 0$ satisfies

$$e^{-l_3} = \frac{1}{2}\sqrt{(e^{hl_1} - 1)(e^{hl_3} - 1)}. \tag{4.1}$$

Proof. The graph is regular with three edges and valency 3. Let us associate the matrix

$$A_{hl} = \begin{pmatrix} e^{-hl_1} & 0 & e^{-hl_1} & 0 & 0 & 0 \\ 0 & e^{-hl_1} & e^{-hl_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-hl_2} & e^{-hl_2} \\ e^{-hl_2} & e^{-hl_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-hl_3} & e^{-hl_2} & 0 \\ 0 & 0 & 0 & e^{-hl_3} & 0 & e^{-hl_3} \end{pmatrix}. \tag{4.3}$$

whose rows and columns are indexed by the 6 directed edges. The entropy $h = h(l) > 0$ is characterised as the value for which the maximal eigenvalue λ_l of M_{hl} is equal to 1. In particular, we can solve

$$0 = \det(I - A_{hl}) = -(1 - e^{-hl_1})(1 - e^{-hl_3})(-1 + e^{-hl_1} + e^{-hl_3} - e^{-h(l_1+l_3)} + 4e^{-h(l_1+2l_2+l_3)}).$$

The last term must vanish, and by multiplying this by $e^{h(l_1+2l_2+l_3)}$ completes the proof of the lemma. □

By shrinking the edge joining the two vertices, we get the homotopy equivalent figure eight graph (i.e., a 2-rose, as in Example I above).

Remark. The three examples above occur very naturally in the context of the study of outer automorphisms of graphs. We recall that the Culler-Vogtmann space (or outer space) $CV(F_2)$ corresponds to metrics (i.e. length functions) on the three types of graph above in the same free homotopy class [CV, Vo].

The graphs in Examples II and III are parameterized by the three lengths $l_1, l_2, l_3 > 0$ of the edges and so, subject to the normalization that $h(l_1, l_2, l_3) = 1$, the corresponding moduli space is a two dimensional simplex. However, these are joined by one dimensional curves corresponding to the graphs in Example I (for $n = 2$). More precisely, simplicies corresponding to Example II are joined to other such simplices along all three edges, while simplices corresponding to Example III have one side joined to an Example I curve with the other two sides not joined (since these degenerations have a different fundamental group).

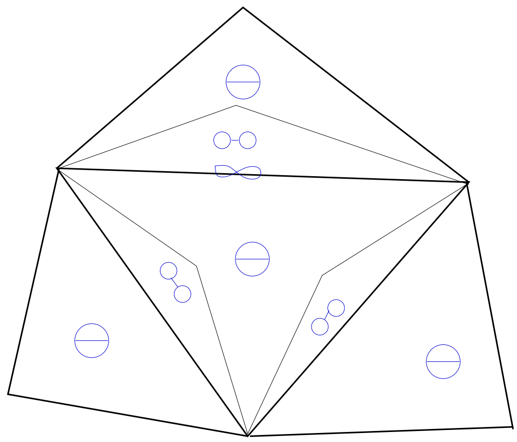

FIGURE 2. The three types of examples of graphs corresponding to $CV(F_2)$

5. Incompleteness and curvature

We can use the examples introduced in the previous section to provide those explicit examples needed for Theorem 2.

5.1. Incompleteness of the metric. To show that the metric can be incomplete we consider the particular case of Example I with $n = 2$ (i.e., where there are two undirected loops) and when $h(l) = 1$. The associated directed graph has a transition matrix and an associated weighted matrix given by

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} \text{ and } A_{hl} = \begin{pmatrix} e^{-hl_1} & e^{-hl_1} & 0 & e^{-hl_1} \\ e^{-hl_2} & e^{-hl_2} & e^{-hl_2} & 0 \\ 0 & e^{-hl_1} & e^{-hl_1} & e^{-hl_1} \\ e^{-hl_2} & 0 & e^{-hl_2} & e^{-hl_2} \end{pmatrix}.$$

Since we restrict to the case $h = 1$, we can write l_1 in terms of l_2 as

$$l_1(l_2) = -\log\left(\frac{1-e^{-l_2}}{1+3e^{-l_2}}\right). \tag{5.1}$$

The normalized left and right eigenvectors p and q of A_l associated to the eigenvalue 1 can be shown to be

$$p = (p_1, p_2, p_3, p_4) = \left(\frac{1+3e^{l_2}}{6+6e^{l_2}}, \frac{1}{3+3e^{l_2}}, \frac{1+3e^{l_2}}{6+6e^{l_2}}, \frac{1}{3+3e^{l_2}}\right)$$

and

$$q = (q_1, q_2, q_3, q_4)^T = \left(\frac{e^{l_2}-1}{2+e^{-l_2}}, \frac{2}{2+e^{-l_2}}, \frac{e^{l_2}-1}{2+e^{-l_2}}, \frac{2}{2+e^{-l_2}}\right)^T.$$

Using (5.1) we can consider the curve $c : (0, 1] \to \mathcal{M}$ defined by $c(l_2) = (l_1(l_2), l_2)$, parameterized by l_2. In particular, the derivative is

$$c'(l_2) := \left(\frac{4e^{l_2}}{(e^{l_2}-1)(3+e^{l_2})}, 1\right) \sim \left(\frac{1}{l_2}, 1\right) \text{ as } l_2 \to 0.$$

Moreover, one easily sees that

$$p_1 = p_3 \sim \frac{1+3e^{l_2}}{6+6e^{l_2}} \sim \frac{1}{3} \text{ and } q_1 = q_3 \sim \frac{e^{l_2}-1}{2+e^{-l_2}} \sim \frac{l_2}{3}, \text{ as } l_2 \to 0.$$

Thus, we see that the μ-measure of the loop of length $l_1 = l_1(l_2)$ is comparable with l_2. In particular, as $l_2 \to 0$ we see from Lemma 2.3 that the corresponding WP-length of the vector can be estimated by

$$\sqrt{\int \|c'\|^2 d\mu} \asymp \sqrt{p_1 q_1 / l_2^2} \asymp l_2^{-1/2}.$$

Finally, since $\int_0^1 x^{-1/2} dx$ is convergent we see that the metric is incomplete, i.e., the curve arrives at $l_2 = 0$ in finite time with respect to the metric.

5.2. Curvature of the metric. We now consider formulae for the Gaussian curvature of the metric for Example II, i.e., the belt buckle with two vertices and 3 edges, with lengths $l_1, l_2, l_3 > 0$.

We first associate to the matrix A_l a stochastic matrix P_l defined by $P_l(i,j) = A(i,j)p_j/p_i$, where $A_h p = p$ is the right eigenvector. In particular, we find that

$$\frac{p_1}{p_2} = \frac{1+e^{-l_3}}{e^{l_1}-e^{-l_3}}, \frac{p_2}{p_3} = \frac{1+e^{-l_1}}{e^{l_2}-e^{-l_1}} \text{ and } \frac{p_3}{p_1} = \frac{1+e^{-l_2}}{e^{l_3}-e^{-l_2}}$$

and then the associated stochastic matrix becomes

$$P_l = \begin{pmatrix} 0 & \frac{1+e^{-l_1}}{1+e^{l_2}} & \frac{-e^{-l_1}+e^{l_2}}{1+e^{l_2}} \\ \frac{1+e^{-l_2}}{1+e^{l_1}} & 0 & \frac{-e^{-l_2}+e^{l_1}}{1+e^{l_1}} \\ \frac{1+e^{l_2}}{2+e^{l_1}+e^{l_2}} & \frac{1+e^{l_1}}{2+e^{l_1}+e^{l_2}} & 0 \end{pmatrix}.$$

The normalized right eigenvector for P_l is

$$q = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \text{ where } \begin{aligned} q_1 &= \frac{e^{l_1}(1+e^{l_2})^2}{2(3e^{l_1+l_2}+e^{2l_1+l_2}+e^{l_1+2l_2}-1)} \\ q_2 &= \frac{e^{l_2}(1+e^{l_1})^2}{2(3e^{l_1+l_2}+e^{2l_1+l_2}+e^{l_1+2l_2}-1)} \\ q_3 &= \frac{(2+e^{l_1}+e^{l_2})(e^{l_1+l_2}-1)}{2(3e^{l_1+l_2}+e^{2l_1+l_2}+e^{l_1+2l_2}-1)}. \end{aligned}$$

In order to give a formula for the Weil-Petersson metric for this graph we want to define an inner product on the tangent space to $\mathcal{M}^1_{\mathcal{G}}$. We can consider the parameterization $\psi : (l_1, l_2) \mapsto (l_1, l_2, l_3) \in \mathcal{M}^1_{\mathcal{G}}$, where $l_3 = \psi_3(l_1, l_2)$ is given by (4.1). The derivatives with respect to l_1 and l_2 are

$$D_1\psi(l_1, l_2, l_3) := \left(1, 0, \frac{\partial \psi_3}{\partial l_1}\right) \text{ and } D_2\psi(l_1, l_2, l_3) := \left(0, 1, \frac{\partial \psi_3}{\partial l_2}\right),$$

respectively, where these triples represent the values of the functions $D_1\psi$ and $D_2\psi$ corresponding to the three edges of the graph. We can then write the metric (in terms of its first fundamental form)

$$ds^2 = E(l_1, l_2)dl_1^2 + F(l_1, l_2)dl_1 dl_2 + G(l_1, l_2)dl_2^2,$$

where

$$\begin{aligned} E(l_1, l_2) &= \sigma^2(D_1\psi, D_1\psi) = q_1(l_1, l_2) + q_3(l_1, l_2)\left(\frac{\partial \psi_3}{\partial l_1}\right)^2, \\ F(l_1, l_2) &= \sigma^2(D_1\psi, D_2\psi) = q_3(l_1, l_2)\left(\frac{\partial \psi_3}{\partial l_1}\right)\left(\frac{\partial \psi_3}{\partial l_2}\right), \\ G(l_1, l_2) &= \sigma^2(D_2\psi, D_2\psi) = q_2(l_1, l_2) + q_3(l_1, l_2)\left(\frac{\partial \psi_3}{\partial l_2}\right)^2. \end{aligned}$$

Explicit computations gives

$$\begin{aligned} E(l_1, l_2) &= \frac{e^{l_1}(1 + e^{l_2})^2(-e^{l_2} + 4e^{l_1+l_2} + e^{2l_1+l_2} + 2e^{l_1+2l_2} - 2)}{2(2 + e^{l_1} + e^{l_2})(e^{l_1+l_2} - 1)(3e^{l_1+l_2} + e^{2l_1+l_2} + e^{l_1+2l_2} - 1)}, \\ F(l_1, l_2) &= \frac{e^{l_1+l_2}(1 + e^{l_1})^2(1 + e^{l_2})^2}{2(2 + e^{l_1} + e^{l_2})(e^{l_1+l_2} - 1)(3e^{l_1+l_2} + e^{2l_1+l_2} + e^{l_1+2l_2} - 1)}, \\ G(l_1, l_2) &= \frac{e^{l_2}(1 + e^{l_1})^2(-e^{l_1} + 4e^{l_1+l_2} + e^{l_1+2l_2} + 2e^{2l_1+l_2} - 2)}{2(2 + e^{l_1} + e^{l_2})(e^{l_1+l_2} - 1)(3e^{l_1+l_2} + e^{2l_1+l_2} + e^{l_1+2l_2} - 1)}. \end{aligned}$$

We can compute the Gaussian curvature of $\mathcal{M}^1_{\mathcal{G}}$ at a point (l_1, l_2, l_3) using the following standard formula [Gr].

Lemma 5.1 (Brioschi formula). *If a metric has local coordinates*

$$ds^2 = E(u, v)du^2 + F(u, v)dudv + G(u, v)dv^2$$

then the curvature is given by

$$\begin{aligned} \kappa(u, v) = &\frac{1}{(EG - F^2)^2} \\ &\times \left(\begin{vmatrix} -\frac{1}{2}E_{vv} + F_{uv} - \frac{1}{2}G_{uu} & \frac{1}{2}E_u & F_u - \frac{1}{2}E_v \\ F_v - \frac{1}{2}G_u & E & F \\ \frac{1}{2}G_v & F & G \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{vmatrix} \right). \end{aligned}$$

We can explicitly compute

$$\kappa(l_1, l_2) = \frac{9}{4}\frac{(e^{l_1+l_2} - 1)(3e^{l_1+l_2} + 2e^{2l_1+l_2} + 2e^{l_1+2l_2} + 1)}{(3e^{l_1+l_2} + e^{2l_1+l_2} + e^{l_1+2l_2} + 1)}$$

The plot of the curvature for $0 < l_1, l_2 < 3$ is presented in the Figure 3, where it is seen to be negative with a minimum at $l_1 = l_2 = \log 2$.

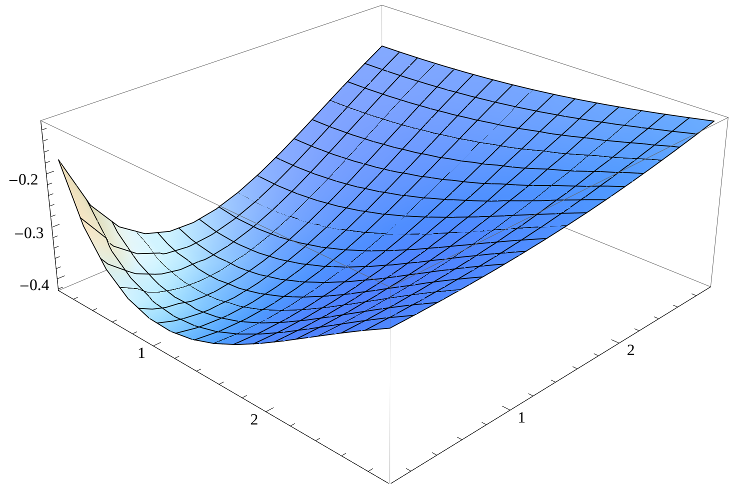

FIGURE 3. The curvature of a portion of $\mathcal{M}^1_{\mathcal{G}}$ for the belt buckle.

Remark. We can expand the curvature at $l_1 = l_2 = 0$ as

$$\kappa(l_1, l_2) = -\frac{9}{8}l_1 - \frac{9}{8}l_2 + \frac{99}{64}l_1^2 + \frac{99}{32}l_1 l_2 + \frac{99}{64}l_2^2 + \cdots .$$

We finally consider formulae for the Gaussian curvature of the metric for Example III, i.e., the dumbbell with two vertices and 3 edges, with lengths $l_1, l_2, l_3 > 0$. If A_l is the weighted matrix in (4.3) then the associated stochastic matrix takes the form

$$P_l = \begin{pmatrix} e^{-l_1} & 0 & 1-e^{-l_1} & 0 & 0 & 0 \\ 0 & e^{-l_1} & 1-e^{-l_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-e^{-l_3} & e^{-l_3} & 0 \\ 0 & 0 & 0 & 1-e^{-l_3} & 0 & e^{-l_3} \end{pmatrix}.$$

The normalized right eigenvector for P_l is

$$q = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{pmatrix} \text{ where } \begin{aligned} q_1 &= \frac{e^{l_1}(e^{l_3}-1)}{4-6e^{l_1}-6e^{l_3}+8e^{l_1+l_3}} \\ q_2 &= \frac{e^{l_1}(e^{l_3}-1)}{4-6e^{l_1}-6e^{l_3}+8e^{l_1+l_3}} \\ q_3 &= \frac{(e^{l_1}-1)(e^{l_3}-1)}{4-6e^{l_1}-6e^{l_3}+8e^{l_1+l_3}} \\ q_4 &= \frac{(e^{l_1}-1)(e^{l_3}-1)}{4-6e^{l_1}-6e^{l_3}+8e^{l_1+l_3}} \\ q_5 &= \frac{e^{l_3}(e^{l_1}-1)}{4-6e^{l_1}-6e^{l_3}+8e^{l_1+l_3}} \\ q_6 &= \frac{e^{l_3}(e^{l_1}-1)}{4-6e^{l_1}-6e^{l_3}+8e^{l_1+l_3}}. \end{aligned}$$

We can again compute the coefficients in the first fundamental form which in this case are

$$E = \frac{e^{l_1}(3e^{l_1}-2)(e^{l_3}-1)}{2(e^{l_1}-1)(2-3e^{l_1}-3e^{l_3}+4e^{l_1+l_3})},$$
$$F = \frac{e^{l_1+l_3}}{2(e^{l_1}-1)(2-3e^{l_1}-3e^{l_3}+4e^{l_1+l_3})},$$
$$G = \frac{e^{l_3}(3e^{l_3}-2)(e^{l_1}-1)}{2(e^{l_1}-1)(2-3e^{l_1}-3e^{l_3}+4e^{l_1+l_3})}.$$

We can then again use the Brioschi formula in Lemma 5.1 to write

$$\kappa(l_1,l_3) = \frac{12-16(e^{l_1}+e^{l_3})+12(e^{2l_1}+e^{2l_3})-9(e^{3l_1}+e^{3l_3})+73(e^{2l_1+l_3}-e^{l_1+2l_3})-15(e^{3l_1+l_3}+e^{l_1+3l_3})}{2(e^{l_1}-1)(e^{l_3}-1)(2-3e^{l_1}-3e^{l_3}+4e^{l_1+l_3})^2}$$
$$+\frac{24(e^{3l_1+2l_3}+e^{2l_1+3l_3})-40e^{l_1+l_3}-110e^{2(l_1+l_3)}}{2(e^{l_1}-1)(e^{l_3}-1)(2-3e^{l_1}-3e^{l_3}+4e^{l_1+l_3})^2}$$

The plot of the curvature is given in Figure 4, where it is seen to be both positive (when l_1, l_3 are small) and negative (when l_1, l_3 are large).

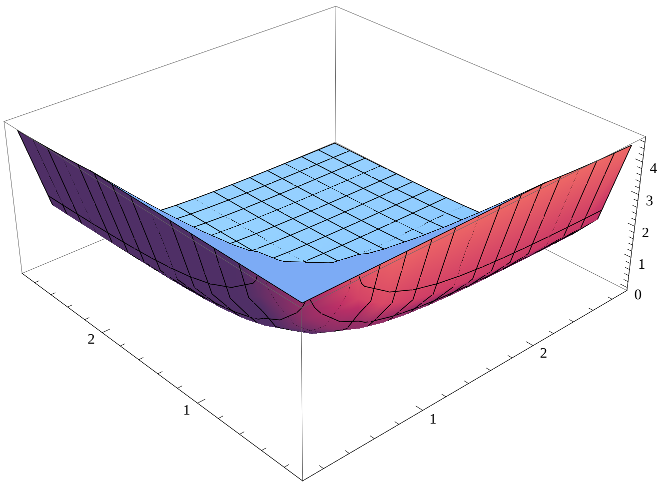

FIGURE 4. The curvature of a portion of $\mathcal{M}^1_{\mathcal{G}}$ for the dumbbell.

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