Weil-Petersson metrics, Manhattan curves and Hausdorff dimension

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1 Introduction

In this note we want to relate the Weil-Petersson metric on Teichmüller space to the boundary correspondence between the actions on the boundary of Fuchsian groups. Consider the space of Riemann metrics g on a compact surface V with negative Euler characteristic. This can be endowed with a number of natural Riemannian metrics. Of particular interest is the Weil-Petersson metric, whose definition was proposed by Weil in 1958 based on earlier work of Petersson, cf. [18]. There is a particularly intuitive equivalent definition of the Weil-Petersson metric using the second derivative of lengths of typical (closed) geodesics due to Thurston and Wolpert [19].

Definition 1.1 (Thurston-Wolpert [19]). *Let g_λ, $-\epsilon < \lambda < \epsilon$, be a (non-constant) smooth family of Riemann metrics. Then Weil-Petersson metric is given by*

$$\left\| \frac{\partial g_\lambda}{\partial \lambda}\Big|_{\lambda=0} \right\|_{\mathrm{WP}}^2 = -\lim_{T\to\infty} \frac{\sum_{l_{g_0}(\gamma)\le T} \left(\int_0^{l_{g_0}(\gamma)} \frac{\partial^2 \log l_{g_\lambda}(\gamma)}{\partial \lambda^2}\Big|_{\lambda=0} dt \right)}{\sum_{l_{g_0}(\gamma)\le T} 1}$$

where the summations are over closed geodesics γ whose lengths $l_{g_0}(\gamma)$ with respect to the metric g_0 are less than or equal to T.

We can associate to the metrics g_λ a family of Fuchsian groups Γ_λ acting on the unit disk $\mathbb{D}$ such that (V, g_λ) is isometric to $\mathbb{D}/\Gamma_\lambda$. Moreover, there exists a family of conjugating homeomorphisms $\pi_\lambda : \partial\mathbb{D} \to \partial\mathbb{D}$ on the boundary satisfying $\gamma_\lambda \circ \pi_\lambda = \pi_\lambda \circ \gamma_0$ for corresponding elements $\gamma_0 \in \Gamma_0$ and $\gamma_\lambda \in \Gamma_\lambda$. In [10], McMullen gave an interesting interpretation of $\|\cdot\|_{\mathrm{WP}}$ in terms of the Hausdorff dimension $\dim_H(\pi_\lambda(l))$ of the image $\pi_\lambda(l)$ of Lebesgue measure l on $\partial\mathbb{D}$.

Theorem 1.2 (McMullen). *Let g_λ, $-\epsilon < \lambda < \epsilon$, be a (non-constant) smooth family of Riemann metrics. Then Weil-Petersson metric is given by*

$$\left\| \frac{\partial g_\lambda}{\partial \lambda}\Big|_{\lambda=0} \right\|_{\mathrm{WP}}^2 = \frac{-1}{3\pi(g-1)} \frac{\partial^2 \dim_H(\pi_\lambda(l))}{\partial \lambda^2}\Big|_{\lambda=0},$$

where $g \ge 2$ is the genus of M.

We will complement this result by giving an alternative definition which depends on the dimension of sets, rather than the dimension of measures. We first recall the well known "rigidity" result that if the boundary map π_λ is not linear fractional, then it must be singular [17] (i.e. there exists a set $E \subset \partial\mathbb{D}$ such that $l(E) = 0$ and $l(\partial\mathbb{D}\backslash\pi_\lambda(E)) = 0$). In fact, there is even the following stronger result of Bishop and Steger (cf. [2], Theorem 3, and [3]).

Proposition 1.3. *For each $\lambda \neq 0$, there exists a set $E \subset \partial\mathbb{D}$ such that*

$$\max\{\dim_H(E), \dim_H(\pi_\lambda(\partial\mathbb{D}\backslash E))\} < 1.$$

Clearly, the boundary map $\pi_\lambda : \partial\mathbb{D} \to \partial\mathbb{D}$ is differentiable almost everywhere, with zero derivative, by monotonicity of the map. In the remaining set of zero measure, we denote by N_λ the set of points of non-differentiability, then these have dimension satisfying $0 < D_\lambda = \dim_H(N_\lambda) < 1$.

Our main result is the following.

Theorem 1.4. *Let g_λ, $-\epsilon < \lambda < \epsilon$, be a (non-constant) smooth family of Riemann metrics. Then Weil-Petersson metric is given by*

$$\left\| \frac{\partial g_\lambda}{\partial \lambda}\bigg|_{\lambda=0} \right\|_{\mathrm{WP}}^2 = \frac{-1}{12\pi(g-1)} \frac{\partial^2 D_\lambda}{\partial \lambda^2}\bigg|_{\lambda=0}$$

where $g \geq 2$ is the genus of M.

Our approach also makes a connection with the Manhattan curve of Burger [6] and the work of Schwartz and Sharp [14] on comparing lengths of closed geodesics with respect to different metrics.

2 The Manhattan curve and the correlation number

Let g_λ, $-\epsilon < \lambda < \epsilon$, be a (non-constant) smooth family of Riemann metrics on the surface V. Given $\lambda > 0$, say, we recall the definition of the Manhattan curve of Burger for the two metrics g_0 and g_λ. Given a free homotopy class γ we let $l_0(\gamma)$ and $l_\lambda(\gamma)$ denote the lengths of the associated closed geodesics for each of these metrics.

Definition 2.1. *We can consider the convex set*

$$\left\{ (a,b) \in \mathbb{R}^2 : \sum_\gamma e^{-(a l_0(\gamma) + b l_\lambda(\gamma))} < +\infty \right\}.$$

The boundary is called the Manhattan curve *for the metrics g_0 and g_λ.*

We can parameterize the Manhatten curve by a curve $\chi_\lambda : \mathbb{R} \to \mathbb{R}$, i.e.,

$$\chi_\lambda(a) = \inf\left\{ b \geq 0 : \sum_\gamma e^{-(a l_0(\gamma) + b l_\lambda(\gamma))} < +\infty \right\}.$$

Remark 2.2. We could equally well define the Manhattan curve in terms of two variable Poincaré series or zeta function associated to the pair of metrics.

Some simple and well known properties of the curve are summarised in the next lemma (cf. [6], [16]). (The final statement follows from the fact that, for any Riemann metric g, $\lim_{T\to+\infty} T^{-1}\log \mathrm{Card}\{\gamma : l_g(\gamma) \le T\} = 1$.)

Lemma 2.3. *The Manhattan curve* $\chi_\lambda : \mathbb{R} \to \mathbb{R}$ *is analytic and convex (and strictly convex unless* $\lambda = 0$*). Furthermore,* $\chi_\lambda(0) = 1$ *and* $\chi_\lambda(1) = 0$*.*

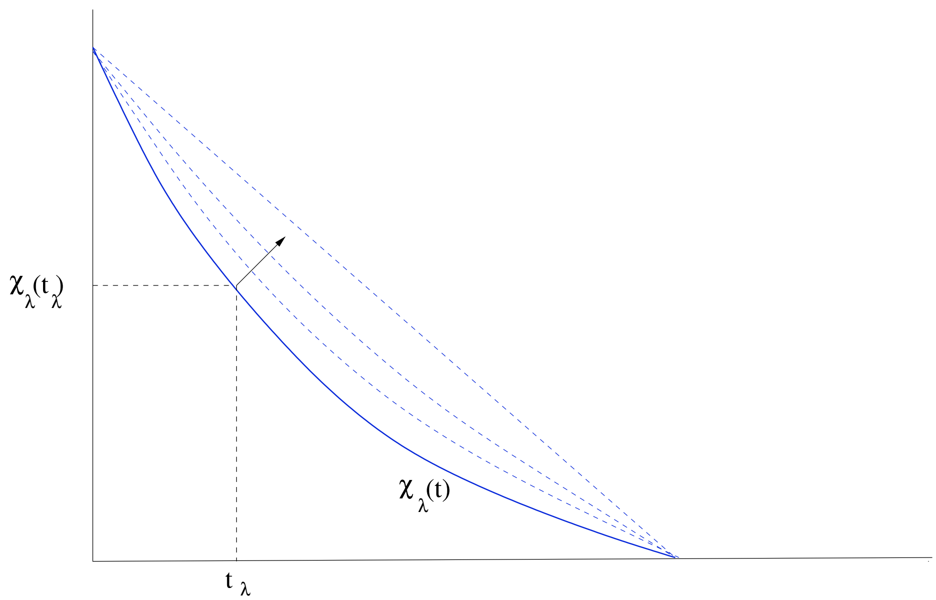

Figure 1: The Manhattan curve restricted to $0 \le t \le 1$. It can be used to define $D_\lambda = \chi_\lambda(t_\lambda) + t_\lambda$ where $t = t_\lambda$ is where the perpendicular to the curve χ_λ has slope 1.

If $g_0 \neq g_\lambda$ then we see by Lemma 2.3 that we can choose a unique value $0 < t_\lambda < 1$ such that $\frac{\partial \chi_\lambda}{\partial t}\big|_{t=t_\lambda} = -1$. The connection with the quantity in Theorem 1.4 comes from the following identification.

Recall that D_λ denotes the Hausdorff dimension of the set where the boundary map $\pi_\lambda : \partial\mathbb{D} \to \partial\mathbb{D}$ which conjugates the actions of Γ_0 and Γ_λ is not differentiable.

Proposition 2.4. *We have that* $D_\lambda = \chi_\lambda(t_\lambda) + t_\lambda$*.*

Proof. This follows from a result of Jordan, Kesseböhmer, Pollicott and Stratmann [8]. The actions of Γ_0 and Γ_λ on $\partial\mathbb{D}$ may be modelled by expanding Markov maps $T_0 : \partial\mathbb{D} \to \partial\mathbb{D}$ and $T_\lambda : \partial\mathbb{D} \to \partial\mathbb{D}$, respectively [1], [15]. These maps are conjugated by $\pi_\lambda : \partial\mathbb{D} \to \partial\mathbb{D}$, which is Hölder continuous. Apart from a finite number of exceptions, there is a natural one-to-one correspondence between the periodic orbits of T_0 and closed geodesics on (V, g_0) (and hence the free homotopy classes on V). Furthermore, if we define Hölder continuous functions $f_0 : \partial\mathbb{D} \to \mathbb{R}$ and $f_\lambda : \partial\mathbb{D} \to \mathbb{R}$ by

$$f_0(x) = -\log|T_0'(x)| \text{ and } f_\lambda(x) = -\log|T_\lambda'(\pi_\lambda(x))|$$

then the numbers $-l_0(\gamma)$ and $-l_\lambda(\gamma)$ are given by $f_0^n(x)$ and $f_\lambda^n(x)$, where $\sigma^n x = x$ corresponds to γ. (Here, $f_0^n(x)$ denotes the sum $f_0(x) + f_0(T_0 x) + \cdots + f_0(T_0^{n-1}x)$.) It now follows from standard thermodynamic formalism that χ_λ is defined implicitly by

$$P(tf_0 + \chi_\lambda(t) f_\lambda) = 0,$$

where P is the pressure function associated to the transformation T_0. Noting that no non-trivial linear combination $af_0 + bf_\lambda$ has the form $u \circ T_0 - u$ for $u : \partial\mathbb{D} \to \mathbb{R}$ continuous (which follows from the Independence Lemma in [14], for example), we can apply Theorem 1.1 of [8] to assert that $D_\lambda = \chi_\lambda(t_\lambda) + t_\lambda$, as required. (Actually, to carry out the analysis in [8], it is convenient to introduce and work with a subshift of finite type which models T_0. We discuss related symbolic dynamics further in the next section.) □

There is an interesting connection between the value D_λ and the relative lengths of corresponding closed geodesics on V which gives another interpretation to D_λ. Let γ denote a free homotopy class on V and denote by $l_0(\gamma)$ and $l_\lambda(\gamma)$ the lengths of the corresponding geodesics with respect to the metrics g_0 and g_λ. Given $\delta > 0$ and $T > 0$ we let

$$\Pi^\lambda(T, \delta) = \mathrm{Card}\{\gamma : T - \delta \leq l_0(\gamma), l_\lambda(\gamma) \leq T\}.$$

Schwartz and Sharp [14] showed that $\Pi^\lambda(T, \delta)$ has a well-defined exponential growth rate independent of the choice of δ. Subsequently, Sharp [16] showed that this growth rate, called the correlation number, is equal to $\chi_\lambda(t_\lambda) + t_\lambda$ and hence it is equal to the dimension D_λ Combing these observations, the main result of Schwartz and Sharp [14] takes the following form.

Proposition 2.5. *For any $\delta > 0$, we have that*

$$D_\lambda := \lim_{T \to +\infty} \frac{1}{T} \log \Pi^\lambda(T, \delta).$$

Moreover, there exists a constant $C = C(\delta) > 0$ such that

$$\Pi^\lambda(T, \delta) \sim C \frac{e^{D_\lambda T}}{T^{3/2}} \text{ as } T \to +\infty.$$

3 Thermodynamic formalism and geodesic flows

In this section we shall introduce the technical machinery which will be used in the sequel to prove Theorem 1.4. Given a Riemann metric $\|\cdot\|_g$ on V, we can model the geodesic flow $\phi_t^{(g)} : T_1 V \to T_1 V$ on the unit tangent bundle $T_1 V = \{v \in TV : \|v\|_g = 1\}$ of V, with respect to the norm $\|\cdot\|_g$, by a suspension flow over a subshift of finite type. Furthermore, for a family of metrics g_λ, $-\epsilon < \lambda < \epsilon$, by structural stability we can use the same subshift for each flow, only varying the suspension function.

We begin by defining subshifts of finite type and suspension flows. Let A be a $k \times k$ matrix with entries 0 and 1. We shall always assume that A is aperiodic (i.e. that A^n has positive entries for some $n \geq 0$). Let

$$X_A = \left\{ \underline{x} = (x_n)_{n \in \mathbb{Z}} \in \{1, \cdots, k\}^{\mathbb{Z}} : A(x_n, x_{n+1}) = 1, \forall n \in \mathbb{Z} \right\}$$

with the metric

$$d(\underline{x}, \underline{y}) = \sum_{n\in\mathbb{Z}} \frac{1-\delta(x_n, y_n)}{2^{|n|}},$$

where $\underline{x} = (x_n)_{n\in\mathbb{Z}}$ and $\underline{y} = (y_n)_{n\in\mathbb{Z}}$. The subshift of finite type $\sigma : X_A \to X_A$ is defined by $(\sigma\underline{x})_n = x_{n+1}$, for $n \in \mathbb{Z}$, and is a homeomorphism with respect to the given metric.

Given a strictly positive Hölder continuous function $r : X_A \to \mathbb{R}^+$ we denote

$$X_A^r = \{(\underline{x}, u) \in X_A \times \mathbb{R} \in \ : 0 \le u \le r(x)\}/\sim,$$

where $\sim$ denotes the identification $(\underline{x}, r(\underline{x})) \sim (\sigma\underline{x}, 0)$. We define the suspension flow $\sigma_t^r : X_A^r \to X_A^r$ by $\sigma_t^r(x,u) = (x, u+t)$, subject to the identification.

Associated to the shift σ and the flow σ_t^r is a body of definitions and results known as thermodynamic formalism. We now introduce what we will need in the sequel. If $f : X_A \to \mathbb{R}$ is a Hölder continuous function then we can define the *pressure* $P(f)$ by

$$P(f) = \sup\left\{h(\mu) + \int f\, d\mu : \mu \text{ is a } \sigma\text{-invariant probability measure}\right\},$$

where $h(\mu)$ is the entropy of σ with respect to the measure μ. There is a unique σ-invariant probability measure realizing the above supremum, denoted by μ_f and called the *Gibbs measure* for f. Fixing the Hölder exponent, we recall some results on the pressure functional $P : C^\alpha(X_A, \mathbb{R}) \to \mathbb{R}$ on the Banach space of α-Hölder continuous functions.

Lemma 3.1 ([11], Propositions 4.10 and 4.11)**.** *The function $P : C^\alpha(X_A, \mathbb{R}) \to \mathbb{R}$ is analytic. Furthermore,*

(i) $D_g P(f) = \int g\, d\mu_f$*; and*

(ii)

$$D_g^2 P(f) = \sigma_f^2(g) := \lim_{n\to+\infty} \frac{1}{n} \int \left(\sum_{i=0}^{n-1} g\circ\sigma^i - \int g\, d\mu_f\right)^2 d\mu_f,$$

where D_g denotes the directional derivative

$$D_g P(f) = \lim_{t\to 0} \frac{P(f+tg) - P(f)}{t}.$$

We now consider the corresponding statements for flows. For a Hölder continuous function $F : X_A^r \to \mathbb{R}$, we define the *pressure* $P(F)$ by

$$P(F) = \sup\left\{h(m) + \int F\, dm : m \text{ is a } \sigma^r\text{-invariant probability measure}\right\}.$$

In particular, there is a unique σ^r-probability measure realizing the above supremum, denoted by m_F and called the *Gibbs measure* for F. The connection between the pressure and Gibbs measure for F and f is given by the following.

Lemma 3.2 (Bowen-Ruelle [5], see also [11], Proposition 6.1). *Given a Hölder continuous function $F : X_A^r \to \mathbb{R}$ we can associate a Hölder continuous function $f : X_A \to \mathbb{R}$ by $f(\underline{x}) = \int_0^{r(\underline{x})} F(\underline{x}, u)\, du$. Then*

(i) $P(f - P(F)r) = 0$; *and*

(ii) $dm_F = d\mu_f \times dt / \int r\, d\mu_f$.

We may also define pressure and Gibbs states with respect to a geodesic flow $\phi_t^{(g)} : T_1V \to T_1V$ for Hölder continuous functions $F : T_1V \to \mathbb{R}$ in an analogous way.

The connection between the geodesic flow and the suspension flow is given by the following.

Lemma 3.3. *We can associate to a smoothly varying family of Riemann metrics g_λ, $-\epsilon < \lambda < \epsilon$, a subshift of finite type $\sigma : X_A \to X_A$ and smoothly varying families of roof functions r_λ and a family of Hölder continuous surjections $p_\lambda : X_A^{r_\lambda} \to T_1V$, $-\epsilon < \lambda < \epsilon$, such that*

1. $p_\lambda \circ \sigma_t^{r_\lambda} = \phi_t^{(g_\lambda)} \circ p_\lambda$, $\forall t \in \mathbb{R}$;
2. *p_λ is bounded-to-one; and*
3. *p_λ is a measure-theoretic isomorphism with respect to the Gibbs measures for $F \circ p_\lambda$ and F, for any Hölder continuous function $F : T_1V \to \mathbb{R}$.*

Proof. For each fixed value of λ, this is a classical result due to Ratner [12] and Bowen [4]. The behaviour as λ varies follows from the structural stability of the geodesic flow [7]. □

Example 3.4. *If we let $f = -r_0$ then $dm_0 = d\mu_{-r_0} \times dt / \int r_0\, d\mu_{-r_0}$ is the measure of maximal entropy for the suspension flow. In particular, $p_0^{-1} m_0$ is the measure of maximal entropy for $\phi_t^{(g_0)} : T_1V \to T_1V$. Moreover, since the metric g_0 has constant curvature this is the same as the normalized Liouville measure (the product of the g_0-area on V and Lebesgue measure on each fibre).*

Given a smooth family g_λ, $-\epsilon < \lambda < \epsilon$, of Riemann metrics on V, we can write

$$g_\lambda = g_0 + \lambda g^{(1)} + (\lambda^2/2) g^{(2)} + o(\lambda^2)$$

(where $g^{(1)}$ lies in the tangent space for Teichmüller space) and

$$r_\lambda = r_0 + \lambda r^{(1)} + (\lambda^2/2) r^{(2)} + o(\lambda^2).$$

The following gives a simple, slightly technical, but useful, symbolic characterization of the tangent space to the Teichmüller space of V and of the Weil-Petersson metric itself.

Proposition 3.5 (McMullen [10]). *We have that*

1. $\int r^{(1)}\, d\mu_{-r_0} = 0$; *and*
2. $$\frac{\|g^{(1)}\|^2_{\mathrm{WP}}}{3\pi(g-1)} = \frac{\sigma^2(r^{(1)})}{\int r_0\, d\mu_{-r_0}} = \frac{\int r^{(2)} d\mu_{-r_0}}{\int r_0\, d\mu_{-r_0}},$$
   where $g \geq 2$ is the genus of the surface.

4 Differentiablility of the dimension function D_λ

In this section we prove Theorem 1.4 (restated below as Theorem 4.3). We begin by characterising the Manhattan curve in terms of the pressure function defined in the preceding section.

Lemma 4.1. *For any $-\epsilon < \lambda < \epsilon$, the Manhattan curve is given by the graph $\{(t, \chi_\lambda(t)) : t \in \mathbb{R}\}$ where*

$$P(-tr_0 - \chi_\lambda(t) r_\lambda) = 0.$$

Moreover, χ_λ has an analytic dependence on $\lambda \in (-\epsilon, \epsilon)$.

Proof. The first part is due Lalley [9]. The second part follows easily by the Implicit Function Theorem. □

We assume for simplicity that the family g_λ, $\lambda \in (-\epsilon, \epsilon)$, is non-degenerate at $\lambda = 0$, i.e. that $g^{(2)} \neq 0$. This ensures that the Manhattan curve is strictly convex.

Lemma 4.2. *We have that*

$$\frac{\partial^2 \chi_\lambda}{\partial \lambda^2}(t)\bigg|_{\lambda=0} = t\,(t-1)\,\frac{\|g^{(1)}\|^2_{\mathrm{WP}}}{3\pi(g-1)}$$

for all $0 \leq t \leq 1$.

Proof. Using the analytic dependence of r_λ and $\chi_\lambda(t)$ on $\lambda \in (-\epsilon, \epsilon)$ we can expand

$$r_\lambda = r_0 + \lambda r^{(1)} + \frac{\lambda^2}{2} r^{(2)} + o(\lambda^2)$$

and

$$\chi_\lambda(t) = \chi_0(t) + \frac{\partial \chi_\lambda}{\partial \lambda}(t)\bigg|_{\lambda=0} \lambda + \frac{\partial^2 \chi_\lambda}{\partial \lambda^2}(t)\bigg|_{\lambda=0} \frac{\lambda^2}{2} + o(\lambda^2),$$

for each $0 < t < 1$. Substituting these expansions into $P(-tr_0 - \chi_\lambda(t) r_\lambda) = 0$ and using the second order expansion of the pressure function gives

$$\begin{aligned}
0 &= P(-tr_0 - \chi_\lambda(t) r_\lambda) \\
&= P\left(-tr_0 - \left(\chi_0(t) + \frac{\partial \chi_\lambda}{\partial \lambda}(t)\bigg|_{\lambda=0} \lambda + \frac{\partial^2 \chi_\lambda}{\partial \lambda^2}(t)\bigg|_{\lambda=0} \frac{\lambda^2}{2} + o(\lambda^2)\right)\left(r_0 + \lambda r^{(1)} + \frac{\lambda^2}{2} r^{(2)} + o(\lambda^2)\right)\right) \\
&= P\Bigg(\underbrace{(-t - \chi_0(t))}_{=-1} r_0 - \lambda\left(\frac{\partial \chi_\lambda}{\partial \lambda}(t)\bigg|_{\lambda=0} r_0 + \chi_0(t) r^{(1)}\right) \\
&\qquad - \frac{\lambda^2}{2}\left(\chi_0(t) r^{(2)} + 2\,\frac{\partial \chi_\lambda}{\partial \lambda}(t)\bigg|_{\lambda=0} r^{(1)} + \frac{\partial^2 \chi_\lambda}{\partial \lambda^2}(t)\bigg|_{\lambda=0} r_0\right) + o(\lambda^2)\Bigg) \\
&= \underbrace{P(-r_0)}_{=0} - \lambda\left(\frac{\partial \chi_\lambda}{\partial \lambda}(t)\bigg|_{\lambda=0} \int r_0\, d\mu_0 + \chi_0(t) \underbrace{\int r^{(1)}\, d\mu_{-r_0}}_{=0}\right)
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\lambda^2}{2}\left(\sigma^2_{-r_0}\left(\frac{\partial \chi_\lambda}{\partial \lambda}(t)\Big|_{\lambda=0} r_0 + \chi_0(t) r^{(1)}\right)\right.\\
&\qquad \left.- \left(\chi_0(t)\int r^{(2)}\, d\mu_{-r_0} + 2\,\frac{\partial \chi_\lambda}{\partial \lambda}(t)\Big|_{\lambda=0} \underbrace{\int r^{(1)}\, d\mu_{-r_0}}_{=0} + \frac{\partial^2 \chi_\lambda}{\partial \lambda^2}(t)\Big|_{\lambda=0}\int r_0\, d\mu_{-r_0}\right)\right) + o(\lambda^2)\\
&= -\lambda\,\frac{\partial \chi_\lambda}{\partial \lambda}(t)\Big|_{\lambda=0}\int r_0\, d\mu_{-r_0}\\
&+ \frac{\lambda^2}{2}\left(\sigma^2_{-r_0}\left(\frac{\partial \chi_\lambda}{\partial \lambda}(t)\Big|_{\lambda=0} r_0 + \chi_0(t) r^{(1)}\right) - \left(\chi_0(t)\int r^{(2)}\, d\mu_{-r_0} + \frac{\partial^2 \chi_\lambda}{\partial \lambda^2}(t)\Big|_{\lambda=0}\int r_0\, d\mu_{-r_0}\right)\right)\\
&+ o(\lambda^2),
\end{aligned}
$$

for all $0 \le t \le 1$.

From the λ term we see that, since $\int r_0\, d\mu_{-r_0} > 0$,

$$
\frac{\partial \chi_\lambda}{\partial \lambda}(t)\Big|_{\lambda=0} = 0 \text{ for all } 0 \le t \le 1. \tag{4.1}
$$

In particular, differentiating (4.1) with respect to t now gives

$$
\frac{\partial^2 \chi_\lambda}{\partial t \partial \lambda}(\lambda)\Big|_{\lambda=0} = 0 \text{ for all } 0 \le t \le 1. \tag{4.2}
$$

We can next consider the λ^2 term and use (4.1) to get

$$
0 = \sigma^2\left(r_0\underbrace{\frac{\partial \chi_\lambda}{\partial \lambda}(t)\Big|_{\lambda=0}}_{=0} + (1-t) r^{(1)}\right) - \left((1-t)\int r^{(2)}\, d\mu_0 + \frac{\partial^2 \chi_\lambda}{\partial \lambda^2}(t)\Big|_{\lambda=0}\int r_0\, d\mu_{-r_0}\right),
$$

which can be rearranged to give

$$
\begin{aligned}
\frac{\partial^2 \chi_\lambda}{\partial \lambda^2}\Big|_{\lambda=0}(t) &= (1-t)^2\frac{\sigma^2_{-r_0}(r^{(1)})}{\int r_0\, d\mu_{-r_0}} - (1-t)\frac{\int r^{(2)}\, d\mu_0}{\int r_0\, d\mu_{-r_0}}\\
&= (t^2 - t)\frac{\|g^{(1)}\|^2_{\mathrm{WP}}}{3\pi(g-1)}.
\end{aligned}
$$

for every $0 \le t \le 1$, using Proposition 3.5. □

Finally we can prove our main in result, Theorem 1.4, which we now restate.

Theorem 4.3. *We have that*

$$
\frac{\partial^2 D_\lambda}{\partial \lambda^2}\Big|_{\lambda=0} = -\frac{\|g^{(1)}\|^2_{\mathrm{WP}}}{12\pi(g-1)}.
$$

Proof. Recall that we can associate to $\lambda \in (-\epsilon, \epsilon)\backslash\{0\}$ a unique value $0 < t_\lambda < 1$ such that

$$\frac{\partial \chi_\lambda}{\partial t}(t_\lambda) = -1. \tag{4.3}$$

By differentiating the definition of D_λ with respect to $\lambda \in (-\epsilon, \epsilon)\backslash\{0\}$, we see that

$$\frac{\partial D_\lambda}{\partial \lambda} = \underbrace{\frac{\partial \chi_\lambda}{\partial t}(t_\lambda)}_{=-1}\frac{\partial t_\lambda}{\partial \lambda} + \frac{\partial t_\lambda}{\partial \lambda} + \frac{\partial \chi_\lambda}{\partial \lambda}(t_\lambda) = \frac{\partial \chi_\lambda}{\partial \lambda}(t_\lambda) \tag{4.4}$$

where we have cancelled the first and second terms, using (4.3). Differentiating (4.4) a second time gives

$$\frac{\partial^2 D_\lambda}{\partial \lambda^2} = \frac{\partial^2 \chi_\lambda}{\partial \lambda^2}(t_\lambda) + \frac{\partial^2 \chi_\lambda}{\partial \lambda \partial t}(t_\lambda)\frac{\partial t_\lambda}{\partial \lambda}(\lambda). \tag{4.5}$$

for $\lambda \neq 0$. By (4.2), we have

$$\lim_{\lambda \to 0} \frac{\partial^2 \chi_\lambda}{\partial \lambda \partial t}(t_\lambda) = 0.$$

Also, using (4.3), we have that

$$\frac{\partial t_\lambda}{\partial \lambda} = \frac{\partial \chi_\lambda}{\partial \lambda}\left(\frac{\partial \chi_\lambda}{\partial t}\bigg|_{t=t_\lambda}\right)^{-1} = -\frac{\partial \chi_\lambda}{\partial \lambda}(t_\lambda),$$

so that $\partial t_\lambda/\partial \lambda$ remains bounded as $\lambda \to 0$. Combining these observations, we conclude from (4.5) that

$$\frac{\partial^2 D}{\partial \lambda^2}(\lambda)\bigg|_{\lambda=0} = \lim_{\lambda \to 0} \frac{\partial^2 \chi_\lambda}{\partial \lambda^2}(t_\lambda) = \lim_{\lambda \to 0}(t_\lambda^2 - t_\lambda)\frac{\|g^{(1)}\|^2_{\mathrm{WP}}}{3\pi(g-1)}.$$

To complete the proof of Theorem 1.4 we need to show that $\lim_{\lambda \to 0} t_\lambda$ exists and is equal to 1/2. Since $\partial t_\lambda/\partial \lambda$ is bounded as $\lambda \to 0$, we have

$$|t_\lambda - t_{\lambda'}| = \left|\int_\lambda^{\lambda'} \frac{\partial t_\xi}{\partial \xi}\, d\xi\right| \leq M|\lambda - \lambda'|,$$

for some $M \geq 0$. Hence, t_λ is a uniformly continuous function of λ, so that $t_0 := \lim_{\lambda \to 0} t_\lambda$ exists, and, furthermore, $t_\lambda = t_0 + O(\lambda)$. For λ sufficiently close to 0 we can approximate $\chi_\lambda(t)$ by

$$(1-t) + \frac{\|g^{(1)}\|_{\mathrm{WP}}}{3\pi(g-1)}(t^2 - t)\frac{\lambda^2}{2} + O(\lambda^3),$$

using (5.2). Differentiating and evaluating at $t = t_\lambda$ gives the equation

$$\begin{aligned} -1 &= -1 + \frac{\|g^{(1)}\|_{\mathrm{WP}}}{3\pi(g-1)}(2t_\lambda - 1)\frac{\lambda^2}{2} + O(\lambda^3) \\ &= -1 + \frac{\|g^{(1)}\|_{\mathrm{WP}}}{3\pi(g-1)}(2t_0 - 1)\frac{\lambda^2}{2} + O(\lambda^3). \end{aligned}$$

Cancelling the constant term and dividing by λ^2, we see that $t_0 = 1/2$, as required.

The result then follows by Proposition 2.4. □

Remark 4.4. Given any Hölder continuous function $F : T_1V \to \mathbb{R}$ we can follow Ruelle in defining a weighted zeta function by

$$\zeta_{g,F}(s) = \prod_{\gamma} \left(1 - e^{l_g^F(\gamma) - sl_g(\gamma)}\right)^{-1}.$$

In particular, when $F = 0$ we see that $\zeta_{g,0}(s) = Z_g(s+1)/Z_g(s)$ where

$$Z_g(s) = \prod_{n=0}^{\infty} \prod_{\gamma} \left(1 - e^{-(s+n)l_g(\gamma)}\right)$$

is the Selberg zeta function which converges for $Re(s) > 1$ and has a simple zero at $s = 1$. Let us now consider the choice $F : T_1V \to \mathbb{R}$ given by $F(v) = \lambda g^{(1)}(v, v)$ then we have a characterization of the Weil-Petersson metric in terms of the second derivative of the residue of $Z_{g_\lambda}(s)$:

$$\|g^{(1)}\|_{\mathrm{WP}}^2 = \lim_{s \to 1} (s-1) \frac{\partial^2 \log \zeta_{g_0, \lambda g^{(1)}}(s)}{\partial^2 \lambda}.$$

The formulation of the $\| \cdot \|_{\mathrm{WP}}^2$ in terms of the Selberg zeta function, combined with Ruelle's Grothendieck determinant approach to $Z_g(s)$ [13] leads, in principle, to a very efficient method for numerically computing the Weil-Petersson metric.

References

[1] R. Adler and L. Flatto, Geodesic flows, interval maps, and symbolic dynamics, Bull. Amer. Math. Soc. 25, 229–334, 1991.

[2] C. Bishop and T. Steger, Three rigidity criteria for $\mathrm{PSL}(2, \mathbb{R})$, Bull. Amer. Math. Soc. 24, 119–123, 1991.

[3] C. Bishop and T. Steger, Representation-theoretic rigidity in $\mathrm{PSL}(2, \mathbb{R})$, Acta Math. 170, 121–149, 1993.

[4] R. Bowen, Symbolic dynamics for hyperbolic flows, Amer. J. Math. 95, 429–460, 1973.

[5] R. Bowen and D. Ruelle, The ergodic theory of Axiom A flows, Invent. math. 29, 181–202, 1975.

[6] M. Burger, Intersection, the Manhattan curve, and Patterson-Sullivan theory in rank 2, Int. Math. Res. Notices 7, 217–225, 1993.

[7] R. de la Llave, J. Marco and R. Moriyón, Canonical perturbation of Anosov systems and regularity results for the Livšic cohomology equation, Ann. of Math. 123, 537–611, 1986.

[8] T. Jordan, M. Kesseböhmer, M. Pollicott and B. Stratmann, Sets of non-differentiability for conjugacies between expanding interval maps, Fund. Math. 206, 161–183, 2009.

[9] S. Lalley, Mostow rigidity and the Bishop-Steger dichotomy for surfaces of variable negative curvature, Duke Math. J. 68, 237–269, 1992.

[10] C. McMullen, Thermodynamics, dimension and the Weil-Petersson metric, Invent. math. 173, 365–425, 2008.

[11] W. Parry and M. Pollicott, Zeta functions and the periodic orbit structure of hyperbolic dynamics, Astérisque, 187–180, 1-268, 1990.

[12] M. Ratner, Markov partitions for Anosov flows on n-dimensional manifolds, Israel J. Math. 15, 92–114, 1973.

[13] D. Ruelle, Zeta-functions for expanding maps and Anosov flows, Invent. math. 24, 231–234, 1976.

[14] R. Schwartz and R. Sharp, The correlation of length spectra of two hyperbolic surfaces, Comm. Math. Phys. 153, 423–430, 1993.

[15] C. Series, Geometrical methods of symbolic coding, in Ergodic Theory, Symbolic Dynamics, and Hyperbolic Spaces (Trieste, 1989), edsT. Bedford, M. Keane and C. Series, 125–151, Oxford University Press, Oxford, 1991.

[16] R. Sharp, The Manhattan curve and the correlation of length spectra on hyperbolic surfaces, Math. Zeit. 228, 745–750, 1998.

[17] D. Sullivan, Discrete conformal groups and measurable dynamics, Bull. Amer. Math. Soc. 6, 57–73, 1982.

[18] A. Weil, On the moduli of Riemann surfaces. In Oeuvres Scient., volume II, 381-389. Springer-Verlag, 1980.

[19] S. Wolpert, Thurston's Riemannian metric for Teichmüller space, J. Diff. Geom. 23, 146–174, 1986.