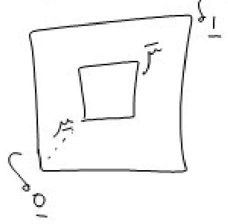
Toom's majority voter

$S = \mathbb{Z}^2$ $X_s = \{0,1\}$
$= \{-,+\}$

$$x_s^{t+1} = \begin{cases} \text{majority of } N \, E \, C \text{ votes at time } t \text{ w/ prob } 1-\lambda \\ \text{opposite} \quad \lambda \end{cases}$$

Has Exponentially attracting sta prob for $|\lambda - \frac{1}{2}| <$ some δ (exercise)

Monotone for $\lambda \leq \frac{1}{2}$

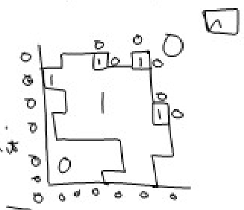


For λ small enough $\underline{\mu} < \bar{\mu}$

Idea of proof: For $\lambda = 0$

no islands of 1 in a sea of 0 survive.

Toom calls the CA an "eroder"



Given an island of 1 in a sea of 0

draw SW envelope: island can't grow beyond it

But every NE corner is eroded $\boxed{1}$

So after finitely many steps the island will disappear

Now turn on λ & show $\underline{\mu}(x_{00}^t = 1) \leq$ some $\Psi(\lambda)$

$\to 0$ as $\lambda \to 0$ by showing

$$P^{\underline{0}}(x_{00}^t = 1) = \sum_{m=0}^{\infty} \sum_{\substack{\text{certain graphs } \Gamma \\ \text{on } S \times T \text{ with} \\ m \text{ edges}}} P(G(\underline{x}) = \Gamma)$$

$$\leq \sum_m \underset{\leq\, (48)^{2m}}{N_m} \; \underset{\leq\, \lambda^{m/4+1}}{P(\text{error at each vertex})} = \frac{\lambda}{1 - 48^2 \lambda^{1/4}}$$

for $\lambda^{1/4} < \frac{1}{48^2}$, $\lambda < \frac{1}{48^8}$

$< \frac{1}{2}$ for λ small enough

$[1 - 48^2 \lambda^{1/4} > 2\lambda]$

So ≥ 2 sta probs (+ their convex combinations)

Maybe more? I call these "ferromagnetic" phases

Variations:

λ near 1 "antimajority" voter

This is equivalent to $\lambda = 1-\lambda$ by recoding

$x^{\text{odd } t} = -x^t$. So if $\delta_{\underline{0}} P_\varepsilon^t \to \underline{\mu}$

then $\delta_{\underline{0}} P_{1-\varepsilon}^t \to$ 2-cycle $\begin{cases} \underline{\mu} \text{ at even } t \\ \bar{\mu} \text{ at odd } t \end{cases}$

So on $\{0,1\}^{S \times T}$ we have ≥ 2 probs

"period-2" phases. This is an example of "non-trivial collective behavior" or "asymptotic periodicity"

- Can also make PCA which have "antiferromagnetic" phases

$$x_s^{t+1} = \text{majority of } (x_s^t, -x_{s+E}^t, -x_{s+N}^t) \text{ w/ prob } 1-\lambda \qquad \begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

Prove by recoding to usual Toom PCA by $(-)^{\text{parity of } s}$ $= s_1 + s_2 \bmod 2$

Similarly by taking λ near 1 get "antiferro per 2" phases

$$\begin{array}{ccc} + & - & + \\ - & + & - \\ + & - & + \end{array} \qquad \begin{array}{ccc} - & + & - \\ + & - & + \\ - & + & - \end{array}$$

even t odd t

- Can break $+ \leftrightarrow -$ symmetry & still get non-unique sta prob
- Can make interaction anisotropic (Bennett–Grinstein) & still get non-unique sta prob.

Contrast 2D Ising model where non-unique phase for low temperature is lost as soon as add a magnetic field.

Open Q: What about adding S & W nbr influence? Toom in conts time?

Note: $\exists$ continuous-time version (of Liggett) but v. different because chooses to duplicate a random nbr's state precisely so $\underset{\sim}{+}$, $\underset{\sim}{-}$ are absorbing.

3. General perspective for phases of PCA

Define a *phase* of a PCA as a limit point of probability on space-time configurations started from an initial space prob in distant past

Any convex combination of phases is a phase, so suffices to consider "extremal" phases

So Stavskaya has 2 extremal phases for λ small enough

$\delta_{\underset{\sim}{1}}$ and that generated by ν_λ

Phases do not have to be time-translation invariant e.g. period-2 phases of Toom PCA for $\lambda = 1 - \varepsilon$.

Emergence : dynamical model $\to$ {phases}

Amount of emergence expressed by a phase μ

$$= D(\mu, \{\text{products of indpt dynamics}\})$$

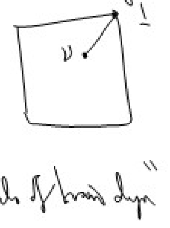
$\cong$ distance from mean-field models

Weak emergence means $D > 0$

Strong emergence : {phases} is not a singleton i.e. non-unique phase

Amount of strong emergence = diam ({phases})

$= D(\nu, \delta_{\underline{1}})$ for Stavskaya



Nonlinearity Dec 2008

Edmund Rolls ~ "Stochastic models of brain dyn"