Topics in Complexity Science

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Tue & Fri 9.15 (start convention?)
except 3 Nov & 11 Dec
Please register by email to (no implied commitment)
graduate.studies @ maths.ox.ac.uk

We'll start at 9.15 sharp!
(aiming to end 10.05)

Complexity Science = study of systems with many interdependent components

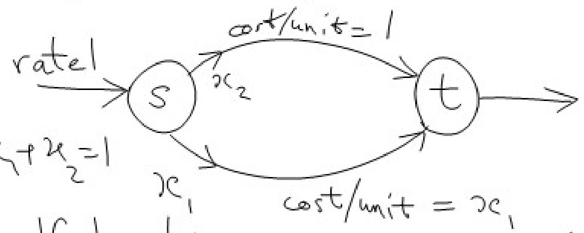
Selfish traffic flow (3)
Space-time phases (13)

Selfish routing

Roughgarden T, Selfish routing and the price of anarchy (MIT, 2005)

Does the free market lead to a desirable situation? NO!

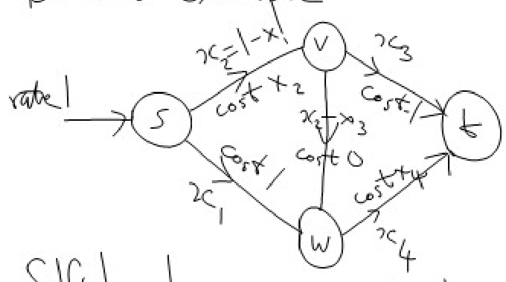
Pigou's example



$x_1 + x_2 = 1$

Selfish solution $x_1 = 1$ has cost 1 for everybody
whereas $x_1 = \frac{1}{2}$ has average cost $\frac{3}{4}$
and cost ≤ 1 for everybody

Braess' example



Selfish soln $x_2 = x_4 = 1$ has cost 2 for everyone
Whereas $x_2 = x_4 = \frac{1}{2} = x_3 = x_4$ has cost $\frac{3}{2}$ for everyone.

1. Basic theory of Nash flows
2. Bounding the price of anarchy
3. Avoiding Braess' paradox & effects of taxes/incentives

Ch 1 Basics for Nash Flows

Finite directed graph G, vertices V, edges E

Commodity i = a pair s_i, t_i of nodes and a traffic rate r_i

Path = route in G with no cycles

$\mathcal{P}_i = \{$ paths from s_i to $t_i \}$

$\mathcal{P} = \bigcup_i \mathcal{P}_i$

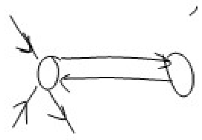
a "flow" is a function $f : \mathcal{P} \to \mathbb{R}_+$ $(f_P)_{P \in \mathcal{P}}$

Induces a "flow on edges", function f (again) : $E \to \mathbb{R}_+$ $(f_e)_{e \in E}$

$$f_e = \sum_{P \in \mathcal{P} \text{ s.t. } e \in P} f_P$$

Say f "feasible" if $\sum_{P \in \mathcal{P}_i} f_P = r_i$

Cost c_e for using edge e, non-neg, continuous, non-decreasing, function of f_e



We suppose no cost from junctions

So $c_P(f) = \sum_{e \in P} c_e(f_e)$

Total cost $C(f) = \sum_{P \in \mathcal{P}} c_P(f) f_P$

$= \sum_{e \in E} c_e(f_e) f_e$.

f^* "optimal" if it minimises $C(f)$ over feasible flows f. $\exists$ optimal flow because C is a cnts fn on a compact set

f is a "Nash flow" (Wardrop equilibrium) iff feasible & $\forall i$, $P, P' \in \mathcal{P}_i$ with $f_P > 0$, $\delta \in (0, f_P]$ then $c_P(f) \leq c_{P'}(\tilde{f})$ where $\tilde{f}$ is obtained from f by switching δ flow from P to P'

i.e. no unilateral improvement is possible

It follows that a feasible f is Nash

$\Leftrightarrow$ $\forall i, P, P' \in \mathcal{P}_i$ with $f_P > 0$

then $c_P(f) \leq c_{P'}(f)$

(all flow travels on min cost paths)

In particular, f Nash $\Rightarrow$ all paths used in $\mathcal{P}_i$ have same cost $c_i(f)$ & $C(f) = \sum c_i(f) r_i$