Summary of last time: <u>Basics for Nash flows</u>

- finite directed graph G, vertices V, edges E
- rates r_i for flow from source s_i to target t_i ("commodity" i)
- cost functions $c_e(f_e)$ of flow on edge e
  continuous, non-negative, non-decreasing
- flow f is <u>Nash</u> if no unilateral improvement possible

<u>Thm 1</u>: Every (G, r, c) admits a Nash flow and "essentially unique": if $f, \tilde{f}$ Nash then $c_e(f_e) = c_e(\tilde{f}_e) \quad \forall e \in E$

To prove it, first characterise optimal flows (minimising $C(f) = \sum_{e \in E} f_e \, c_e(f_e)$) under stronger condns on c then prove Nash flows are optimal for a modified cost fn satisfying the stronger conditions.

Reminder: start promptly at 09.15
copy of starboard files on my website under Topics in Complexity Science tag

So temporarily suppose c_e "semi-convex"
i.e. $h_e(x) = x\, c_e(x)$ convex
$\left(h_e(\lambda x + (1-\lambda) y) \leq \lambda h_e(x) + (1-\lambda) h_e(y) \quad \forall \lambda \in (0,1) \right)$
& suppose h_e is C^1.
Then C is convex. For a path P
let $h_P'(f) = \sum_{e \in P} h_e'(f_e)$ = derivative of $h_P = \sum_e h_e$ wrt f_P

<u>Thm 2</u> f^* optimal $\Leftrightarrow$ $h_P'(f^*) \leq h_{P'}'(f^*) \quad \forall i, P, P' \in \mathcal{P}_i$ with $f_P^* > 0$.

<u>Proof</u>: $\Rightarrow$ Suppose f^* optimal, $P, P' \in \mathcal{P}_i$, $f_P^* > 0$. Transferring $\lambda > 0$ flow from P to P' gives cost $\sum_e h_e(f_e^*) + \lambda \left(\sum_{e \in P'} h_e'(f_e^*) - \sum_{e \in P} h_e'(f_e^*) \right) + o(\lambda) \geq C(f^*)$ for $\lambda > 0$ so

So $h_P'(f^*) \leq h_{P'}'(f^*)$

$\Leftarrow$ Suppose $h_P'(f^*) \leq h_{P'}'(f^*) \ \forall P, P' \in \mathcal{P}_i$, $f_P^* > 0$. By convexity of h_e (Roughgarden 2005)

$$C(f) = \sum_e h_e(f_e) \geq \sum h_e(f_e^*) + h_e'(f_e^*)(f_e - f_e^*)$$
$$= C(f^*) + H(f) - H(f^*) \quad \text{where}$$
$$H(f) = \sum_e h_e'(f_e^*) f_e = \sum_P h_P'(f^*) f_P$$

A feasible f minimises H iff all $P \in \mathcal{P}$ with $f_p > 0$ minimises $h'_p(f^*)$ over $P' \in \mathcal{P}_i$

So f^* minimises H. Thus $C(f) \geq C(f^*)$. $\square$

<u>Cor 3</u> f is optimal for c with $h_e(x) = x c_e(x)$ convex C' $\iff$ f is Nash for $c_e^* = h'_e$

i.e. $c_e^*(x) = c(x) + x c'(x)$. "marginal cost fn"

In particular, Nash flows optimise $\sum_e f_e c_e^*(f)$ but not necessarily C. And can make optimal flow for c arise as Nash flow if add "marginal cost taxes" $f_e c'_e(f_e)$ [can also subtract a constant c_i for each commodity i] [true even if h is not convex]

<u>Proof of Thm 1</u> $\exists$ Nash: Let

$$k_e(x) = \frac{1}{x} \int_0^x c_e(t)\,dt$$

which has $x k_e(x)$ convex & C'. Then $\exists$ opt flow f for $K = \sum_e x_e k_e(x_e)$ [because K cont. fn on a compact set] and thus f is Nash for $[x k_e(x)]' = c_e(x)$

2 Nash flows have equal edge costs: K convex

$f, \tilde{f}$ opt flows for K $\Rightarrow$ K constant on $\lambda f + (1-\lambda)\tilde{f}$ $\quad \lambda \in [0,1]$

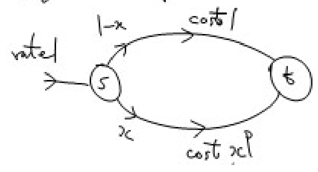
Each $x k_e(x)$ convex $\Rightarrow$ each affine on this line $\Rightarrow$ $c = (xk)'$ constant between f_e & $\tilde{f}_e$. $\square$

<u>Cor 4</u>: f is Nash iff $\sum_e c_e(f_e) f_e \leq \sum_e c_e(f_e) \tilde{f}_e$ for all feasible $\tilde{f}$

<u>Proof</u>: Thm 2 applied to $h_e(x) = \int_0^x c_e(t)\,dt$.

2. <u>Bounding price of anarchy</u> $\quad \rho = \dfrac{C(f_{Nash})}{C(f^*)}$

Pigou example (modified)



Nash $x = 1$

Optimal $x = (p+1)^{-1/p}$

(equality of h'_p for both paths where $h(x) = x c(x)$)

So $\rho = \dfrac{C(f_{Nash})}{C(f^*)} = \dfrac{1}{1 - p(p+1)^{-(p+1)/p}} \to \infty$ as $p \to \infty$

<u>Affine cost functions</u> $\quad c_e(x) = a_e x + b_e$

We'll prove $\rho \leq \frac{4}{3}$ [note: equality was attained for Pigou ($p=1$) & Braess example of last time]