Today I'll start main chapter on Space-Time Phases

Recall notes available on my website via Topics in CX Sci tab

Recall no lecture next Tues 3? Nov

Statistical Phases of dynamics on large networks:

I Stochastic case

Probabilistic Cellular Automata

e.g. Stavskaya PCA (aka directed percolation, asymmetric contact process, Reggeon field)

State $x = (x_s)_{s \in \mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$ 1 = healthy, 0 = infected

Reading: Toom A.L. (Manchester UP, 1990)

Parameter $\lambda \in (0,1)$ = prob of recovery in one step = prob of avoiding infection from nbr

At each time $t \in \mathbb{Z}_+$, for each site $s \in \mathbb{Z}$

if $(x_{s-1}^t, x_s^t) = (1,1)$ then $x_s^{t+1} = 1$

indept of all other $x_{s'}^{t'}$ $\neq (1,1)$ then $x_s^{t+1} = \begin{cases} 1 & \text{with prob } \lambda \\ 0 & 1-\lambda \end{cases}$

$\underset{\sim}{1}$ = All 1 is an absorbing state

For finite system i.e. on $\mathbb{Z}_N = \mathbb{Z} \bmod N$

then $P\{x^{t+1} = \underset{\sim}{1} \mid x^t\} \geqslant \lambda^N$

for all x^t, so $P\{\text{not absorbed by time } t\}$
$\leqslant (1-\lambda^N)^t \to 0$ as $t \to \infty$, so
$P\{\text{eventually } \underset{\sim}{1}\} = 1$.

Hides an important feature: $\exists \lambda_c \in (0,1)$

s.t. for $\begin{cases} \lambda > \lambda_c & \text{time to absorption} \sim \log_\gamma N \\ \lambda < \lambda_c & \sim e^{\gamma N} \end{cases}$ some $\gamma(\lambda) > 0$

Best understood by considering ∞ system:

for $\begin{cases} \lambda > \lambda_c & \text{stationary prob. measure } \delta_{\underset{\sim}{1}} \text{ attracts all initial prob. measures} \\ \lambda < \lambda_c & \exists \text{ additional sta prob } \nu_\lambda \end{cases}$

with $\nu_\lambda(x_s = 0) = c(\lambda) > 0$ and (at least for small λ) $\forall$ initial prob ν

$$\nu P^t \to \sigma \delta_{\underset{\sim}{1}} + (1-\sigma)\nu_\lambda \quad \text{with}$$

$\sigma = \nu\{\text{eventual absorption}\}$

ν_λ = "endemic infection", λ_c = epidemic threshold

Treat λ near 1

Introduce a nice metric on spaces of multivariate probs to prove (exponential) attraction of νP^t to $\delta_{\underset{\sim}{1}}$

See my paper "Robustness of Markov processes on large networks" (on my website)

Setting: S countable set, "network", can have a metric on this but not reqd

$s \in S$ "site"

$\forall s \in S$ (X_s, d_s) complete separable metric space of diameter $\leqslant \Omega$

e.g. $S=\mathbb{Z}$, $X_s = \{0,1\}$ $\quad d_s(0,1)=1$

$X = \prod_{s\in S} X_s$ with product topology

$\mathcal{P}$ = Borel probs on X

Z = zero-charge measures on X

Transition probs $p_s(x'_s \mid \underset{\sim}{x})$ (or $p_s(dx'_s \mid \underset{\sim}{x})$ if X_s is not discrete)

Product $p(d\underset{\sim}{x}' \mid \underset{\sim}{x})$

C = bdd conts fns $X \to \mathbb{R}$

Transition operator P on C: $(Pf)(x) = \int f(x')\,p(dx' \mid x)$

Note $P\mathbb{1} = \mathbb{1}$ where $\mathbb{1}(x) = 1 \ \forall x\in X$

Transition operator P on $\rho\in\mathcal{P}$ $\quad (\rho P)f = \rho(Pf)$

(where $\rho(f)$ = mean of f wrt ρ)

Stationary prob $\rho P = \rho$

For $f\in C$, $s\in S$, $\Delta_s(f) = \sup \dfrac{f(x)-f(x')}{d_s(x_s,x'_s)}$

over $x_s \neq x'_s$, $x_r = x'_r \ \forall r\neq s$

i.e. Lipschitz constant of f wrt x_s

$$F = \{ f\in C : \|f\|_F = \sum_{s\in S} \Delta_s(f) < \infty \}$$

normed linear space $\quad$ addition of cts

For $\mu\in Z$ let $\|\mu\|_Z = \sup\limits_{f\in F\setminus 0} \dfrac{\mu(f)}{\|f\|_F}$

where $\mu(f) = \int f\,d\mu$. $\ Z, \|\ \|_Z$ Banach space

$D(\rho,\rho')$ $\ \rho,\rho'\in\mathcal{P}$ $= \|\rho-\rho'\|_Z$ makes $\mathcal{P}$ a complete metric space

This is based on theory by Dobrushin

For $L: Z \circlearrowleft$ linear e.g. restriction of P to Z

let $\|L\|_Z = \sup\limits_{\mu\in Z\setminus 0} \dfrac{\|\mu L\|_Z}{\|\mu\|_Z}$

The point of $\|\ \|_Z$ is to make transition operators P, P' which "look" close be close, uniformly in size of S.