Fun with Leibniz sums

Benedict Sewell

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Abstract

A Leibniz sum is an alternating sum with the absolute value of the terms decreasing to zero, well loved and well understood. Here we prove a small statement regarding such sums whose terms decrease exponentially.

Theorem. *Suppose* $(a_n)_{n=1}^{\infty}$ *is a non-negative sequence together with* $\lambda \geq \mu \geq 0$ *satisfying for all* $n \geq 1$,

$$\mu a_n \leq a_{n+1} \leq \lambda a_n \tag{1}$$

then

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n \leq a_1 \frac{1-\mu}{1-\lambda\mu}. \tag{2}$$

Proof. First we establish some notation. Let $\lambda \geq \mu \geq 0$ and a_1 be fixed and let S denote the set of non-negative sequences which satisfy (1) and have first term equal to a_1. Let $f : S \to \mathbb{R}$ denote the map taking sequences onto their alternating sum:

$$(a_n) \longmapsto \sum_{n=1}^{\infty} (-1)^{n+1} a_n.$$

As for the proof itself: for obvious reasons, take $a_1 \neq 0$. Then

$$\begin{aligned} f(a_1, a_2, a_3, \ldots) = \sum_{n=1}^{\infty} (-1)^{n+1} a_n \quad &= \quad a_1 - \overbrace{\sum_{n=1}^{\infty} (-1)^{n+1} a_{n+1}}^{\text{positive}} && (3) \\ &\leq \quad a_1 - \frac{\mu a_1}{a_2} \sum_{n=1}^{\infty} (-1)^{n+1} a_{n+1} && (4) \\ &= \quad \underbrace{f(a_1, \mu a_1, \frac{\mu a_1 a_3}{a_2}, \ldots)}_{\text{still in } S} && (5) \end{aligned}$$

with a strict inequality if $a_2 > \mu a_1$.

Likewise, the value of f can be increased by a perturbation within S, if for any $n \geq 1$, either $a_{2n} > \mu a_{2n-1}$ or $a_{2n+1} < \lambda a_{2n}$ holds. It is then immediate that the maximal value of f on S is obtained by taking the almost-geometric series,

$$f(a_1,\, \mu a_1,\, \lambda\mu a_1,\, \lambda\mu^2 a_1,\, \lambda^2\mu^2 a_1,\, \ldots) \;=\; a_1 \frac{1-\mu}{1-\lambda\mu}. \qquad \square$$