2.3 LDP in Topological Vector Spaces

We have seen earlier (see chapter 1 Cramér's theorem) that when a limiting logarithmic moment generating function exists for a family/sequence of real-valued random variables, then its Fenchel-Legendre transform is the natural candidate rate function for the LDP associated with these variables. The goal of this chapter is to extend this result to topological vector space.

Throughout this chapter, E is a Hausdorff (real) topological vector space. Furthermore, even when E is not itself a vector space, it often turns out that it is a convex subset of one. For this reason we shall formulate the following somewhat cumbersome hypothesis about E:

(A) E is a closed convex subset of the locally convex, Hausdorff topological (real) vector space, and E is a Polish space with respect to the topology that it inherits as a subset of X.

Remark: The two examples which should be kept in mind are when E is itself a Banach space (in which case we take $X = E$) and when $E = \mathcal{M}_1(\Sigma)$, where Σ is a Polish space. In the latter case, we take $X = M(\Sigma)$ to be the space of all finite signed measures on Σ (this is a vector space) and endow $M(\Sigma)$ with the topology generated by sets

$$\left\{ \beta \in M(\Sigma) : \left| \int \phi(x)\, d(\beta(x) - \alpha(x)) \right| < r \right\},$$

where $\alpha \in M(\Sigma)$, $\phi \in C_b(\Sigma)$, and $r > 0$.

The norm on $M(\Sigma)$ is the total variation, i.e.,

$$\|\alpha\|_{\text{var}} = \sup\left\{ \int \phi(x)\, \alpha(dx) : \phi \in C_b(\Sigma) \text{ with } \|\phi\|_\infty \le 1 \right\}, \; \alpha \in M(\Sigma).$$

$\|\cdot\|_{\text{var}}$ is lower semi-continuous on $M(\Sigma)$ and therefore certainly measurable on $M(\Sigma)$; and clearly $\|\cdot\|_{\text{var}}$ is bounded on $\mathcal{M}_1(\Sigma)$.

The Lévy metric on $\mathcal{M}_1(\Sigma)$ is a complete separable metric, which is consistent with the restriction of the topology on $M(\Sigma)$ to $\mathcal{M}_1(\Sigma)$.

Following Lévy and Prohorov, define the Lévy metric

$$d(\alpha, \nu) = \inf\Big\{ \delta > 0 : \alpha(F) \le \nu(F^{(\delta)}) + \delta \text{ and } \nu(F) \le \alpha(F^{(\delta)}) + \delta \;\; \forall \text{ closed } F \subset \Sigma \Big\}$$

for $\alpha, \nu \in \mathcal{M}_1(\Sigma)$, where $F^{(\delta)}$ is defined relative to a complete

metric on Σ (here $F^{(\delta)}$ is the open δ-hull of F).
Since it is clear that $d(\alpha,\nu) \leq \|\nu-\alpha\|_{var}$,
all that remains is to show that d is compatible with the weak topology and that $(M_1(\Sigma), d)$ is Polish.
Recall the following facts for a sequence $(\alpha_n)_{n\geq 1}$ of probability measures $\alpha_n \in M_1(\Sigma)$:

(i) $\alpha_n \Rightarrow \nu$ (weakly) as $n\to\infty \iff$

$$\overline{\lim_{n\to\infty}} \alpha_n(F) \leq \nu(F) \quad \forall \text{ closed } F\subset\Sigma$$

(ii) $\Gamma \subset M_1(\Sigma)$ is relatively compact $\iff \forall\ \delta>0$ there is a $K\subset\subset\Sigma$ (compact subset) such that $\alpha(K)\geq 1-\delta$ for every $\alpha\in\Gamma$. (Such a subset Γ is said to be tight).

(iii) If $\mathcal{F}\subset C_b(\Sigma)$ is uniformly bounded on all of Σ and is equicontinuous on each compact subset of Σ, then $\alpha_n \Rightarrow \nu$ implies that

$$\sup\left\{\left|\int\phi(x)\alpha_n(dx) - \int\phi(x)\nu(dx)\right| : \phi\in\mathcal{F}\right\} \to 0 \text{ as } n\to\infty.$$

(see books on measure theory or Billingsley: "Conv. of prob. measures").

<u>Lemma 5</u>: (Lévy & Prohorov). The metric d (defined above) is compatible with the weak topology on $M_1(\Sigma)$, and $(M_1(\Sigma),d)$ is Polish.

After this brief excursion on important facts we return to set up the setting in topological vector spaces.
The dual space of E, namely, the space of all continuous linear functionals on E, is denoted throughout by E^*. Let $(X_n)_{n\geq 1}$ be a sequence of E-valued random variables X_n, and let $\mu_n \in \mathcal{M}_1(E)$ denote the probability measure associated with X_n.
By analogy with the $\mathbb{R}$ case presented in chapter 1, the logarithmic moment generating function
$\Lambda_{\mu_n}: E^* \to (-\infty, \infty]$ is defined to be

$$\Lambda_{\mu_n}(\lambda) = \log \mathbb{E}[e^{\langle \lambda, \cancel{X_n} \rangle}] = \log \int_E e^{\lambda(x)} \mu_n(dx), \quad \lambda \in E^*;$$

where for $x \in E$ and $\lambda \in E^*$, $\langle \lambda, x \rangle$ denotes the value of $\lambda(x) \in \mathbb{R}$.
Let $\overline{\Lambda}(\lambda) = \limsup_{n\to\infty} \frac{1}{n} \Lambda_{\mu_n}(n\lambda)$, using the notation $\Lambda(\lambda)$ whenever the limit exists.
The Legendre (Fenchel-Legendre) transform of a function $f: E^* \to [-\infty, \infty]$ is defined as

$$f^*(x) := \sup_{\lambda \in E^*} \{ \langle \lambda, x \rangle - f(\lambda) \}, \quad x \in E.$$

<u>**Theorem 8:**</u> Let $(\mu_n)_{n\geq 1}$ be a sequence in $\mathcal{M}_1(E)$ and assume that

$$\Lambda(\lambda) := \lim_{n\to\infty} \frac{1}{n} \Lambda_{\mu_n}(n\lambda) \quad \in [-\infty, \infty]$$

exists for every $\lambda \in E^*$. Then Λ is a convex function on E^*. Moreover, if the Legendre transform is defined by
$$\Lambda^*(f) = \sup_{\lambda\in E^*} \{ \langle \lambda, f\rangle - \Lambda(\lambda) \} ,$$ then

Λ^* is a non-negative, lower semi-continuous, convex function; and for any $F \subset\subset E$,

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \log \mu_n(F) \leq - \inf_{x\in F} \Lambda^*(x) .$$

Finally, if in addition, $(\mu)_{n\geq 1}$ is exponentially tight, then the latter inequality continues to holds for all closed subsets F of E.

<u>Remark:</u> $\overline{\Lambda}, \overline{\Lambda}^*$ are convex and $\forall\ F \subset\subset E$

$$\limsup_{n\to\infty} 1/n \log \mu_n(F) \leq -\inf_{x\in F} \overline{\Lambda}^*(x)$$

<u>Proof:</u> Convexity of Λ follows from that of the Λ_{μ_n}'s, which in turn is a consequence of Hölder's inequality. To see $\Lambda^*(f) \geq 0$ for every $f\in E$, simply note that $\Lambda(0)=0$. Λ^* is supremum (point-wise) over continuous affine functions on E, henceforth

it is lower semi-continuous and convex.

To show the upper bound, fix a compact set $F \subset\subset E$, $x \in F$, and $\delta > 0$. There is a $\lambda_x \in E^*$ such that

$$\langle \lambda_x, x \rangle - \Lambda(\lambda_x) \geq \begin{cases} 1 + 1/\delta & \text{if } \Lambda^*(x) = \infty, \\ \Lambda^*(x) - \delta/2 & \text{if } \Lambda^*(x) < \infty. \end{cases}$$

As λ_x is continuous, there exists $r > 0$ such that

$$\inf_{y \in B_r(x)} \{ \langle \lambda_x, y \rangle - \langle \lambda_x, x \rangle \} \geq -\delta .$$

For any $\theta \in E^*$, by Chebycheff's inequality,

$$\mu_n(\overline{B_r(x)}) \leq \mathbb{E}[e^{\langle \theta, Z_n \rangle - \langle \theta, x \rangle}] \exp\{ - \inf_{y \in \overline{B_r(x)}} \{ \langle \theta, y \rangle - \langle \theta, x \rangle$$

Choosing $\theta = n \lambda_x$ yields

$$\frac{1}{n} \log \mu_n(\overline{B_r(x)}) \leq \delta - \{ \langle \lambda_x, x \rangle - n \Lambda_{\mu_n}(n \lambda_x) \} .$$

There is a finite cover, $\bigcup_{i=1}^{N} A_{x_i}$, of the compact set F, where we choose $A_{x_i} = \overline{B_{r_i}(x_i)}$, $x_i \in F$.

Thus

$$1/n \log \mu_n(F) \leq \frac{1}{n} \log N + \delta - \min_{i=1,\dots,N} \{ \langle \lambda_{x_i}, x_i \rangle - 1/n \Lambda_{\mu_n}(\lambda_{x_i} n) \}$$

and hence

$$\limsup_{n\to\infty} \frac{1}{n}\log\mu_n(F) \le \delta - \min_{i=1,\dots,N} \underbrace{\{\langle\lambda_{x_i}, x_i\rangle - \Lambda(\lambda_{x_i})\}}_{=: I^\delta(x_i)}$$

Moreover, $x_i \in F$ for each $i \in \{1,\dots,N\}$, yielding the inequality

$$\limsup_{n\to\infty} 1/n \log\mu_n(F) \le \delta - \inf_{x\in F} I^\delta(x)$$

with $I^\delta(x) = \langle\lambda_x, x\rangle - \Lambda(\lambda_x) \geq \begin{cases} 1/\delta & \Lambda^*(x)=\infty \\ \Lambda^*(x) - \delta/2 \end{cases}$.

The proof is complete by taking $\delta \to 0$.

Finally, the extension to all closed F when $(\mu_n)_{n\geq 1}$ is exponentially tight is precisely the assertion of Lemma 4. ■

Although the preceding indicates that, when Λ exists, its Fenchel-Legendre transform Λ^* is a good candidate for the rate function governing the large deviations of $(\mu_n)_{n\geq 1}$, we know that, in general, Λ^* will not be the correct rate function.

Before proceeding with the attempt to identify the rate function of the LDP as Λ^*, note that while Λ^* is always convex, the rate function may well be non-convex.

Lemma 6: (Duality Lemma)

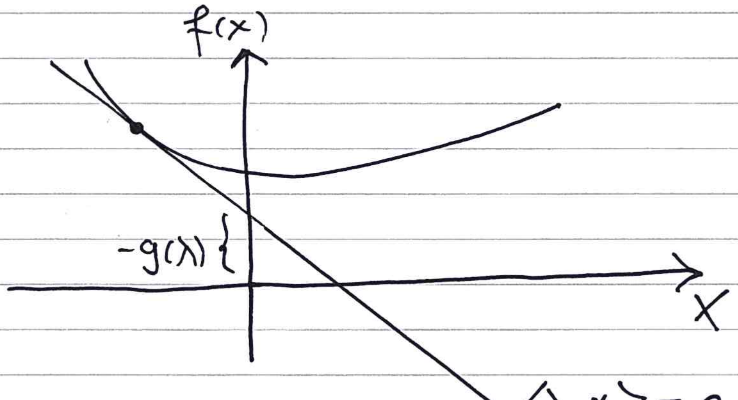
Let E be a locally convex Hausdorff topological vector space. Let $f: E \to (-\infty, \infty]$ be a lower semi-continuous convex function, and define

$$g(\lambda) = \sup_{x \in E} \{ \langle \lambda, x \rangle - f(x) \}, \quad \lambda \in E^*.$$

Then f is the Fenchel-Legendre transform of g, i.e.,

$$f(x) = \sup_{\lambda \in E^*} \{ \langle \lambda, x \rangle - g(\lambda) \}, \quad x \in E.$$

Remark: For every hyperplane defined by $\lambda \in E^*$, $g(\lambda)$ is the largest amount one may push up the tangent before it hits f and becomes a tangent hyperplane.



Before we proof this important lemma we show the first application of it in the following theorem.

Theorem 9: Let E be locally convex Hausdorff topological vector space. Assume that $(\mu_n)_{n\geq 1}$ satisfies the LDP with good rate function I. Suppose in addition that

$$\overline{\Lambda}(\lambda) = \limsup_{n\to\infty} \frac{1}{n} \Lambda_{\mu_n}(n\lambda) < \infty \quad \forall \lambda \in E^*.$$

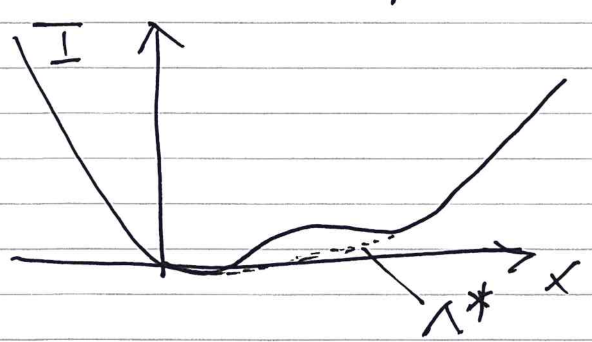
(a) $\forall \lambda \in E^*$, the limit $\Lambda(\lambda) = \lim_{n\to\infty} 1/n\, \Lambda_{\mu_n}(n\lambda)$ exists, is finite, and satisfies

$$\Lambda(\lambda) = \sup_{x\in E}\{\langle \lambda, x\rangle - I(x)\}.$$

(b) If I is convex, then it is the Fenchel-Legendre transform of Λ, namely,

$$I(x) = \Lambda^*(x) = \sup_{\lambda\in E^*}\{\langle\lambda, x\rangle - \Lambda(\lambda)\}.$$

(c) If I is not convex, then Λ^* is the affine regularisation of I, i.e., $\Lambda^* \leq I$, and for any convex function f, $f \leq I$ implies $f \leq \Lambda^*$.



Proof: exercise (combine all previous results) ∎

Proof of Lemma 6:

If f is identically ∞, then g is identically $-\infty$ and the lemma trivially holds. Assume otherwise and define

$$\mathcal{E} = \{(x,\alpha) \in E \times \mathbb{R} : f(x) \leq \alpha\},$$

$$\mathcal{E}^* = \{(\lambda,\beta) \in E^* \times \mathbb{R} : g(\lambda) \leq \beta\}.$$

For any $(\lambda,\beta) \in E^*$ and any $x \in E$,

$f(x) \geq \langle \lambda, x \rangle - \beta$. Therefore, it also holds that

$$f(x) \geq \sup_{(\lambda,\beta)\in\mathcal{E}^*} \{\langle \lambda, x\rangle - \beta\} = \sup_{\lambda\in E^*} \{\langle \lambda, x\rangle - g(\lambda)\}.$$

It thus suffices to show that for any $(x,\alpha) \notin \mathcal{E}$ (i.e., $f(x) > \alpha$), there exists a $(\lambda,\beta) \in \mathcal{E}^*$ such that

(+) $\langle \lambda, x\rangle - \beta > \alpha$, in order to complete the proof of the lemma.

The set $\mathcal{E}$ is closed. This follows from the fact that f is lower semi-continuous. Indeed, whenever $f(x) > \gamma$, there exists a neighbourhood V of x such that $\inf_{y\in V} f(y) > \gamma$, and thus $\mathcal{E}^c$ contains a neighbourhood of (x,γ).

Since f is convex and $f \not\equiv \infty$, the set $\mathcal{E}$ is non-empty convex subset of $E \times \mathbb{R}$.

Fix $(x, \alpha) \notin \mathcal{E}$. $E \times \mathbb{R}$ is locally convex and therefore, by the Hahn-Banach theorem, there exists a hyperplane in $E \times \mathbb{R}$ that strictly separates the non-empty, closed, and convex set $\mathcal{E}$ and the point (x, α) in its complement. (dual of $E \times \mathbb{R}$ is $E^* \times \mathbb{R}$)
Hence, for some $\mu \in E^*$, $\rho \in \mathbb{R}$, and $\gamma \in \mathbb{R}$

$$\sup_{(y, \xi) \in \mathcal{E}} \{ \langle \mu, y \rangle - \rho \xi \} \leq \gamma < \langle \mu, x \rangle - \rho \alpha .$$

$f \not\equiv \infty \Rightarrow \rho \geq 0$.
Pick $(y, \xi) = (x, f(x))$ to conclude that $\rho > 0$ whenever $f(x) < \infty$.

We first suppose that $\rho > 0$. Then $(\mu/\rho, \gamma/\rho)$ satisfies inequality $(+)$, i.e., $\langle \mu/\rho, x \rangle - \gamma/\rho > \alpha$.
This point must be in $\mathcal{E}^*$, for otherwise there exists a $y_0 \in E$ such that $\langle \mu, y_0 \rangle - \rho f(y_0) > \gamma$, contradicting the construction of the separating hyperplane.
Since $f(x) < \infty$ for some $x \in E$ it follows that $\mathcal{E}^*$ is non-empty.
Now suppose that $\rho = 0$ so that

$$\sup_{\{y : f(y) < \infty\}} \{ \langle \mu, y \rangle - \gamma \} \leq 0 \qquad \text{while } \langle \mu, x \rangle - \gamma > 0.$$

Consider $(\lambda_\delta, \beta_\delta) = \left(\mu/\delta + \lambda_0, \gamma/\delta + \beta_0\right)$, $\forall \delta > 0$,

and some $(\lambda_0, \beta_0) \in \mathcal{E}^*$. Then, for all $y \in E$,

$$\langle \lambda_\delta, y \rangle - \beta_\delta = 1/\delta \left(\langle \mu, y\rangle - \gamma\right) + \left(\langle \lambda_0, y \rangle - \beta_0\right) \leq f(y).$$

Therefore, $(\lambda_\delta, \beta_\delta) \in \mathcal{E}^*$ $\forall \delta > 0$. Moreover,

$$\lim_{\delta \to 0} (\langle \lambda_\delta, x\rangle - \beta_\delta) = \lim_{\delta \to 0} \left\{ \frac{1}{\delta}(\langle \mu, x \rangle - \gamma) + (\langle \lambda_0, x\rangle - \beta_0)\right\} = \infty.$$

Thus, for any $\alpha < \infty$, there exists $\delta > 0$ small enough so that $\langle \lambda_\delta, x \rangle - \beta_\delta > \alpha$. ∎

We finish the section with a version of an abstract Gärtner-Ellis Theorem.
To this end, recall that a point $x \in E$ is called an exposed point of $\overline{\Lambda}^*$ if there exists an exposing hyperplane $\lambda \in E^*$ such that

$$\langle \lambda, x \rangle - \overline{\Lambda}^*(x) > \langle \lambda, z \rangle - \overline{\Lambda}^*(z), \forall z \neq x.$$

<u>Theorem 10</u>: (Baldi) Suppose $(\mu_n)_{n \geq 1}$ are exponentially tight probability measures on E.

(a) For every closed set $F \subset E$,

$$\limsup_{n \to \infty} 1/n \log \mu_n(F) \leq - \inf_{x \in F} \overline{\Lambda}^*(x).$$

(b) Let $\mathcal{F}$ be the set of exposed points of $\bar{\Lambda}^*$ with an exposing hyperplane λ for which
$$\Lambda(\lambda) = \lim_{n\to\infty} \frac{1}{n} \Lambda_{\mu_n}(n\lambda) \text{ exists and } \bar{\Lambda}(\gamma\lambda) < \infty$$
for some $\gamma > 1$.
Then, for every open set $G \subset E$,
$$\liminf_{n\to\infty} \frac{1}{n} \log \mu_n(G) \geq -\inf_{x\in G\cap\mathcal{F}} \bar{\Lambda}^*(x).$$

(c) If for every open set $G \subset E$,
$$\inf_{x\in G\cap\mathcal{F}} \bar{\Lambda}^*(x) = \inf_{x\in G} \bar{\Lambda}^*(x),$$
then $(\mu_n)_{n\geq 1}$ satisfies the LDP with the good rate function $\bar{\Lambda}^*$.

<u>Proof:</u> (be omitted if timing does not allow)
(a) The upper bound follows with Theorem 8 in conjunction with the exponential tightness.

(b) If $\bar{\Lambda}(\lambda) = -\infty$ for some $\lambda \in E^*$, then $\bar{\Lambda}^* \equiv \infty$ and the large deviations lower bound trivially holds.
W.l.o.g $\bar{\Lambda} : E^* \to (-\infty, \infty]$.
Fix an open set G, an exposed point $y \in G \cap \mathcal{F}$, and

$\delta > 0$ arbitrarily small.

Let $\eta \in E^*$ be an exposing hyperplane for $\bar{\Lambda}^*$ at y such that

$\Lambda(\eta)$ exists and $\bar{\Lambda}(\gamma\eta) < \infty$ for some $\gamma > 1$.

There exists an open subset of G, denoted by B_δ, such that $y \in B_\delta$ and

$$\sup_{z \in B_\delta} \{ \langle \eta, z - y \rangle \} < \delta \quad \text{(continuity of } \eta\text{)}.$$

Observe that $\Lambda(\eta) < \infty$ and thus $\Lambda_{\mu_n}(n\eta) < \infty$ for sufficiently large n. Thus, for n sufficiently large, define $\tilde{\mu}_n$ via

$$\frac{d\tilde{\mu}_n}{d\mu_n}(z) = \exp\left\{ \langle \eta n, z \rangle - \Lambda_{\mu_n}(n\eta) \right\}.$$

Henceforth

$$1/n \log \mu_n(B_\delta) = \frac{1}{n} \Lambda_{\mu_n}(n\eta) - \langle \eta, y \rangle + \frac{1}{n} \log \int_{B_\delta} \exp\{ \langle n\eta, y - z \rangle \} \tilde{\mu}_n(dz)$$

$$\geq \frac{1}{n} \Lambda_{\mu_n}(\eta n) - \langle \eta, y \rangle - \delta + 1/n \log \tilde{\mu}_n(B_\delta)$$

(change of measure method). Therefore, by our assumptions for (b),

$$\liminf_{n\to\infty} 1/n \log \mu_n(G) \geq \lim_{\delta \to 0} \liminf_{n\to\infty} \frac{1}{n} \log \mu_n(B_\delta)$$

$$\geq \Lambda(\eta) - \langle \eta, y \rangle + \lim_{\delta \to 0} \liminf_{n \to \infty} \frac{1}{n} \log \tilde{\mu}_n(B_\delta)$$

$$\geq -\overline{\Lambda}^*(y) \qquad + \qquad \text{"} \qquad .$$

$(\mu_n)_{n \geq 1}$ are exponentially tight, so for each $\alpha < \infty$, there exists a compact set K_α such that

$$\limsup_{n \to \infty} 1/n \log \mu_n(K_\alpha^c) < -\alpha .$$

If for all $\delta > 0$ and all $\alpha < \infty$,

$$\limsup_{n \to \infty} 1/n \log \tilde{\mu}_n(B_\delta^c \cap K_\alpha) < 0, \qquad (*)$$

and for all α large enough,

$$\limsup_{n \to \infty} 1/n \log \tilde{\mu}_n(K_\alpha^c) < 0, \qquad (**)$$

then $\tilde{\mu}_n(B_\delta) \longrightarrow 1$ when $n \longrightarrow \infty$ and part (b) follows since $y \in G \cap \mathcal{F}$ is arbitrary.

Proof of $(*)$: Note that for every $\theta \in E^*$,

$$1/n \Lambda_{\tilde{\mu}_n}(n\theta) = 1/n \Lambda_{\mu_n}(n(\theta + \eta)) - 1/n \Lambda_{\mu_n}(n\eta),$$

and, by assumption (b),

$$\tilde{\Lambda}(\theta) := \limsup_{n \to \infty} 1/n \Lambda_{\tilde{\mu}_n}(n\theta) = \overline{\Lambda}(\theta + \eta) - \Lambda(\eta).$$

$$\tilde{\Lambda}^*(z) = \bar{\Lambda}^*(z) + \Lambda(\eta) - \langle \eta, z \rangle \geq$$
$$\bar{\Lambda}^*(z) - \bar{\Lambda}^*(y) - \langle \eta, z - y \rangle \qquad \forall\, z \in E.$$

Since η is an exposing hyperplane for $\bar{\Lambda}^*$ at $y \in B_\delta$, this implies that $\tilde{\Lambda}^*(z) > 0 \quad \forall\, z \neq y$.

We apply Theorem 8 to the measures $(\tilde{\mu}_n)_{n \geq 1}$ and the compact set $B_\delta^c \cap K_\alpha$ to conclude with

$$\limsup \tfrac{1}{n} \log \tilde{\mu}_n (B_\delta^c \cap K_\alpha) \leq -\inf_{z \in B_\delta^c \cap K_\alpha} \tilde{\Lambda}^*(z) < 0,$$

where the strict inequality follows because $\tilde{\Lambda}^*$ is a lower semi-continuous function and $y \in B_\delta$.

Proof of (**):

Consider the half-spaces (open) $H_\delta = \{z \in E : \langle \eta, z \rangle - \delta < 0\}$.

By Chebycheff's inequality, for any $\beta > 0$

$$\tfrac{1}{n} \log \tilde{\mu}_n (H_\delta^c) = \tfrac{1}{n} \log \int_{\{z : \langle \eta, z \rangle \geq \delta\}} \tilde{\mu}_n(dz)$$

$$\leq \tfrac{1}{n} \log \int_E \exp(n\beta \langle \eta, z \rangle)\, \tilde{\mu}_n(dz) - \beta\delta$$

$$= \tfrac{1}{n} \Lambda_{\tilde{\mu}_n}(n\beta\eta) - \beta\delta.$$

Hence,
$$\limsup_{n \to \infty} \tfrac{1}{n} \log \tilde{\mu}_n (H_\delta^c) \leq \inf_{\beta > 0} \{\tilde{\Lambda}(\beta\eta) - \beta\delta\}.$$

Due to assumption (b), $\tilde{\Lambda}(\beta\eta) < \infty$ for some $\beta > 0$, implying that for large enough ρ

$$\limsup_{n\to\infty} 1/n \log \tilde{\mu}_n (H_\rho^c) < 0 .$$

Now, for every α and every $\rho > 0$,

$$\limsup_{n\to\infty} 1/n \log \tilde{\mu}_n (K_\alpha^c \cap H_\rho)$$

$$= \limsup_{n\to\infty} 1/n \log \int_{K_\alpha^c \cap H_\rho} \exp\{\langle n\eta, z\rangle - \Lambda_{\mu_n}(n\eta)\} \mu_n(dz)$$

$< \rho - \Lambda(\eta) - \alpha$. (**) follows from the last two inequalities.

(c) follows easily with (a) and (b).