THE UNIVERSITY OF WARWICK

THIRD YEAR EXAMINATION: Summer 2010

MARKOV PROCESSES AND PERCOLATION THEORY

Time Allowed: **3 hours**

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not needed and are not permitted in this examination.

ANSWER 4 QUESTIONS.

If you have answered more than the required 4 questions in this examination, you will only be given credit for your 4 best answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

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1. (a) Define the simple random walk on $\mathbb{Z}^d$, the n-step transition function, and express the corresponding transition function for the first visit to some lattice site in terms of the transition function. **[6]**

(b) Give a mathematical criterion when a random walk given by a transition function $P\colon \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}$ is transient and recurrent respectively. **[3]**

(c) Consider the simple Bernoulli random walk on $\mathbb{Z}$ with $p \in [0,1]$. Prove that it is recurrent if and only if $p = \frac{1}{2}$. (Hint: You may use $\binom{2n}{n} = (-1)^n 4^n \binom{-1/2}{n}$). **[8]**

(d) Consider the simple Bernoulli random walk on $\mathbb{Z}$ starting at the origin 0 in which each step is to the right with probability $p = 1 - q \in (0,1)$. Let T_b be the number of steps until the walk first reaches b where $b > 0$. Show that $\mathbb{E}(T_b | T_b < \infty) = \frac{b}{|p-q|}$. (Hint: $\mathbb{P}(T_b = n) = \frac{b}{n}\mathbb{P}(X_n = b)$ and $\mathbb{P}(T_b < \infty) = \left(\frac{p}{q}\right)^b$; you may use $\mathbb{E}(T_b) = b\mathbb{E}(T_1)$). **[8]**

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2. (a) (i) Define the jump chain (and its transition matrix) of a continuous-time Markov process $X = (X_t)_{t \geq 0}$ on a countable state space I. **[2]**

(ii) Let $X = (X_t)_{t \geq 0}$ be a minimal right-continuous process with Q-matrix Q on a countable state space I. State two different conditions which ensure that $X = (X_t)_{t \geq 0}$ is a Markov process. **[4]**

(b) (i) Give the definition of positive recurrence and null-recurrence of a state for a Markov process $X = (X_t)_{t \geq 0}$ on a countable state space I. **[4]**

(ii) State when a vector $\lambda = (\lambda(i))_{i \in I}$ is an invariant distribution of a Markov process. **[2]**

(c) Let $X = (X_t)_{t \geq 0}$ be a simple birth-death process, with $\lambda_j = j\lambda, \mu_j = j\mu$, for $j \geq 1$, and $\lambda_0 = \lambda$, where $\lambda, \mu \in (0, \infty)$. Find the invariant distribution of X, and also of the jump chain, when they exist. Explain the difference in your answers. **[9]**

(d) Let $X = (X_t)_{t \geq 0}$ be an irreducible positive recurrent birth-death process. Show that $X = (X_t)_{t \geq 0}$ is reversible. **[4]**

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3. (a) Let a continuous-time Markov process $X = (X_t)_{t \geq 0}$ with countable state space I be given.

(i) Define the hitting time and the hitting probability of a subset $A \subset I$ and give the system of equations for which the hitting probability (vector) is the minimal non-negative solution. **[4]**

(ii) State the system of equations for which the expected hitting times of a subset $A \subset I$ is the minimal non-negative solution. Justify that the hitting times solve these equations, i.e., prove the statement without proving the minimality and the positivity of the solution. **[6]**

(b) A continuous time Markov process $X = (X_t)_{t \geq 0}$ with state space $I = \{1, \ldots, 5\}$ is governed by the following Q-matrix

$$Q = \begin{pmatrix} -3 & 1 & 0 & 1 & 1 \\ 1 & -3 & 1 & 0 & 1 \\ 0 & 1 & -3 & 1 & 1 \\ 1 & 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(i) In the case $X_0 = 1$, find the probability that $X_t = 2$ for some $t \geq 0$, i.e., find $h_1 = \mathbb{P}_1(\text{hit } 2)$. **[7]**

Question 3 continued overleaf

(ii) Assuming that $X_0 \neq 5$, find the probability that $X = (X_t)_{t\geq 0}$ eventually visits every state. Hint:

$$\mathbb{P}_1(\text{hit } 2,3,4) = 1 - \mathbb{P}_1(\{\text{avoid } 2\} \cup \{\text{avoid } 3\} \cup \{\text{avoid } 4\}),$$

and use the inclusion-exclusion principle. **[8]**

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4. Consider bond percolation on $\mathbb{Z}^d$.

(a) Define increasing/decreasing events and state the FKG (Fortuin, Kasteleyn, Ginibre) inequality. **[4]**

(b) State and prove Harris's Lemma for events depending only on finitely many bonds. **[7]**

(c) Let A and B be events. Define the BK (van den Berg, Kesten) operation $A\Box B$ for bond percolation (on a finite set of bonds) and state the BK inequality. Give an example for $A\Box B$ and an example for an event in $A \cap B \setminus (A\Box B)$. **[6]**

(d) Consider bond percolation on $\mathbb{Z}^2$. Denote by $R_{n,L}(p)$ the probability of an open horizontal crossing of an nL by L rectangle with $L > 1, n \in \mathbb{N}$. Pick $c = \frac{1}{16}$ and $\lambda \in (0,1)$. Prove the following statement: If $R_{2,L}(p) \geq 1 - c\lambda$ then $R_{2,2L}(p) \geq 1 - c\lambda^2$. **[8]**

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5. (a) Consider bond percolation on $\mathbb{Z}^2$.

(i) Prove that $\mathbb{P}_{\frac{1}{2}}(H(Q)) \geq \frac{1}{2}$ and $\mathbb{P}_{\frac{1}{2}}(H(R)) = \frac{1}{2}$, where Q is any square and R is an $(n+1)\times n$ rectangle. Use the fact that exactly one of the events $H(R)$ (horizontal crossing using open bonds of R) and $V(R^{\mathrm{h}})$ (vertical crossing of the dual R^{h}; where for a rectangle $R = [a,b] \times [c,d]$ in $\mathbb{Z}^2$ the horizontal dual is the rectangle $R^{\mathrm{h}} = [a+\frac{1}{2}, b-\frac{1}{2}] \times [c-\frac{1}{2}, d+\frac{1}{2}]$ in the dual lattice $(\mathbb{Z}^2)^*$) always hold. **[5]**

(ii) Formulate Harris's theorem. **[3]**

(iii) Give the main ideas for the proof of Harris Theorem (you may assume an appropriate lower bound on probabilities of left-right crossings of rectangles). **[5]**

(b) Consider bond percolation on a rooted tree $\mathbb{T}_k$, $k \geq 2$, (i.e. a k-branching tree with a root v_0 in which each vertex has k children).

(i) Prove that $p_{\mathrm{c}}(\mathbb{T}_k) = \frac{1}{k}$ (i.e., write and prove a formula for the probability $\theta(p)$ of an open infinite cluster containing the root for $p > \frac{1}{k}$). **[6]**

(ii) Prove that $p_{\mathrm{T}}(\mathbb{T}_k) = \sup\{p \in [0,1] \colon \chi(p) < \infty\} = \frac{1}{k}$. **[6]**

(Hint: Derive an equation for the probability π_n that there is an open path of length n from the root to a leaf).
