Percolation on trees

k-branching tree T_k

- rooted tree in which each vertex has k children, so all sites but one have degree $k+1$. Let v_0 be the root of T_k

  In the following $T_{k,n}$ is the section of T_k up to height (depth) n.

- bonds are open independently with probability p

Let π_n be the probability that $T_{k,n}$ contains an open path of length n from the root to a leaf.

Such an open path exists if and only if, for some child v_1 of v_0, the bond $v_0 v_1$ is open and there is an open path of length $n-1$ from v_1 to a leaf, we have

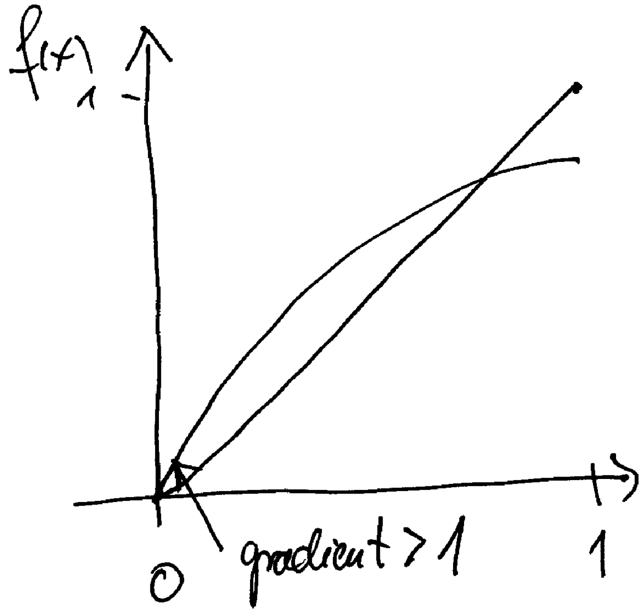
$$\pi_n = 1 - \underbrace{\underbrace{(1 - \underbrace{p\,\pi_{n-1}}_{\substack{\text{prob. bond } v_0v_1 \text{ open} \\ \text{and open path } v_1 \text{ to} \\ \text{a leaf}}})}_{\text{no open path}}{}^{k}}_{\text{'' \quad for all } k \text{ children of } v_0} =: f(\pi_{n-1})$$

f is defined on $[0,1]$

clearly, $f(0) = 0$ and $f(1) < 1$. f concave

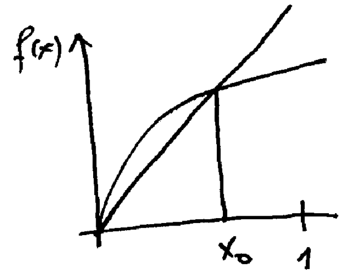
Thus $f(x_0) = x_0$ for some $0 < x_0 < 1$

iff $f'(0) = kp > 1$ (fixpoint is unique when it exists)



If $p > \frac{1}{k}$ we have $f'(x) > 1 \Rightarrow \exists\, x_0 \in (0,1)$

with $f(x_0) = x_0$

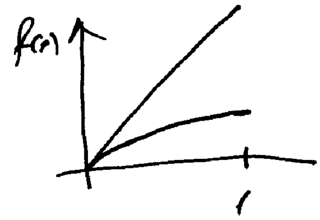


Then $\pi_{n-1} \geq x_0$

implies $f(\pi_{n-1}) = \pi_n \geq x_0$
(see figure).

Since $\pi_0 = 1$ we have $\pi_n \geq x_0$ for all $n \geq 1$

and thus $\underbrace{\Theta(T_k, p)}_{\mathbb{P}_p(|C_{v_0}| = \infty)} \geq x_0 > 0$



$p \leq \frac{1}{k} \Rightarrow f'(0) \leq 1$

$f(0) = 0$

$\pi_n = 1 - (1 - p\pi_{n-1})^k$ $\quad \pi_n \to$ fixpoint $= 0$

and so $\Theta(T_k, p) = 0$

$\Rightarrow p_c(T_k) = \frac{1}{k}$

recall

$p_T = \sup\{p : \mathbb{E}_p(|C_{v_0}|) < \infty\}$

A site y at graph distance l from the root v_0 prob. that

belongs to C_{v_0} is p^l

$$\mathbb{E}(|C_{v_0}|) = \sum_{y \in T_k} \mathbb{P}(y \in C_{v_0}) = \sum_{l=0}^{\infty} k^l p^l$$

sum is finite for $p < \frac{1}{k}$ and infinite for $p \geq 1/k$

$\Rightarrow p_T(T_k) = \frac{1}{k}$.