MA3H2 Markov Processes and Percolation theory Example Sheet 3

2010, term 2
Stefan Adams

Students should hand in solutions by 1pm Monday of week 7 to the maths pigeonloft.

Exercise 9: Let $\lambda\mu > 0$ and let $X = (X_t)_{t\ge 0}$ be a Markov process on $I = \{1, 2\}$ with Q-matrix

$$Q = \begin{pmatrix} -\mu & \mu \\ \lambda & -\lambda \end{pmatrix}.$$

(a) Write down the forward equations and solve them for the transition probabilities $P_t(i, j), i, j \in I$.

(b) Calculate Q^n and hence find $\sum_{n=0}^{\infty}(t^n/n!)Q^n$. Compare your answer with that of part (a).

(c) Solve the equation $\pi Q = 0$ in order to find the stationary distribution. Verify that $P_t(i, j) \to \pi_j$ as $t \to \infty$, $i \in I$.

(d) Find (i) $\mathbb{P}(X(t) = 2|X(0) = 1, X(3t) = 1)$ and (ii) $\mathbb{P}(X(t) = 2|X(0) = 1, X(3t) = 1, X(4t) = 1)$.

Exercise 10: In each of the following cases, the state space I and non-zero rates $q_{i,j}$ $(i \neq j)$ of a continuous-time Markov chain are given. Determine in which cases the chain is explosive.

(a) $I = \mathbb{N}$ and $q_{i,i+1} = i^2, i \in I$.

(b) $I = \mathbb{Z}$ and $q_{i,i+1} = q_{i,i-1} = 2^i, i \in I$.

Exercise 11: Children arrive at a see-saw according to a Poisson process of rate 1. Initially there are no children. The first child to arrive waits at the see-saw. When the second child arrives, they play on the see-saw. When the third child arrives, they all decide to go and play on the merry-go-around. The cycle then repeats. Show that the number of children at the see-saw evolves as a Markov process and determine its Q-matrix (generator matrix). Find the probability that there are no children at the see-saw at time t. Hence obtain the identity

$$\sum_{n=0}^{\infty} \mathrm{e}^{-t}\frac{t^{3n}}{(3n)!} = \frac{1}{3} + \frac{2}{3}\mathrm{e}^{-3t/2}\cos\big(\frac{\sqrt{3}}{2}t\big).$$

Hint: The Q-matrix is a 3×3*-matrix, and you need to do some elementary linear algebra computations.*

Exercise 12:

(a) Describe the jump chain for a birth-death process with rates λ_n and μ_n.

(b) Consider an immigration-death process $X = (X(t))_{t\geq 0}$, being a birth-death process with birth rates $\lambda_n = \lambda$ and death rates $\mu_n = n\mu$. Find the transition matrix of the jump chain Z, and show that it has as stationary distribution

$$\pi_n = \frac{1}{2(n!)}\big(1 + \frac{n}{\rho}\big)\rho^n \mathrm{e}^{-\rho}, \quad \rho = \lambda/\mu.$$

Explain why this differs from the stationary distribution of X.