MA3H2 Markov Processes and Percolation theory Example Sheet 4

2010, term 2
Stefan Adams

This sheet is an extra exercise and henceforth not for credit in assignments. The TA will discuss solutions during the support class in week 8. It is strongly recommended that you work on it (at least partially). The next sheet for assignments (sheet 5) will be issued Monday week 8 with deadline Monday week 10.

Exercise 13: Suppose that $X = (X_t)_{t\geq 0}$ is a continuous time Markov process having values in $\mathbb{N}_0$. Define the jump chain. Describe a method of using the jump chain to construct a sample path of $t \mapsto X_t$. A population of individuals is subject to immigration and threat of total extermination. The number of individuals in the population at time t is a continuous time Markov process. For n individuals in the population at time t then, in the short time interval $(t, t+h)$:

(i) The probability that they are joined by a new member is $\frac{h}{n+2} + o(h)$;

(ii) the probability that they are exterminated is $\frac{h}{(n+2)(n+1)} + o(h)$;

(iii) the probability that more than one incident of either kind happens is $o(h)$.

Is state 0 recurrent? Are the other states recurrent? Justify your answers.

Exercise 14:

(a) Let S and T be independent exponential random variables of parameters λ and μ respectively. Set $M = \min\{S, T\}$. Determine the distribution of M and show that M is independent of the event $\{S < T\}$.

(b) Let $X = (X_t)_{t\geq 0}$ be an irreducible non-explosive Markov process with invariant measure ν and Q-matrix $Q = (q_{i,j})_{i,j\in\mathbb{Z}}$, with $q_{i,i-1} = i^2 + 1, q_{i,i} = -2(i^2 + 1), q_{i,i+1} = i^2 + 1$. Determine whether $(X_t)_{t\geq 0}$ is positive recurrent.

Exercise 15: The Quality Assurance Agency for Higher Education has sent a team to investigate the teaching of mathematics at the University of Warwick. As the visit progresses, the team keeps a count of the number of complaints which it has received. We assume that, during the time interval $(t, t+h)$, a new complaint is made with probability $\lambda h + o(h)$, while any given existing complaint is found groundless and removed from the list with probability $\mu h + o(h)$. Under reasonable conditions to be stated, show that the number $C(t)$ of active complaints at time t constitutes a birth-and-death process with birth rates $\lambda_n = \lambda$ and death rates $\mu_n = n\mu$. Derive, but do not solve, the forward system of equations for the probabilities $p_n(t) = \mathbb{P}(C(t) = n)$. Show that $M(t) = \mathbb{E}(C(t))$ satisfies the differential equation $M'(t) = \lambda - \mu M(t)$, and find $M(t)$ subject to the initial condition $M(0) = 1$. Find the invariant distribution of the process.

Exercise 16: The rubbish bins of a certain Canadian campsite are renowned amongst bears for their tasty morsels. Bears arrive in the campsite at times of a Poisson process with rate λ. After arrival, the m-th bear spends a time R_m roaming for a bin, followed by a time S_m raiding it. The vectors (R_m, S_m), $m \geq 1$, are independent random vectors with the same (joint) distribution. Let $U(t)$ and $V(t)$ be the numbers of roaming and raiding bears, respectively, at time t, and assume that $U(0) = V(0) = 0$.

Let α (respectively β) be the probability that a bear arriving at some time T, chosen uniformly at random from the interval $(0, t)$, is roaming (respectively, raiding) at time t. Compute $\mathbb{P}(U(t) = u, V(t) = v)$ in terms of α and β, and hence show that $U(t)$ and $V(t)$ form independent random variables, each with a Poisson distribution. Show that $\mathbb{E}(U(t)) \to \lambda \mathbb{E}(R_1)$ as $t \to \infty$.