Example sheet 6 – hints for solution

Question 1:

(a) The simple r.w. on $\mathbb{Z}^d$ is the family $X = (X_n)_{n\in\mathbb{N}_0}$ of $\mathbb{Z}^d$-valued r.v.s X_n such that $\mathbb{P}(X_0 = x_0) = 1$ for some $x_0 \in \mathbb{Z}^d$ and

$$\mathbb{P}(X_{n+1} = y \mid X_n = x) = P(x,y) \qquad (x,y\in\mathbb{Z}^d)$$

with $$P(x,y) = \begin{cases} \frac{1}{2d} &, \text{if } |x-y| = 1 \\ 0 &, \text{otherwise} \end{cases}$$

$\forall x,y \in \mathbb{Z}^d$ and $n\in\mathbb{N}_0$. The n-step transition function $P_n(x,y)$ is defined by $P_n(x,y) = \sum_{\substack{x_i\in\mathbb{Z}^d,\\ i=1,\dots,n-1}} P(x,x_1)P(x_1,x_2)\cdots P(x_{n-1},y)$.

By $F_n(x,y)$ one denotes the probability that the random walk starting at x will visit site y for the first time n.

$$F_n(x,y) = \sum_{\substack{x_i\in\mathbb{Z}^d\setminus\{y\}\\ i=1,\dots,n-1}} P(x,x_1)\cdots P(x_{n-1},y)$$

(b) Let $G(x,y) = \sum_{n=0}^{\infty} P_n(x,y)$; respectively $F(x,y) = \sum_{n=1}^{\infty} F_n(x,y)$

and write $G := G(0,0)$ and $F := F(0,0)$. Then the r.w. $X = (X_n)_{n\in\mathbb{N}_0}$ with transition function P is recurrent if $F = 1$ and it is transient if $F < 1$.

(d) $P(0,1) = p$, $P(0,-1) = q = 1-p$, $p\in[0,1]$ (see also lecture notes page 11 ff.) Details!

From the lecture we know that

$$\mathbb{P}(S_{2n}=0) = P_{2n}(0,0) = (pq)^n \binom{2n}{n}$$

$$= (-1)^n (4pq)^n \binom{-1/2}{n}$$

where we used $\binom{2n}{n} = (-1)^n 4^n \binom{-1/2}{n}$

Newton's generalised Binomial theorem

$$(x+y)^r = \sum_{k=0}^{\infty} \binom{r}{k} x^{r-k} y^k$$

we get (noting that $0 \le p = 1-q$ implies $4pq \le 1$)

$$\sum_{n=0}^{\infty} t^n P_{2n}(0,0) = (1-4pqt)^{-1/2} \quad , |t|<1$$

Thus

$$\lim_{\substack{t\to 1\\ t<1}} \sum_{n=0}^{\infty} t^n P_{2n}(0,0) = \sum_{n=0}^{\infty} P_{2n}(0,0) = \sum_{n=0}^{\infty} P_n(0,0) = G \le \infty$$

$$P_{2n+1}(0,0) = 0 \quad \forall n$$

Henceforth

$$G = \begin{cases} (1-4pq)^{-1/2} < \infty & \text{if } p \neq q \\ +\infty & \text{if } p = q . \end{cases}$$

Question 2:

(a) (i) Denote by $J_0, J_1, \ldots$ the jump times of X obtained from X by $J_0 = 0$,

$J_{n+1} = \inf\{t \geq J_n : X_t \neq X_{J_n}\}$

See lecture notes

Then jump chain $Y = (Y_n)_{n \in \mathbb{N}}$ by $Y_n = X_{J_n}$ with transition matrix

$\Pi = (\pi_{i,j})_{i,j \in I}$ where $\pi_{i,i} = 0$ if $q_i \neq 0$ and $\pi_{i,i} = 1$ if $q_i = 0$ and for $i \neq j$

$$\pi_{i,j} = \begin{cases} \frac{q_{ij}}{q_i} & q_i \neq 0 \\ 0 & \text{if } q_i = 0 \end{cases}$$

(ii) Let $(X_t)_{t \geq 0}$ be a minimal right-continuous process with Q-matrix Q on I and jump matrix Π and semi-group $(P_t)_{t \geq 0}$, then the following conditions are equivalent (and if X satisfies any of these then X is Markov)

See lecture notes

(*) Cond. on $X_0 = i$, the jump chain $(Y_n)_{n \in \mathbb{N}_0}$ is a discrete time Markov (δ_i, Π) chain and for each $n \geq 1$, cond. on $Y_0, \ldots, Y_{n-1}$, the holding times $E_1, \ldots, E_n$ are independent exponential random variables of parameters $q(Y_0), \ldots, q(Y_{n-1})$ respectively

(**) For all $n \in \mathbb{N}_0$, all times $0 \leq t_0 \leq t_1 \leq \cdots \leq t_{n+1}$ and all states $i_0, i_1, \ldots, i_{n+1} \in I$

$$\mathbb{P}(X_{t_{n+1}} = i_{n+1} \mid X_{t_0} = i_0, \ldots, X_{t_n} = i_n) = P_{t_{n+1} - t_n}(i_n, i_{n+1}).$$

(b) (i) Let $T_i = \inf\{t \geq J_1 : X_t = i\}$ for $i \in I$. Then i is called positive recurrent if $m_i = \mathbb{E}_i(T_i) < \infty$, and $i \in I$ is called null-recurrent if $\mathbb{P}_i(T_i < \infty) > 0$ but $m_i = \infty$

(ii) $\lambda = (\lambda(i))_{i \in I}$ prob. vector i.e. $\sum_{i \in I} \lambda(i) = 1$

$\mathbb{P}(X_t = j) = \lambda(j) \quad \forall\, t \geq 0 \ \forall j \in I$ i.e. $\lambda P_t = \lambda$

or alternatively, $\lambda Q = 0$.

(c) The state space is $I = \{0, 1, 2, 3\}$ representing the number of lines not in action.

mean holding times are $\frac{1}{7}, \frac{1}{20}, \frac{1}{32}, \frac{1}{42}$

$$Q = \begin{pmatrix} -7 & 3 & 3 & 1 \\ 14 & -20 & 4 & 2 \\ 0 & 28 & -32 & 4 \\ 0 & 0 & 42 & -42 \end{pmatrix}$$

To see this recall Prop 2.20: For example suppose 3 lines are no inaction, so to jump to the state with only two not in action you must wait until the first of three is repaired that is you have to ~~[illegible]~~ wait the inf of 3 exp. r.v. each with parameter 14, hence $q_{3,2} = 42$.

to get the long run behaviour find invariant distribution $\pi = (\pi_0, \pi_1, \pi_2, \pi_3)$ with $\pi Q = 0$

$$-7\pi_0 + 14\pi_1 = 0$$
$$3\pi_0 - 20\pi_1 + 28\pi_2 = 0$$
$$3\pi_0 + 4\pi_1 - 32\pi_2 + 42\pi_3 = 0$$
$$\pi_0 + 2\pi_1 + 4\pi_2 - 42\pi_3 = 0$$

This gives $\pi_0 = 28/51$, $\pi_1 = 14/51$, $\pi_2 = 7/51$, $\pi_3 = 2/51$ and $\pi_0 + \pi_1 = 14/17$ for the long-run prop. of time when all pairs communicate.

Question 3:

(a)(i) $A \subset I$ hitting time $D^A = \inf\{t \geq 0 : X_t \in A\}$

hitting prob. $h^A = (h_i^A)_{i \in I}$ $h_i^A = \mathbb{P}_i(D^A < \infty)$

$$\begin{cases} h_i^A = 1 \text{ if } i \in A \\ \sum_{j \in I} q_{i,j} h_j^A = 0 \text{ for } i \notin A \end{cases}$$

(ii) $k^A = (k_i^A)_{i \in I}$, $k_i^A = \mathbb{E}_i(D^A)$ expected hitting time

$$\begin{cases} k_i^A = 0 & \text{if } i \in A \\ -\sum_{j \in I} q_{ij} k_j^A = 1 & \text{if } i \notin A \end{cases}$$

Proof: If $X_0 = i \in A$ then $D^A = 0$ so $k_i^A = 0$.
If $X_0 = i \notin A$ then $D^A \geq J_1$ where J_1 is the first jump time.
$(Y_n)_{n \in \mathbb{N}_0}$ corresponding jump chain with transition matrix $\Pi = (\pi_{i,j})_{i,j \in I}$
Using the Markov property we get that
$\mathbb{E}_i(D^A - J_1 \mid Y_1 = j) = \mathbb{E}_j(D^A)$. Henceforth
(recall $q_i = -q_{i,i}$)

$$k_i^A = \mathbb{E}_i(D^A) = \mathbb{E}_i(J_1) + \sum_{j \neq i} \underbrace{\mathbb{E}_i(D^A - J_1 \mid Y_1 = j)}_{= \mathbb{E}_j(D^A) = k_j^A} \mathbb{P}_i(Y_1 = j)$$

$$= q_i^{-1} + \sum_{j \neq i} \pi_{i,j} k_j^A, \text{ and so}$$

$$-\sum_{j \in I} q_{i,j} k_j^A = 1$$

(b) (i) Put $h_j := h_j^{\{3\}}$. Clearly $h_3 = 1$

Then

$$-h_1 + 1/2\, h_2 + 1/2 = 0$$
$$1/4\, h_1 - 1/2\, h_2 + 1/4\, h_4 = 0$$
$$1/6\, h_1 - 1/3 + 1/6\, h_4 = 0$$

giving $h_1 = \frac{1}{2} h_2 + 1/2$ and $h_4 = 3h_2 - 1/2$ and

$h_2 = 4/7$ and henceforth $h_1 = \frac{2}{7} + 1/2 = 11/14$

(ii) Put $k_j = k_j^{\{4\}}$. Clearly $k_4 = 0$.

$$\left.\begin{aligned} -(-k_1 + 1/2\, k_2 + 1/2\, k_3) &= 1 \\ -(1/4\, k_1 - 1/2\, k_2 \qquad) &= 1 \\ -(1/6\, k_1 - 1/3\, k_3\,) &= 1 \end{aligned}\right\} \text{this gives } k_3 = 3 + \tfrac{1}{2} k_1$$

$k_2 = 2(1 + 1/4\, k_1)$

and finally $k_1 = 7$.

Question 4:

(a) An event A is increasing if $1\!\!1_A(\omega) \leq 1\!\!1_A(\omega')$ whenever $\omega \leq \omega'$ (i.e. $\omega(e) \leq \omega'(e)$)

A is decreasing if A^c is increasing. A random variable N is increasing if $N(\omega) \leq N(\omega')$ whenever $\omega \leq \omega'$.

FKG (i) If X and Y are increasing r.v.s with $\mathbb{E}_p(X^2) < \infty$ and $\mathbb{E}_p(Y^2) < \infty$, then $\mathbb{E}_p(XY) \geq \mathbb{E}_p(X)\mathbb{E}_p(Y)$.

(ii) (Harris) If A and B are increasing events then $\mathbb{P}_p(A \cap B) \geq \mathbb{P}_p(A)\mathbb{P}_p(B)$.

Harris's Lemma (for events depending only on finitely many bonds/edges)

Let Λ be a finite set of bonds/edges. Let A and B two events which depend only on the states in Λ.

(i) If both (A and B) are increasing or both are decreasing then $\mathbb{P}_p(A\cap B) \geq \mathbb{P}_p(A)\mathbb{P}_p(B)$

(ii) If A is increasing and B is decreasing then
$$\mathbb{P}_p(A\cap B) \leq \mathbb{P}_p(A)\mathbb{P}_p(B).$$

Proof: Identify power set of Λ with the cube $\{0,1\}^n =: Q^n$ (where $|\Lambda| = n$, i.e. Λ ~~contain~~ has n edges)

$A \in Q^n$ and $t=0,1$ write
$$A_t = \{ (a_i)_{i=1}^{n-1} : (a_1, a_2, \dots, a_{n-1}, t) \in A \} \in Q^{n-1}$$

$$(+)\quad \mathbb{P}_p(A) = (1-p)\mathbb{P}_p(A_0) + p\,\mathbb{P}_p(A_1) \qquad \forall\, A \in Q^n$$

Suppose (i) holds for $(n-1)$ ($n=1$ is obvious).
We get
$$\left(\mathbb{P}_p(A_0) - \mathbb{P}_p(A_1)\right)\left(\mathbb{P}_p(B_0) - \mathbb{P}_p(B_1)\right) \geq 0$$
(because both are either increasing or decreasing)

With (+) we get therefore

$$\mathbb{P}_p(A\cap B) = (1-p)\mathbb{P}_p(A_0 \cap B_0) + p\,\mathbb{P}_p(A_1\cap B_1)$$
$$\overset{\text{assumption for } n-1}{\geq} (1-p)\mathbb{P}_p(A_0)\mathbb{P}_p(B_0) + p\,\mathbb{P}_p(A_1)\mathbb{P}_p(B_1)$$
$$\geq \left((1-p)\mathbb{P}_p(A_0) + p\,\mathbb{P}_p(A_1)\right)\left((1-p)\mathbb{P}_p(B_0) + p\,\mathbb{P}_p(B_1)\right)$$
$$= \mathbb{P}_p(A)\mathbb{P}_p(B)$$

(ii) Apply (i) to A and B^c. Hence, $\mathbb{P}_p(A\cap B) \leq \mathbb{P}_p(A) - \mathbb{P}_p(A)\mathbb{P}_p(B^c)$
$= \mathbb{P}_p(A)(1-(1-\mathbb{P}_p(B)) = \mathbb{P}_p(A)\mathbb{P}_p(B)$.

(b) An event A is said to occur on the subset $\Lambda \subset \mathbb{E}^d$ of bonds/edges if the conf. ω of A occurs using only ~~the~~ bonds in Λ, independent of the values of the bonds in Λ^C.

The collection of such ω is

$$A|_\Lambda = \{\omega : \forall \tilde{\omega},\ \tilde{\omega} = \omega \text{ on } \Lambda \ (\text{i.e. } \tilde{\omega}(e) = \omega(e)\ \forall e \in \Lambda) \Longrightarrow \tilde{\omega} \in A\}$$

BK operation

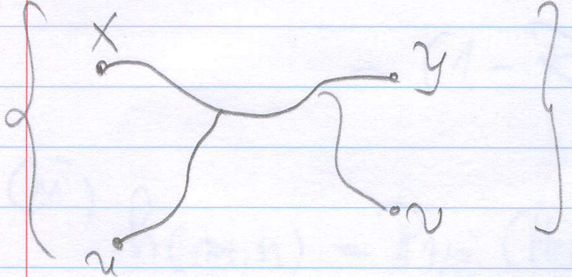
$$A \square B = \{\omega : \exists\, \Lambda_1, \Lambda_2 \subset \mathbb{E}^d,\ \Lambda_1 \cap \Lambda_2 = \emptyset,\ \omega \in A|_{\Lambda_1} \cap B|_{\Lambda_2}\}$$

ex. $A = \{x \leftrightarrow y\}$ $\quad B = \{u \leftrightarrow v\}$

$x \neq y \neq u \neq v$

$$\Longrightarrow A \square B = \left\{\begin{matrix} x \rightsquigarrow y \\ u \rightsquigarrow v \end{matrix}\right\}$$

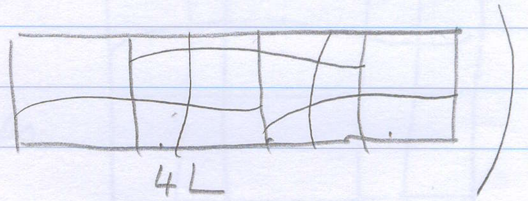
$A \cap B \setminus (A \square B)$



$\subset A \cap B \setminus A \square B$.

(c) (i) Let $R_{2,L}(p) \geq 1 - c\lambda$.

We have

$R_{4,L}(p) \geq \mathbb{P}_p\Big($  $\Big)$

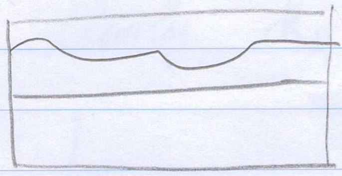
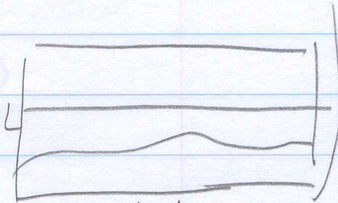
from the vertical crossings in the last figure

$$\overset{\text{FKG}}{\geq} R_{2,L}(p)^{3}\, \overbrace{R_{1,L}(p)^{2}}$$

$$\geq R_{2,L}(p)^4$$ (any crossing of a $2L \times L$ rectangle can be decomposed into a separate crossing of disjoint $L \times L$ B-K inequality

$$\geq (1-c\lambda)^4$$ assumption

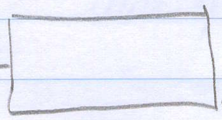
$$\geq (1-4c\lambda)$$

But $R_{2,2L}(p) \geq \mathbb{P}_p\Big($ 2L  or 2L 

4L   4L

henceforth by independence of the two crossings above

← complement

$1 - R_{2,2L}(p) \leq \mathbb{P}_p\Big(\Big\{$ 2L  $\Big\}^c\Big)^2$

4L

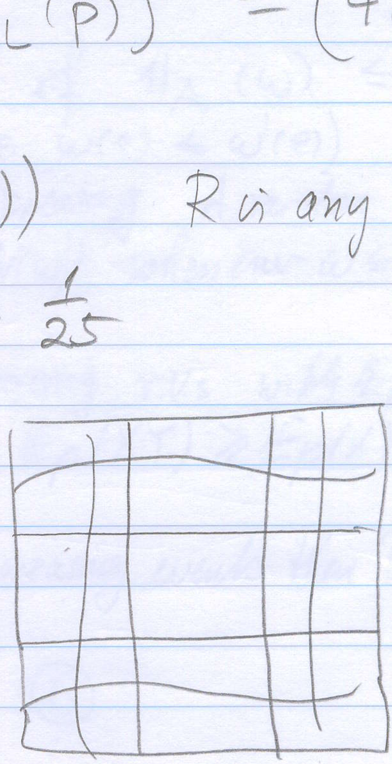
$$= (1 - R_{4,L}(p))^2 \leq (4c\lambda)^2 = c\lambda^2$$

(ii) $h(m,n) = \mathbb{P}_{1/2}(H(R))$ R is any m by n rectangle

RSW gives $h(6n, 2n) \geq \frac{1}{25}$

'ring' fencing of origin

two 6n by 2n
two 2n by 6n } rectangles



With prob. at least $2^{-100} > 0$ each rectangle is crossed to long way (2 times horizontal / 2 times vertical) by an open (dual path)
The union of these open paths contains an open dual cycle surrounding the centre of the annulus.

One can estimate $\mathbb{P}_{1/2}(E_k) \geq 2^{-100}$ where E_k is some annulus centred on $(1/2, 1/2)$ with outer and inner radii $(3 \times 4^k, 4^k)$

then $\prod_{k=1}^{\ell} \mathbb{P}_{1/2}(E_k^c) \leq (1-2^{-100})^{\ell}$

and finally $\theta(1/2) = 0$