MA3H2 Markov Processes and Percolation theory Example Sheet 1

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Students should hand in solutions to questions by 1pm Monday of week 3 to the maths pigeonloft.

Exercise 1: Discrete random variables

(a) *Bernoulli trials*. A random variable takes values 1 and -1 with probabilities $p \in [0,1]$ and $q = 1-p$, respectively. Compute $\mathbb{E}(X)$ and $\mathrm{var}(X)$.

(b) *Binomial distribution*. We perform n independent Bernoulli trials $X_1, \dots,$ X_n and count the total number of successes $Y = X_1 + X_2 + \cdots + X_n$. Give the probability mass function, $\mathsf{B}_{n,p}(\{k\}) = \mathbb{P}(Y = k), k \in \mathbb{N}_0$, and compute $\mathbb{E}(Y)$ and $\mathrm{var}(Y)$.

(c) *Poisson distribution*. Pick $\lambda > 0$. A Poisson random variable X has values in $\mathbb{N}_0$ and probability mass function $\mathsf{P}_\lambda(\{k\}) = \frac{\lambda^k}{k!}\mathrm{e}^{-\lambda}, k \in \mathbb{N}_0$. Compute $\mathbb{E}(X)$ and $\mathrm{var}(X)$. Let Y be a $\mathsf{B}_{n,p}$ random variable, and let $n \to \infty$ and $p \to 0$ in such a way that $\mathbb{E}(Y) = np$ approaches a non-zero constant λ. Show that $\mathsf{B}_{n,p} \to \mathsf{P}_\lambda$ as $n \to \infty$.

Exercise 2: Symmetric random walk (or 'Gambler's ruin') A man is saving up to buy a new Jaguar at a cost of N units of money. He starts with k units where $0 < k < N$, and tries to win the remainder by the following gamble with his bank manager. He tosses a fair coin repeatedly; if it comes up heads then the manager pays him one unit, but if it comes up tails then he pays the manager one unit. He plays his game repeatedly until one of two events occurs: either he runs out of money and is bankrupted or he wins enough to buy the Jaguar.

(a) What is the probability that he is ultimately bankrupted?

(b) Suppose now that the coin is not fair, that is $\mathbb{P}(\{\text{head}\}) = p$ for $p \in (0,1], p \neq \frac{1}{2}$. What is the probability that he is ultimately bankrupted in this case?

Exercise 3: Consider a symmetric simple random walk $S = (S_n)_{n \in \mathbb{N}_0}$ on $\mathbb{Z}$ with $S_0 = 0$. Let $T = \min\{n \geq 1 \colon S_n = 0\}$ be the time of the first return of the walk to its starting point. Show that

$$\mathbb{P}(T = 2n) = \frac{1}{2n-1}\binom{2n}{n}2^{-2n},$$

and deduce that $\mathbb{E}(T^\alpha) < \infty$ if and only if $\alpha < \frac{1}{2}$. You may need Stirling's formula: $n! \sim n^n \mathrm{e}^{-n}\sqrt{2\pi n}$.

Exercise 4: For a symmetric simple random walk $S = (S_n)_{n\in\mathbb{N}_0}$ on $\mathbb{Z}$ starting at 0, show that the probability of the maximum $M_n = \max\{S_i \colon 0 \le i \le n\}$ satisfies $\mathbb{P}(M_n = k) = \mathbb{P}(S_n = k) + \mathbb{P}(S_n = k+1)$ for $k \ge 0$.

Information, sheets, and lecture notes at

http://www2.warwick.ac.uk/fac/sci/maths/people/staff/stefan_adams/ma3h2_2011/