MA3H2 Markov Processes and Percolation theory Example Sheet 4

2011, term 2
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Students should hand in solutions by 3pm Tuesday of week 10 to the maths pigeonloft.

Exercise 13: Bond percolation on $\mathbb{Z}^d$.

(a) Explain and justify the following facts (recall that $p_c(d) = p_c(\mathbb{Z}^d)$):
(i) $p_c(1) = 1$; (ii) $p_c(d+1) \leq p_c(d)$ for all $d \geq 1$; (iii) $\lambda(d) \leq 2d - 1$; (iv) $p_c(d) \sim (2d)^{-1}$ as $d \to \infty$.

(b) Consider now $\mathbb{Z}^2$: Prove that $\mathbb{P}_{\frac{1}{2}}(H(Q)) \geq \frac{1}{2}$ and $\mathbb{P}_{\frac{1}{2}}(H(R)) = \frac{1}{2}$, where Q is any square and R is an $(n+1) \times n$ rectangle. Use the fact that exactly one of the events $H(R)$ (horizontal crossing using open bonds of R) and $V(R^{\mathrm{h}})$ (vertical crossing of the dual R^{h}; where for a rectangle $R = [a, b] \times [c, d]$ in $\mathbb{Z}^2$ the horizontal dual is the rectangle $R^{\mathrm{h}} = [a + \frac{1}{2}, b - \frac{1}{2}] \times [c - \frac{1}{2}, d + \frac{1}{2}]$ in the dual lattice $(\mathbb{Z}^2)^*$) always hold.

(c) Formulate Harris Theorem.

Exercise 14: Bond percolation on rooted trees. Consider bond percolation on a rooted tree $\mathbb{T}_k$, $k \geq 2$, (i.e. a k-branching tree with a root in which each vertex has k children).

(a) Prove that $p_c(\mathbb{T}_k) = p_{\mathrm{T}}(\mathbb{T}_k) = \frac{1}{k}$.

(b) Write and prove a formula (respectively an equation) for the probability $\theta(p)$ of infinite cluster containing the root for $p > p_c(\mathbb{T}_k) = \frac{1}{k}$, and the expectation $\chi(p)$ of the size of the cluster containing the root for $p < p_c(\mathbb{T}_k)$.

(c) Find an explicit formula for $\theta(p)$ for $k = 2$.

Exercise 15:

(a) Define the BK (van den Berg, Kesten) operation $A \Box B$ for bond percolation (on a finite set of bonds).

(b) Prove that $A \Box B \subset A \cap B$ always and $A \Box B = A \cap B$ whenever A is increasing and B decreasing.

(c) Consider bond percolation on $\mathbb{Z}^d$. Formulate Russo's Lemma and prove Russo's Lemma.

Exercise 16:

(a) Define increasing/decreasing events and state the FKG (Fortuin, Kasteleyn, Ginibre) inequality. State and prove Harris's Lemma for events depending only on finitely many bonds.

(b) Let A and B be events. Define the BK (van den Berg, Kesten) operation $A \Box B$ for bond percolation (on a finite set of bonds) and state the BK inequality. Give an example for $A \Box B$ and an example for an event in $A \cap B \setminus (A \Box B)$.

(d) Consider bond percolation on $\mathbb{Z}^2$.

(i) Denote by $R_{n,L}(p)$ the probability of an open horizontal crossing of an nL by L rectangle with $L > 1, n \in \mathbb{N}$. Pick $c = \frac{1}{16}$ and $\lambda \in (0, 1)$. Prove the following statement: If $R_{2,L}(p) \geq 1 - c\lambda$ then $R_{2,2L}(p) \geq 1 - c\lambda^2$.

(ii) Give the main ideas for the proof of Harris Theorem (you may assume an appropriate lower bound on probabilities of left-right crossings of rectangles (RSW - Theorem)).