Scaling limits for weakly pinned Gaussian random fields under the presence of two possible candidates

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Abstract

We study the scaling limit and prove the law of large numbers for weakly pinned Gaussian random fields under the critical situation that two possible candidates of the limits exist at the level of large deviation principle. This paper extends the results of [3], [7] for one dimensional fields to higher dimensions: $d \geq 3$, at least if the strength of pinning is sufficiently large.

1 Introduction and main result

This paper is concerned with weakly pinned Gaussian random fields which are microscopically defined on a d-dimensional region D_N of large size N. We study its macroscopic limit by scaling down its size to $O(1)$ as $N \to \infty$ under the critical situation that two possible candidates of the limits exist at the level of rough large deviations. We work out which one really appears in the limit assuming that $d \geq 3$ and the strength $\varepsilon > 0$ of the pinning is sufficiently large.

1.1 Weakly pinned Gaussian random fields

We work on the d-dimensional square lattice $D_N = \{0, 1, 2, \dots, N\} \times \mathbb{T}_N^{d-1}$ and denote its elements by $i = (i_1, i_2, \dots, i_d) \equiv (i_1, \underline{i}) \in D_N$, where $\mathbb{T}_N^{d-1} = (\mathbb{Z}/N\mathbb{Z})^{d-1}$ is the $(d-1)$-dimensional lattice torus. In other words, we consider the lattice under periodic boundary conditions for the coordinates except the first one. The left and right boundaries of D_N

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are denoted by $\partial_L D_N = \{0\} \times \mathbb{T}_N^{d-1}$ and $\partial_R D_N = \{N\} \times \mathbb{T}_N^{d-1}$, respectively. We set $\partial D_N = \partial_L D_N \cup \partial_R D_N$ and $D_N^\circ = D_N \setminus \partial D_N$.

The Hamiltonian is associated with an $\mathbb{R}$-valued field $\phi = (\phi_i)_{i\in D_N} \in \mathbb{R}^{D_N}$ over D_N by

$$H_N(\phi) = \frac{1}{2} \sum_{\langle i,j\rangle \subset D_N} (\phi_i - \phi_j)^2, \tag{1.1}$$

where the sum is taken over all undirected bonds $\langle i,j\rangle$ in D_N, i.e., all pairs $\{i,j\}$ such that $i,j \in D_N$ and $|i-j| = 1$. We sometimes denote ϕ_i by $\phi(i)$. For given $a,b > 0$, we impose the Dirichlet boundary condition for ϕ at ∂D_N by

$$\phi_i = aN \quad \text{for} \quad i \in \partial_L D_N, \qquad \phi_i = bN \quad \text{for} \quad i \in \partial_R D_N. \tag{1.2}$$

For $\varepsilon \geq 0$, the strength of the pinning force toward 0 acting on the field ϕ, we introduce the Gibbs probability measure on $\mathbb{R}^{D_N^\circ}$:

$$\mu_N^{aN,bN,\varepsilon}(d\phi) = \frac{1}{Z_N^{aN,bN,\varepsilon}} \mathrm{e}^{-H_N^{aN,bN}(\phi)} \prod_{i\in D_N^\circ} \left[\varepsilon\delta_0(d\phi_i) + d\phi_i\right], \tag{1.3}$$

where $Z_N^{aN,bN,\varepsilon}$ is the normalizing constant (partition function) and $H_N^{aN,bN}(\phi)$ is the Hamiltonian $H_N(\phi)$ with the boundary condition (1.2). We sometimes regard $\mu_N^{aN,bN,\varepsilon}$ as a probability measure on $\mathbb{R}^{D_N}$ by extending it over ∂D_N due to the condition (1.2).

1.2 Scaling and large deviation rate functional

Let $D = [0,1] \times \mathbb{T}^{d-1}$ be the macroscopic region corresponding to D_N, where $\mathbb{T}^{d-1} = (\mathbb{R}/\mathbb{Z})^{d-1}$ is the $(d-1)$-dimensional unit torus. We associate a macroscopic height field $h^N : D \to \mathbb{R}$ with the microscopic one $\phi \in \mathbb{R}^{D_N}$ as a step function defined by

$$h^N(t) = \frac{1}{N}\phi(i), \quad t = (t_1, \underline{t}) \in B\big(\frac{i}{N}, \frac{1}{N}\big) \cap D,\ i \in D_N, \tag{1.4}$$

where $B\big(\frac{i}{N}, \frac{1}{N}\big)$ denotes the box $\big[\frac{i-\frac{1}{2}}{N}, \frac{i+\frac{1}{2}}{N}\big)^d$ with the center $\frac{i}{N}$ and sidelength $\frac{1}{N}$ considered periodically in the direction of $\underline{t}$. It is sometimes convenient to introduce another macroscopic filed h^N, denoted by h_{PL}^N, as a polilinear interpolation of $\frac{1}{N}\phi(i)$:

$$h_{\mathrm{PL}}^N(t) = \frac{1}{N} \sum_{v\in\{0,1\}^d} \left[\prod_{\alpha=1}^d \big(v_\alpha\{Nt_\alpha\} + (1-v_\alpha)(1-\{Nt_\alpha\})\big)\right] \phi([Nt]+v), \tag{1.5}$$

where $[\cdot]$ and $\{\cdot\}$ stand for the integer and the fractional parts, respectively, see (1.17) in [5]. Note that $h_{\mathrm{PL}}^N \in C(D,\mathbb{R})$. We will prove that h^N and h_{PL}^N are close enough in a superexponential sense; see Lemma 6.7 below. Our goal is to study the asymptotic behavior of h^N distributed under $\mu_N^{aN,bN,\varepsilon}$ as $N \to \infty$.

We will prove that a large deviation principle (LDP) holds for h^N under $\mu_N^{aN,bN,\varepsilon}$, roughly stating

$$\mu_N^{aN,bN,\varepsilon}(h^N \sim h) \sim \mathrm{e}^{-N^d\Sigma^*(h)},$$

as $N \to \infty$ with an unnormalized rate functional

$$\Sigma(h) = \frac{1}{2} \int_D |\nabla h(t)|^2 dt - \xi^\varepsilon \left| \{t \in D; h(t) = 0\} \right|, \tag{1.6}$$

for $h : D \to \mathbb{R}$; see (1.13). The functional Σ^* is the normalization of Σ such that $\min \Sigma^* = 0$ by adding a suitable constant, i.e., $\Sigma^*(h) = \Sigma(h) - \min \Sigma$. The non-negative constant ξ^ε is the free energy determined by

$$\xi^\varepsilon = \lim_{\ell \to \infty} \frac{1}{|\Lambda_\ell|} \log \frac{Z^{0,\varepsilon}_{\Lambda_\ell}}{Z^0_{\Lambda_\ell}}, \tag{1.7}$$

where $\Lambda_\ell = \{1, 2, \dots, \ell\}^d \Subset \mathbb{Z}^d$, $|\Lambda_\ell| = \ell^d$, and $Z^{0,\varepsilon}_{\Lambda_\ell}$ and $Z^0_{\Lambda_\ell}$ are the partition functions on Λ_ℓ with 0-boundary conditions with and without pinning, respectively. It is known that ξ^ε exists, and that the field is localized by the pinning effect (even if $d = 1, 2$), meaning that $\xi^\varepsilon > 0$ for all $\varepsilon > 0$ (and all $d \ge 1$); see, e.g., Section 7 of [6] or Remark 6.1 of [8].

1.3 Minimizers of the rate functional

The functional Σ is defined for functions h on D, which satisfy the (macroscopic) boundary conditions:

$$h(0, \underline{t}) = a, \quad h(1, \underline{t}) = b. \tag{1.8}$$

We denote $t = (t_1, \underline{t}) \in D = [0, 1] \times \mathbb{T}^{d-1}$. Since the boundary conditions (1.8) and the functional Σ are translation-invariant in the variable $\underline{t}$, the minimizers of Σ are functions of t_1 only and the minimizing problem can be reduced to the 1D case; see Lemma 1.1 below. Thus the candidates of the minimizers of Σ are of the forms:

$$\hat{h}(t) = \hat{h}^{(1)}(t_1), \quad \bar{h}(t) = \bar{h}^{(1)}(t_1),$$

where $\hat{h}^{(1)}$ and $\bar{h}^{(1)}$ are the candidates of the minimizers in the one-dimensional problem under the condition $h(0) = a, h(1) = b$, that is, $\bar{h}^{(1)}(t_1) = (1 - t_1)a + t_1 b, t_1 \in [0, 1]$, and, when $a + b < \sqrt{2\xi^\varepsilon}$,

$$\hat{h}^{(1)}(t_1) = \begin{cases} (s_1^L - t_1)a/s_1^L, & t_1 \in [0, s_1^L], \\ 0, & t_1 \in [s_1^L, s_1^R], \\ (t_1 - s_1^R)b/(1 - s_1^R), & t_1 \in [s_1^R, 1], \end{cases}$$

where $0 < s_1^L < s_1^R < 1$ are determined by $a/s_1^L = b/(1 - s_1^R) = \sqrt{2\xi^\varepsilon}$; see Section 3.1 below, Section 1.3 and Appendix B of [3] or Section 6.4 of [6].

Lemma 1.1 *The set of the minimizers of the functional Σ is contained in $\{\hat{h}, \bar{h}\}$.*

Proof. Consider the functional

$$\Sigma^{(1)}(g) = \frac{1}{2} \int_0^1 \dot{g}(t_1)^2 dt_1 - \xi^\varepsilon \left| \{t_1 \in [0, 1]; g(t_1) = 0\} \right|$$

for functions $g = g(t_1)$ with a single variable $t_1 \in [0,1]$. Then, for $h = h(t) \equiv h(t_1, \underline{t})$, one can rewrite $\Sigma(h)$ as

$$\Sigma(h) = \int_{\mathbb{T}^{d-1}} \Sigma^{(1)}(h(\cdot, \underline{t}))\, d\underline{t} + \frac{1}{2} \int_D |\nabla_{\underline{t}} h(t_1, \underline{t})|^2 \, dt, \tag{1.9}$$

where

$$\nabla_{\underline{t}} h = \left(\frac{\partial h}{\partial t_2}, \dots, \frac{\partial h}{\partial t_d} \right), \quad \underline{t} = (t_2, \dots, t_d).$$

However, since the minimizers of $\Sigma^{(1)}$ are $\hat{h}^{(1)}$ or $\bar{h}^{(1)}$ (see [3], [6]), we see that

$$\Sigma^{(1)}(h(\cdot, \underline{t})) \geq \Sigma^{(1)}(\hat{h}^{(1)}) \wedge \Sigma^{(1)}(\bar{h}^{(1)}),$$

and this inequality integrated in $\underline{t}$ combined with (1.9) implies

$$\Sigma(h) \geq \Sigma(\hat{h}) \wedge \Sigma(\bar{h}) \tag{1.10}$$

for all $h = h(t)$. Moreover, from (1.9) again, the identity holds in (1.10) if and only if

$$\int_D |\nabla_{\underline{t}} h(t_1, t)|^2 \, dt = 0,$$

which implies that h is a function of t_1 only. ■

1.4 Main result

We are concerned with the critical situation where $\Sigma(\hat{h}) = \Sigma(\bar{h})$ holds with $\hat{h} \neq \bar{h}$, which is equivalent to $\sqrt{a} + \sqrt{b} = (2\xi^{\varepsilon})^{1/4}$, see Proposition B.1 of [3]. Note that this condition implies $0 < s_1^L < s_1^R < 1$ for $\hat{h}^{(1)}$. Otherwise, from (1.13) below, h^N converges to the unique minimizer of Σ ($\hat{h}$ in case $\Sigma(\hat{h}) < \Sigma(\bar{h})$ and $\bar{h}$ in case $\Sigma(\bar{h}) < \Sigma(\hat{h})$) as $N \to \infty$ in probability. Our main result is

Theorem 1.2 *We assume $\Sigma(\hat{h}) = \Sigma(\bar{h})$. Then, if $d \geq 3$ and if $\varepsilon > 0$ is sufficiently large, we have that*

$$\lim_{N \to \infty} \mu_N^{aN, bN, \varepsilon} \left(\|h^N - \hat{h}\|_{L^1(D)} \leq \delta \right) = 1,$$

for every $\delta > 0$.

Remark 1.3 *One can even take $\delta = N^{-\alpha}$ with some $\alpha > 0$.*

We conjecture that neither the conditions on the dimension d, nor the one on ε being large, are necessary for the result. For $d = 1$, the convergence to $\hat{h}$ was proved in [3], [7]. The largeness of ε is used here in an essential way to prove the lower bound (1.11). The other parts of the proof don't use it. The condition $d \geq 3$ is used at a number of places where it is convenient that the random walk on $\mathbb{Z}^d$ is transient. We believe, however, that a proof for $d = 2$ would only be technically more involved.

1.5 Outline of the proof

The proof of Theorem 1.2 will be completed in the following three steps. In the first step, we show the following lower bound: For every $\alpha < 1$ and $1 \le p \le 2$,

$$\frac{Z_N^{aN,bN,\varepsilon}}{Z_N^{aN,bN}} \mu_N^{aN,bN,\varepsilon}(\|h^N - \hat{h}\|_{L^p(D)} \le N^{-\alpha}) \ge e^{cN^{d-1}} \tag{1.11}$$

with $c = c_\varepsilon > 0$ for $N \ge N_0$ if $\varepsilon > 0$ is sufficiently large, where $Z_N^{aN,bN} = Z_N^{aN,bN,0}$ (i.e., $\varepsilon = 0$). The second step establishes an upper bound for the probability of the event that the surface stays near $\bar{h}$:

$$\frac{Z_N^{aN,bN,\varepsilon}}{Z_N^{aN,bN}} \mu_N^{aN,bN,\varepsilon}(\|h^N - \bar{h}\|_{L^p(D)} \le (\log N)^{-\alpha_0}) \le 2 \tag{1.12}$$

with some $\alpha_0 > 0$ and $N \ge N_0$. In the last step, we prove a large deviation type estimate:

$$\lim_{N\to\infty} \mu_N^{aN,bN,\varepsilon}\left(\mathrm{dist}_{L^1}(h^N, \{\hat{h}, \bar{h}\}) \ge N^{-\alpha_1}\right) = 0 \tag{1.13}$$

for some $\alpha_1 > 0$. These three estimates (1.11)–(1.13) conclude the proof of Theorem 1.2. In fact, choosing α such that $0 < \alpha < (\alpha_1 \wedge 1)$, (1.11) together with (1.12) implies

$$\lim_{N\to\infty} \frac{\mu_N^{aN,bN,\varepsilon}(\|h^N - \hat{h}\|_{L^1(D)} \le N^{-\alpha})}{\mu_N^{aN,bN,\varepsilon}(\|h^N - \bar{h}\|_{L^1(D)} \le N^{-\alpha})} = \infty,$$

since $N^{-\alpha} \le (\log N)^{-\alpha_0}$ for N large, and at the same time the sum of the numerator and the denominator converges to 1 from (1.13) since $\alpha < \alpha_1$.

A difficulty is stemming from the fact that for $d \ge 2$ a statement like (1.13) cannot be correct with the L^1-distance replaced by the L^∞-distance. If (1.13) would be correct in sup-norm, then h^N would stay, with large probability, either L^∞-close to $\bar{h}$ or $\hat{h}$. However, if it would stay close to $\bar{h}$ in sup-norm, the field ϕ would nowhere be 0, and therefore (1.12) would be trivial, with the bound 1.

Remark 1.4 *An estimate weaker than* (1.11)*:*

$$\frac{Z_N^{aN,bN,\varepsilon}}{Z_N^{aN,bN}} \ge \mathrm{e}^{cN^{d-1}} \tag{1.14}$$

is enough to conclude the proof of Theorem 1.2. In fact, this combined with (1.12) *implies that* $\mu_N^{aN,bN,\varepsilon}(\|h^N - \bar{h}\|_{L^p(D)} \le (\log N)^{-\alpha_0})$ *tends to* 0 *as* $N \to \infty$.

The three estimates (1.11), (1.12) and (1.13) will be proved in Sections 4, 5 and 6, respectively. Section 2 gathers some necessary estimates on the partition functions and Green's functions. Section 3 contains an analytic stability result which is important in Section 6. The capacity plays a role in Section 5. The arguments in Section 6 are similar to those in [4], but there is an additional complication here due to the non-zero boundary conditions. To overcome this, we introduce fields on an extended set with zero boundary conditions.

2 Estimates on partition functions and Green's functions

2.1 Reduction to 0-boundary conditions, the case without pinning

Let $E_n = \{1, 2, \dots, n\} \times \mathbb{T}_N^{d-1} \subset D_N^\circ$ for $1 \le n \le N-1$. For $A \subset D_N^\circ$, we denote $\partial A = \{i \in D_N \backslash A : |i-j| = 1 \text{ for some } j \in A\}$ and $\bar{A} = A \cup \partial A$. For A such that $E_n \subset A$ with some $n \ge 1$ and for $\alpha, \beta \in \mathbb{R}$, the partition function $Z_A^{\alpha,\beta}$ without pinning is defined by

$$Z_A^{\alpha,\beta} = \int_{\mathbb{R}^A} \mathrm{e}^{-H_A^{\alpha,\beta}(\phi)} \prod_{i \in A} d\phi_i, \tag{2.1}$$

where $H_A^{\alpha,\beta}(\phi)$ is the Hamiltonian (1.1) with the sum taken over all $\langle i,j \rangle \subset \bar{A}$ under the boundary condition

$$\phi_i = \alpha \text{ for } i \in \partial_L A, \ \phi_i = \beta \text{ for } i \in \partial_R A, \tag{2.2}$$

where $\partial_L A = \partial_L D_N$ and $\partial_R A = \partial A \backslash \partial_L A (= \partial A \cap \{i : i_1 \ge 2\})$. For general $A \subset D_N^\circ$, we denote Z_A^0 the partition function without pinning defined by (2.1) under the boundary condition $\phi_i = 0$, $i \in \partial A$.

Lemma 2.1 (1) *We have*

$$Z_{E_{n-1}}^{\alpha,\beta} = \mathrm{e}^{-\frac{N^{d-1}}{2n}(\alpha-\beta)^2} Z_{E_{n-1}}^{0,0}.$$

In particular,

$$Z_N^{aN,bN} = \mathrm{e}^{-\frac{N^d}{2}(a-b)^2} Z_N^{0,0}. \tag{2.3}$$

(2) *If $A \supset E_{n-1}$ for some $n \ge 2$, we have*

$$Z_A^{\alpha,\beta,0} \ge \mathrm{e}^{-\frac{N^{d-1}}{2n}(\alpha-\beta)^2} Z_A^{0,0}. \tag{2.4}$$

Proof. We first recall the summation by parts formula for the Hamiltonian $H_A^\psi(\phi)$ for $A \subset D_N^\circ$ with the general boundary condition $\psi = (\psi_i)_{i \in \partial A}$:

$$H_A^\psi(\phi) = -\frac{1}{2}\left((\phi - \bar{\phi}^{A,\psi}), \Delta_A(\phi - \bar{\phi}^{A,\psi})\right)_A + (\mathrm{BT}),$$

where $(\phi^1, \phi^2)_A = \sum_{i \in A} \phi_i^1 \phi_i^2$ stands for the inner product of $\phi^1, \phi^2 \in \mathbb{R}^A$, $\Delta_A \equiv \Delta$ is the discrete Laplacian on A, $\bar{\phi} = \bar{\phi}^{A,\psi}$ is the solution of the Laplace equation:

$$\begin{cases} \left(\Delta\bar{\phi}\right)_i = 0 & i \in A \\ \bar{\phi}_i = \psi_i & i \in \partial A \end{cases} \tag{2.5}$$

and the boundary term (BT) is given by

$$(\mathrm{BT}) = \frac{1}{2} \sum_{i \in A, j \in \partial A : |i-j|=1} \psi_j \{\psi_j - \bar{\phi}_i^{A,\psi}\},$$

see the proof of Proposition 3.1 of [6] (which is stated only for $A \Subset \mathbb{Z}^d$, but the same holds for $A \subset D_N^\circ$).

When $A = E_{n-1}$ and the boundary condition ψ is given as in (2.2), the Laplace equation (2.5) has an explicit solution $\bar{\phi} = \bar{\phi}^{E_{n-1},\psi}$:

$$\bar{\phi}_i = \frac{1}{n}\left(\beta i_1 + \alpha(n - i_1)\right), \quad i \in \bar{E}_{n-1}. \tag{2.6}$$

Thus, in this case, the boundary term is given by

$$(\mathrm{BT}) = \frac{N^{d-1}}{2n}(\alpha - \beta)^2,$$

which shows the first assertion in (1). In particular, (2.3) follows by noting that $Z_N^{aN,bN} = Z_{E_{N-1}}^{aN,bN}$.

To prove (2), we may assume $\alpha > 0$ by symmetry. Let $\bar{\phi}^A$ be the solution of the Laplace equation (2.5) on A with ψ given by (2.2) and set $\bar{\phi}^{n-1} := \bar{\phi}^{E_{n-1}}$. Then, we have

$$\bar{\phi}_i^A \geq \bar{\phi}_i^{n-1} \quad \text{for all } \; i \in \bar{E}_{n-1}. \tag{2.7}$$

Indeed, since $\alpha > 0$, the maximum principle implies that $\bar{\phi}^A \geq 0$ on $\partial_R E_{n-1}$ and, in particular, two harmonic functions $\bar{\phi}^A$ and $\bar{\phi}^{n-1}$ on E_{n-1} satisfy $\bar{\phi}^A \geq \bar{\phi}^{n-1}$ on ∂E_{n-1}. Therefore, by the comparison principle, we obtain (2.7).

Consider now the boundary term (BT) of $H_A^{\alpha,0}(\phi)$. Then, the contribution from the pair $\langle i,j\rangle$ such that $j \in \partial_R A$ vanishes, since $\psi_j = 0$ for such j. On the other hand, for $i \in A, j \in \partial_L A$ such that $|i-j| = 1$, we see from (2.7) and then by (2.6),

$$\psi_j\{\psi_j - \bar{\phi}_i^{A,\psi}\} \leq \alpha\{\alpha - \bar{\phi}_i^{n-1}\} = \frac{1}{n}\alpha^2.$$

This completes the proof of (2). ■

Remark 2.2 *If $A \subset E_{n-1}$, one can similarly show an upper bound on $Z_A^{\alpha,0}$ (i.e. an inequality opposite to* (2.4)*), but this will not be used.*

2.2 Estimates on the partition functions with 0-boundary conditions without pinning

In the subsequent part of Section 2, we will only consider the partition functions under the 0-boundary conditions. The superscripts "$RW^{d,N}$" and "RW^d" refer to simple random walks $\{\eta_n\}_{n=0,1,2,\ldots}$ on $\mathbb{Z} \times \mathbb{T}_N^{d-1}$ and $\mathbb{Z}^d$, respectively, and k in $P_k^{RW^d}$ or $P_k^{RW^{d,N}}$ refers to the starting point of the random walk. We introduce three quantities:

$$q = \sum_{n=1}^{\infty} \frac{1}{2n} P_0^{RW^d}(\eta_{2n} = 0),$$

$$q^N = \sum_{n=1}^{\infty} \frac{1}{2n} P_0^{RW^{d,N}}(\eta_{2n} = 0),$$

$$r = \sum_{n=1}^{\infty} \frac{1}{2n} E_0^{RW^d}\left[\max_{1\leq m\leq 2n} |\eta_m| \cdot 1_{\{\eta_{2n}=0\}}\right].$$

Note that $q < \infty$ for all $d \ge 1$ and $r < \infty$ for $d \ge 2$ (the case that $d \ge 3$ is easy, while the case that $d = 2$ is discussed in [4], p.543). Indeed, if $d \ge 3$, $r < \bar{c} = G(0,0)$, the Green's function defined below in Sections 2.3 and 2.4.

The next lemma, in particular its assertion (1), is shown similarly to the proof of Proposition 4.2.2 or Lemma 2.3.1-a) in [4], only keeping in mind the fact that our random walk "$RW^{d,N}$" is periodic in the second to the dth components.

Lemma 2.3 (1) *Assume that $d \ge 2$ and N is even, and let $A \subset D_N^\circ$. Then, we have that*

$$\frac{1}{2}\left(\log\frac{\pi}{d} + q^N\right)|A| - r \max_{n=1,2,\dots} |\partial A_n| \le \log Z_A^0 \le \frac{1}{2}\left(\log\frac{\pi}{d} + q^N\right)|A|,$$

where $|A| = \sharp\{i \in A\}$ is the number of points in A and $A_n = \{i \in A; \min_{j \in D_N \setminus A} |i-j| \ge n\}$.
(2) *We have the estimate*

$$0 \le q^N - q \le CN^{-d},$$

with some $C > 0$ for every $d \ge 2$.

Proof. We recall the random walk representation for the partition function Z_A^0 from [4], (4.1.1) and (4.1.3) noting that $\Delta_A = 2d(P_A - I)$ in our setting:

$$\log Z_A^0 = \frac{1}{2}\left(|A|\log\frac{\pi}{d} + I\right), \tag{2.8}$$

where

$$I = \sum_{k \in A}\sum_{n=1}^{\infty} \frac{1}{2n} P_k^{RW^{d,N}}(\eta_{2n} = k, \tau_A > 2n) \tag{2.9}$$

and τ_A is the first exit time of η from A; note that, since N is even, $P_k^{RW^{d,N}}(\eta_{2n-1} = k) = 0$. The upper bound for $\log Z_A^0$ in (1) is immediate by dropping the event $\{\tau_A > 2n\}$ from the probability. To show the lower bound, we follow the calculations subsequent to (4.2.8) in the proof of Proposition 4.2.2 of [4]:

$$I = q^N|A| - \sum_{t=1}^{N-1}\sum_{k \in \partial A_t}\sum_{n=1}^{\infty} \frac{1}{2n} P_k^{RW^{d,N}}(\eta_{2n} = k, \tau_A \le 2n),$$

note that $\partial A_t = \emptyset$ for $t \ge N$. Let $\tilde{A} \subset \mathbb{Z}^d$ be the periodic extension of A in the second to the dth coordinates. Then, since τ_A under $RW^{d,N}$ is the same as $\tau_{\tilde{A}}$ under RW^d and $\tau_{\tilde{A}} \ge \tau_{k+S_t}$ for $k \in \partial A_t$, we have

$$P_k^{RW^{d,N}}(\eta_{2n} = k, \tau_A \le 2n) \le P_0^{RW^d}(\eta_{2n} = 0, \tau_{S_t} \le 2n),$$

where $S_t = [-t,t]^d \cap \mathbb{Z}^d$ is a box in $\mathbb{Z}^d$. The rest is the same as in [4].

We finally show the assertion (2). In the representation

$$q^N - q = \sum_{n=1}^{\infty} \frac{1}{2n} P_0^{RW^d}\big(\eta_{2n} \in \{0\} \times (N\mathbb{Z}^{d-1} \setminus \{0\})\big),$$

by applying the Aronson's type estimate for the random walk on $\mathbb{Z}^d$:

$$P_0^{RW^d}(\eta_{2n}=k) \le \frac{C_1}{n^{d/2}} e^{-|k|^2/C_1 n}, \quad k \in \mathbb{Z}^d,$$

with some $C_1>0$, we obtain that

$$0 \le q^N - q \le \frac{C_1}{2} \sum_{n=1}^{\infty} \frac{1}{n^{(d+2)/2}} \sum_{\underline{\ell}\in\mathbb{Z}^{d-1}\setminus\{0\}} e^{-N^2|\underline{\ell}|^2/C_1 n}.$$

However, the last sum in $\underline{\ell}$ can be bounded by

$$C_2\left(1+\frac{\sqrt{n}}{N}\right) e^{-N^2/C_2 n}$$

with some $C_2>0$. Indeed, the sum over $\{\underline{\ell}: 1\le|\underline{\ell}|\le 10\}$ is bounded by $\sharp\{\underline{\ell}: 1\le|\underline{\ell}|\le 10\}\times e^{-N^2/C_1 n}$, while the sum over $\{\underline{\ell}: |\underline{\ell}|\ge 11\}$ can be bounded by the integral:

$$C_3\int_{\{x\in\mathbb{R}^{d-1}:|x|\ge 10\}} e^{-N^2|x|^2/C_1 n}\, dx$$

with some $C_3>0$ and this proves the above statement. Thus, we have

$$0 \le q^N - q \le \frac{C_1 C_2}{2} \sum_{n=1}^{\infty} \frac{1}{n^{(d+2)/2}} \left(1+\frac{\sqrt{n}}{N}\right) e^{-N^2/C_2 n}.$$

Again, estimating the sum in the right hand side by the integral:

$$C_4\int_1^{\infty} \frac{1}{t^{(d+2)/2}} \left(1+\frac{\sqrt{t}}{N}\right) e^{-N^2/C_2 t}\, dt,$$

with some $C_4>0$ and then changing the variables: $t=N^2/u$ in the integral, the conclusion of (2) follows immediately. ∎

2.3 Estimates on the Green's functions

Let $G_N(i,j), i,j\in D_N$ be the Green's function on D_N with Dirichlet boundary condition at ∂D_N:

$$G_N(i,j)=\sum_{n=0}^{\infty} P_i(\eta_n=j, n<\sigma) \left(=E_i^{RW^{d,N}}\left[\sum_{n=0}^{\infty} 1_{\{\eta_n=j,n<\sigma\}}\right]\right), \tag{2.10}$$

where η_n is the random walk on D_N (or on $\mathbb{Z}\times\mathbb{T}_N^{d-1}$) and

$$\sigma=\inf\{n\ge 0; \eta_n\in\partial D_N\}.$$

Let $\tilde{G}_N(i,j), i,j\in\tilde{D}_N := \{0,1,2,\dots,N\}\times\mathbb{Z}^{d-1}$ be the Green's function on $\tilde{D}_N$ with Dirichlet boundary condition at $\partial\tilde{D}_N=\{0,N\}\times\mathbb{Z}^{d-1}$, which has a similar expression to (2.10) with the random walk $\tilde{\eta}_n$ on $\tilde{D}_N$ and its hitting time $\tilde{\sigma}$ to $\partial\tilde{D}_N$. For i or

$j \notin D_N^\circ := D_N \setminus \partial D_N$, we put $G_N(i,j) := 0$, and similarly for $\tilde{G}_N$. We also denote the Green's function of the random walk on the whole lattice $\mathbb{Z}^d$ by $G(i,j), i,j \in \mathbb{Z}^d$, which exists because we assume $d \geq 3$.

Then, we easily see that

$$G_N(i,j) = \sum_{k \in \mathbb{Z}^{d-1}} \tilde{G}_N(i, j+kN), \quad i,j \in D_N, \tag{2.11}$$

where D_N is naturally embedded in $\tilde{D}_N$ and kN is identified with $(0,kN) \in \mathbb{Z}^d$. In fact, the sum in the right hand side of (2.11) does not depend on the choice of $j \in \tilde{D}_N$, in the equivalent class to the original $j \in D_N$ in modulo N in the second to nth components.

The function $\tilde{G}_N$ has the following estimates. For e with $|e| = 1$, we denote $\nabla_{j,e}\tilde{G}_N(i,j) = \tilde{G}_N(i,j+e) - \tilde{G}_N(i,j)$ and similar for $\nabla_{j,e}G_N(i,j)$.

Lemma 2.4 (1) *For $i,j \in \tilde{D}_N$, we have*

$$|\nabla_{j,e}\tilde{G}_N(i,j)| \leq \frac{C}{1+|i-j|^{d-1}} + E_i\left[\frac{C}{1+|\tilde{\eta}_{\tilde{\sigma}} - j|^{d-1}}\right] \tag{2.12}$$

with some $C > 0$.
(2) *With the natural embedding of $D_N \subset \tilde{D}_N$, we have*

$$\sup_{i \in \tilde{D}_N} \sum_{j \in kN + D_N} |\nabla_{j,e}\tilde{G}_N(i,j)| \leq CN.$$

(3) *We have*

$$\tilde{G}_N(i,j) \leq \frac{C}{N^{d-2}} e^{-c|i-j|/N}, \quad \text{if } |i-j| \geq 5N, \tag{2.13}$$

with some $C, c > 0$.

Proof. To show (2.12), we rewrite $\tilde{G}_N(i,j)$ with the random walk $\tilde{\eta}_n$ on $\mathbb{Z}^d$ and its hitting time $\tilde{\sigma}$ to $\partial\tilde{D}_N$ as

$$\begin{aligned}\tilde{G}_N(i,j) &= \sum_{n=0}^{\infty} P_i(\tilde{\eta}_n = j, n < \tilde{\sigma}) \\ &= \sum_{n=0}^{\infty} P_i(\tilde{\eta}_n = j) - \sum_{n=0}^{\infty} P_i(\tilde{\eta}_n = j, n \geq \tilde{\sigma}) \\ &= G(i,j) - E_i[G_N(\tilde{\eta}_{\tilde{\sigma}}, j)],\end{aligned} \tag{2.14}$$

by the strong Markov property of $\tilde{\eta}_n$. Therefore, we have

$$|\nabla_{j,e}\tilde{G}_N(i,j)| \leq |\nabla_{j,e}G(i,j)| + E_i[|\nabla_{j,e}G_N(\tilde{\eta}_{\tilde{\sigma}}, j)|],$$

and we obtain (2.12) from the well-known estimate on the Green's function G on $\mathbb{Z}^d$ (e.g., [11], Theorem 1.5.5, p.32). This proves (1). (2) is an immediate consequence of (1), as

$$\sup_i \sum_{j \in kN + D_N} \frac{1}{1+|i-j|^{d-1}} \leq CN.$$

The next task is to show (2.13). We assume $i \in D_N$ and $j = j_0 + kN$ with $j_0 \in D_N$ and $k \in \mathbb{Z}^{d-1}$. We denote $\Gamma_N(0) = \{\underline{i} = (i_2, \dots, i_d) \in \mathbb{Z}^{d-1}, 0 \le i_\ell < N, \ell = 2, \dots, d\}$ the box in $\mathbb{Z}^{d-1}$ with side length N and divide $\mathbb{Z}^{d-1}$ into a disjoint union of boxes $\{\Gamma_N(\underline{i}) = \underline{i} + \Gamma(0); \underline{i} \equiv 0 \text{ modulo } N\}$. For $k \in \mathbb{Z}^{d-1}$, let $\Gamma_{3N}(k)$ be the box with side length $3N$ with $\Gamma_N(\underline{i})$ as its center, where $\underline{i}$ is determined in such a manner that $k \in \Gamma_N(\underline{i})$. We set $\bar{\sigma} := \inf\{n \ge 0; (\tilde{\eta}_n^{(2)}, \dots, \tilde{\eta}_n^{(d)}) \in \Gamma_{3N}(k)\}$. Note that i and $\Gamma_{3N}(k)$ are separate enough by the condition $|i - j| \ge 5N$. Then, by the strong Markov property,

$$
\begin{aligned}
\tilde{G}_N(i,j) &= E_i\left[\sum_{n=0}^{\infty} 1_{\{\tilde{\eta}_n = j_0 + kN, n \le \tilde{\sigma}\}}\right] \\
&= E_i\left[E_{\tilde{\eta}_{\bar{\sigma}}}\left[\sum_{n=0}^{\infty} 1_{\{\tilde{\eta}_n = j_0 + kN, n \le \tilde{\sigma}\}}\right], \bar{\sigma} < \tilde{\sigma}\right] \\
&= E_i\left[\tilde{G}_N(\tilde{\eta}_{\bar{\sigma}}, j_0 + kN), \bar{\sigma} < \tilde{\sigma}\right] \\
&\le \frac{C}{N^{d-2}} P_i(\bar{\sigma} < \tilde{\sigma}),
\end{aligned}
$$

since $|\tilde{\eta}_{\bar{\sigma}} - (j_0 + kN)| \ge N$ and $\tilde{G}_N(i,j) \le G(i,j) \le \frac{C}{1+|i-j|^{d-2}}$. The event $\{\bar{\sigma} < \tilde{\sigma}\}$ means that the 2nd–dth components of the random walk $\underline{\tilde{\eta}}_n := (\tilde{\eta}_n^{(2)}, \dots, \tilde{\eta}_n^{(d)})$ hits $\{i \in \tilde{D}_N; \underline{i} \in \partial\Gamma_{3N}(k)\}$ before the 1st component of the random walk $\tilde{\eta}_n^{(1)}$ hits $\{0, N\}$ (namely, the random walk $\tilde{\eta}$ hits $\partial\tilde{D}_N$). In other words, $\underline{\tilde{\eta}}$ passes at least $|k| - 2$ boxes $\Gamma_N(\underline{i})$ before $\tilde{\eta}_n^{(1)}$ reaches the boundary of one box of the same size. Such probability can be bounded by the geometric distribution so that we obtain the desired estimate. ■

The following lemma will be used in the proof of Proposition 6.6.

Lemma 2.5 *We have that*

$$
\sup_{i \in D_N} \sum_{j \in D_N} |\nabla_{j,e} G_N(i,j)| \le CN.
$$

Proof. For $k \in \mathbb{Z}^{d-1}$, we write $D_N^{(k)}$ for $D_N + kN$ enlarged by "one layer", so that for any j, e, we can find k with $j, j \in D_N^{(k)}$. Let τ_k for the first entrance time of the random walk $\{\tilde{\eta}_n\}$ into $D_N^{(k)}$. ($\tau_k = 0$ if $\tilde{\eta}_n \in D_N^{(k)}$). Remember that $\tilde{\sigma}$ was the first hitting time of $\partial\tilde{D}_N$. Using the strong Markov property, we have for $j, j+e \in D_N^{(k)}$,

$$
\tilde{G}_N(i,j) - \tilde{G}_N(i,j+e) = E_i[(G_N(\tilde{\eta}_{\tau_k}, j) - G_N(\tilde{\eta}_{\tau_k}, j+e))\, 1_{\tau_k < \tilde{\sigma}}].
$$

We use the representation (2.11) which leads to

$$
\sum_{j \in D_N} |\nabla_{j,e} G_N(i,j)| = \sum_{j \in \tilde{D}_N} \left|\nabla_{j,e} \tilde{G}_N(i,j)\right| \le \sum_{k \in \mathbb{Z}^{d-1}} \sum_{j \in D_N^{(k)}} \left|\nabla_{j,e} \tilde{G}_N(i,j)\right|.
$$

Using Lemma 2.4-(2), we have

$$
\sum_{j \in D_N^{(k)}} \left|\nabla_{j,e} \tilde{G}_N(i,j)\right| \le CNP_i\left(\tau_k < \tilde{\sigma}\right),
$$

implying

$$\sum_{j\in D_N} |\nabla_{j,e} G_N(i,j)| \le CN \sum_k P_i\left(\tau_k < \tilde\sigma\right).$$

It is however easy to see that for $i \in D_N$, $P_i\left(\tau_k < \tilde\sigma\right)$ is exponentially decaying in $|k|$, so the sum on the left hand side is finite, with a bound which is independent of $i \in D_N$. ■

2.4 Decoupling estimate, the case without pinning

The next lemma, which corresponds to Lemma 2.3.1-c) in [4], is prepared for the next subsection. We set

$$\begin{aligned} c_N :=& \sup_{k\in D_N^\circ} \sum_{n=1}^\infty P_k^{RW^{d,N}}(\eta_{2n} = k, 2n < \sigma) \\ =& \sup_{k\in D_N^\circ} G_N(k,k), \end{aligned}$$

and, recalling $d \ge 3$,

$$\bar c := G(0,0) = \sum_{n=1}^\infty P_0^{RW^d}(\eta_{2n} = 0).$$

Lemma 2.6 *Assume $d \ge 3$. Then, we have the following two assertions.*
(1) *c_N is bounded: $c_N \le C$.*
(2) *For two disjoint sets $A, C \subset D_N^\circ$, if N is even, we have*

$$0 \le \log \frac{Z^0_{A\cup C}}{Z^0_A Z^0_C} \le \frac{c_N}{2} |\partial_A C|,$$

where $\partial_A C = \partial A \cap C$.

Proof. For (1), from (2.11), we have that

$$G_N(k,k) = \sum_{\ell\in\mathbb{Z}^{d-1}} \tilde G_N(k, k+\ell N). \tag{2.15}$$

From (2.14), we see that $\tilde G_N(i,j) \le G(i,j)$. Since $G(i,j)$ is bounded, the sum in the right hand side of (2.15) over $\ell : |\ell| \le 5$ is bounded in N. To show the sum over $|\ell| \ge 6$ is also bounded, we can apply the estimate (2.13):

$$\sum_{\ell\in\mathbb{Z}^{d-1}:|\ell|\ge 6} \tilde G_N(k, k+\ell N) \le C \sum_{|\ell|\ge 6} e^{-c|\ell|} < \infty.$$

For (2), we follow the arguments in the middle of p.544 of [4]. From (2.8) and (2.9), we have

$$\begin{aligned} 2\log \frac{Z^0_{A\cup C}}{Z^0_A Z^0_C} =& \sum_{k\in A} \sum_{n=1}^\infty \frac{1}{2n} P_k^{RW^{d,N}}(\eta_{2n} = k, \tau_A < 2n < \tau_{A\cup C}) \\ &+ \sum_{k\in C} \sum_{n=1}^\infty \frac{1}{2n} P_k^{RW^{d,N}}(\eta_{2n} = k, \tau_C < 2n < \tau_{A\cup C}), \end{aligned}$$

note that "$\tau_A = 2n$" does not occur under "$\eta_{2n} = k \in A$". The lower bound in (1) is now clear. To show the upper bound, as in [4], noting that $\tau_{A\cup C} \le \sigma$ for $A, C \subset D_N^\circ$, we further estimate the right hand side by

$$
\begin{aligned}
&\le \sum_{k\in\partial_A C} \sum_{n=1}^{\infty} P_k^{RW^{d,N}}(\eta_{2n} = k, \tau_{A\cup C} > 2n) \\
&\le |\partial_A C| \sum_{n=1}^{\infty} P_0^{RW^{d,N}}(\eta_{2n} = 0, 2n < \sigma) = c_N |\partial_A C|,
\end{aligned}
$$

which concludes the proof of the assertion (2). ■

2.5 Estimates on the partition functions with pinning

For $A \subset D_N^\circ$, we set

$$
Z_A^{0,\varepsilon} = \int_{\mathbb{R}^A} \mathrm{e}^{-H_A^0(\phi)} \prod_{i\in A} (\varepsilon\delta_0(d\phi_i) + d\phi_i)\,.
$$

The next lemma, which corresponds to Lemma 2.3.1-b) in [4], is proved based on Lemma 2.6.

Lemma 2.7 *Assume $d \ge 3$. Then, there exists a constant $\hat{q}^\varepsilon > 0$ such that*

$$
\hat{q}^\varepsilon |A| - \frac{\bar{c}}{4}\left(|\partial A| + 4\ell_1(A)N^{d-2}\right) \le \log Z_A^{0,\varepsilon} \le \hat{q}^\varepsilon |A| + c_N \ell_1(A) N^{d-2},
$$

for every rectangles $A \subset D_N^\circ$, where $\ell_1(A)$ denotes the side length of A in the first coordinate's direction.

Proof. We follow the arguments from the bottom of p.544 to p.545 of [4] noting that we are discussing under the periodic boundary condition for the second to the dth coordinates. We first observe that

$$
\log Z_B^{0,\varepsilon} + \log Z_{B'}^{0,\varepsilon} \le \log Z_{B\cup B'}^{0,\varepsilon} \le \log Z_B^{0,\varepsilon} + \log Z_{B'}^{0,\varepsilon} + \frac{c_N}{2}|\partial_B B'|, \tag{2.16}
$$

for every disjoint $B, B' \subset D_N^\circ$. In fact, since

$$
Z_{B\cup B'}^{0,\varepsilon} = \sum_{A\subset B} \sum_{C\subset B'} \varepsilon^{|B\setminus A| + |B'\setminus C|} Z_{A\cup C}^0,
$$

the lower bound in (2.16) follows from $Z_{A\cup C}^0 \ge Z_A^0 Z_C^0$ (see the lower bound in Lemma 2.6-(2)), while the upper bound follows from

$$
Z_{A\cup C}^0 \le Z_A^0 Z_C^0 \mathrm{e}^{\frac{1}{2}c_N|\partial_A C|} \le Z_A^0 Z_C^0 \mathrm{e}^{\frac{1}{2}c_N|\partial_B B'|}.
$$

In a similar way, we have that

$$
\log Z_B^{0,\varepsilon} + \log Z_{B'}^{0,\varepsilon} \le \log Z_{B\cup B'}^{0,\varepsilon} \le \log Z_B^{0,\varepsilon} + \log Z_{B'}^{0,\varepsilon} + \frac{\bar{c}}{2}|\partial_B B'|, \tag{2.17}
$$

for every disjoint $B, B' \Subset \mathbb{Z}^d$ (or $B, B' \subset D_N^\circ$ which do not contain loops in periodic directions).

For $p = (p_1, \dots, p_d) \in \mathbb{N}^d$, let $S_p = \prod_{\alpha=1}^d [1, p_\alpha] \cap \mathbb{Z}^d$ be the rectangle in $\mathbb{Z}^d$ with volume $|S_p| = \prod_{\alpha=1}^d p_\alpha$ and set $Q(p) = \frac{1}{|S_p|} \log Z_{S_p}^{0,\varepsilon}$. Then, one can show that the limit

$$\hat{q}^\varepsilon = \lim_{m\to\infty} Q(2^m p)$$

exists (independently of the choice of p) and

$$\hat{q}^\varepsilon - \frac{\bar{c}}{4} \frac{|\partial S_p|}{|S_p|} \le Q(p) \le \hat{q}^\varepsilon \tag{2.18}$$

holds for every $p \in \mathbb{N}^d$. Indeed, as in [4], (2.17) implies that

$$Q(p) \le \cdots \le Q(2^{m-1}p) \le Q(2^m p) \le Q(2^{m-1}p) + \frac{\bar{c}}{4} \frac{|\partial S_{2^m p}|}{|S_{2^m p}|}.$$

By letting $m \to \infty$, we obtain that

$$Q(p) \le \hat{q}^\varepsilon \le Q(p) + \frac{\bar{c}}{4} \sum_{m=1}^\infty \frac{|\partial S_{2^m p}|}{|S_{2^m p}|} = Q(p) + \frac{\bar{c}}{4} \frac{|\partial S_p|}{|S_p|},$$

which implies (2.18).

The conclusion of the lemma follows from (2.18) if $A = S_p \subset D_N^\circ$ does not contain loops in periodic directions. In fact, for such A, better inequalities hold:

$$\hat{q}^\varepsilon |A| - \frac{\bar{c}}{4} |\partial A| \le \log Z_A^{0,\varepsilon} \le \hat{q}^\varepsilon |A|.$$

If the rectangle $A \subset D_N^\circ$ is periodically connected, we divide it into two rectangles: $A = A_1 \cup A_2$, where $A_1 = A \cap \{i \in D_N^\circ; 0 \le i_2 \le \frac{N}{2} - 1\}$ and $A_2 = A \cap \{i \in D_N^\circ; \frac{N}{2} \le i_2 \le N-1\}$. Then, noting that $|\partial_{A_1} A_2| = 2\ell_1(A) N^{d-2}$, the conclusion follows from (2.16) (with $B = A_1, B' = A_2$) and (2.18) (applied for each of A_1 and A_2). ■

Remark 2.8 (1) *The ε-dependent quantity is only $\hat{q}^\varepsilon$; c_N and $\bar{c}$ are independent of ε.*
(2) *Lemmas 2.3, 2.7 (for $A \Subset \mathbb{Z}^d$) and* (1.7) *imply that $\xi^\varepsilon = \hat{q}^\varepsilon - \hat{q}^0$ and $\hat{q}^0 = \frac{1}{2}(\log \frac{\pi}{d} + q)$.*

3 Stability result

3.1 Stability at macroscopic level

Recall that the macroscopic energy (LD unnormalized rate functional) $\Sigma(h)$ of $h : D \to \mathbb{R}$ is given by (1.6). We set $\Sigma^*(h) = \Sigma(h) - \min \Sigma$. Note that $\Sigma(\bar{h}) = \frac{1}{2}(a-b)^2$ and $\Sigma(\hat{h}) = \sqrt{2\xi^\varepsilon}(a+b) - \xi^\varepsilon$, see p.446 of [3], and $\Sigma(\bar{h}) = \Sigma(\hat{h}) = \min \Sigma$ from our assumption.

Proposition 3.1 *If $\delta_1 > 0$ is sufficiently small, $\Sigma^*(h) < \delta_1$ implies $d_{L^1}(h, \{\bar{h}, \hat{h}\}) < \delta_2$ with $\delta_2 = c\delta_1^{1/4}$ and some $c > 0$.*

Remark 3.2 *The metric d_{L^1} can be extended to d_{L^p} with $p \in [1, \frac{2d}{d-2})$, but with different rates for δ_2.*

We begin with the stability in one-dimension under a stronger L^∞-topology.

Lemma 3.3 *If $\delta_1 > 0$ is sufficiently small, for $g : [0,1] \to \mathbb{R}$, $\Sigma^*(g) < \delta_1$ implies $d_{L^\infty}(g, \{\bar{h}^{(1)}, \hat{h}^{(1)}\}) < \delta_2$ with $\delta_2 = \sqrt{\delta_1}$.*

Proof. Let us assume $d_{L^\infty}(g, \{\bar{h}^{(1)}, \hat{h}^{(1)}\}) \geq \delta_2$, that is, $d_{L^\infty}(g, \bar{h}^{(1)}) \geq \delta_2$ and $d_{L^\infty}(g, \hat{h}^{(1)}) \geq \delta_2$. First, we consider the case where g does not touch 0, more precisely, $|\{t_1 \in [0,1]; g(t_1) = 0\}| = 0$. Then, the condition $d_{L^\infty}(g, \bar{h}^{(1)}) \geq \delta_2$ implies

$$\Sigma^*(g) \geq 2\delta_2^2. \tag{3.1}$$

Indeed, since the straight line has the lowest energy among curves which have the same heights at both ends and do not touch 0, we consider piecewise linear functions g^{t_0} with $t_0 \in (0,1)$ defined by

$$g^{t_0}(t_1) = \begin{cases} a + \left\{(b-a) \pm \frac{\delta_2}{t_0}\right\} t_1 & \text{for } t_1 \in [0, t_0] \\ b + \left\{(b-a) \pm \frac{\delta_2}{t_0 - 1}\right\} (t_1 - 1) & \text{for } t_1 \in [t_0, 1] \end{cases}.$$

These functions satisfy $d_{L^\infty}(g^{t_0}, \bar{h}^{(1)}) = \delta_2$. Thus, for g satisfying $d_{L^\infty}(g, \bar{h}^{(1)}) \geq \delta_2$ and not touching 0, we see that

$$\begin{aligned} \Sigma^*(g) &\geq \inf_{t_0 \in (0,1)} \Sigma^*(g^{t_0}) \\ &= \inf_{t_0 \in (0,1)} \frac{\delta_2^2}{2} \left(\frac{1}{t_0} + \frac{1}{1-t_0} \right) = 2\delta_2^2, \end{aligned}$$

by a simple computation, which proves (3.1).

Next, we consider the case where g touches 0, i.e., $|\{t_1 \in [0,1]; g(t_1) = 0\}| > 0$. Then, the condition $d_{L^\infty}(g, \hat{h}^{(1)}) \geq \delta_2$ implies

$$\Sigma^*(g) \geq \min\{2\sqrt{2\xi^\varepsilon}\delta_2, \delta_2^2\} \ (= \delta_2^2 \ \text{ if } \delta_2 \leq 2\sqrt{2\xi^\varepsilon}). \tag{3.2}$$

Indeed, it is known that the interval $[s_1^L, s_1^R] = \{t_1 \in [0,1]; \hat{h}^{(1)}(t_1) = 0\}$ of zeros of $\hat{h}^{(1)}$ is determined by the so-called Young's relation:

$$\frac{a}{s_1^L} = \frac{b}{1 - s_1^R} = \sqrt{2\xi^\varepsilon}, \tag{3.3}$$

see [6], p.176, (6.26). Here we assume $a, b > 0$ for simplicity. First, consider the case where the discrepancy at least of size δ_2 of g from $\hat{h}^{(1)}$ occurs at $t_0 \in [s_1^L, s_1^R]$. For such g, the energy $\Sigma_{[0,t_0]}$ on the interval $[0, t_0]$ has a lower bound:

$$\begin{aligned} \Sigma_{[0,t_0]}(g) &\geq \Sigma_{[0,t_0]}(\hat{g}_{[0,t_0]}) \\ &= \frac{a^2}{2s_1^L} - \sqrt{2\xi^\varepsilon}(t_0 - s_1^L - \theta) + \frac{\delta_2^2}{2\theta} \\ &= (a + \delta_2)\sqrt{2\xi^\varepsilon} - \xi^\varepsilon t_0, \end{aligned}$$

where θ is determined by $\frac{\delta_2}{\theta} = \sqrt{2\xi^\varepsilon}$, and $\hat{g}_{[0,t_0]} : [0,t_0] \to \mathbb{R}$ is the minimizer of $\Sigma_{[0,t_0]}$ among the curves $g : [0,t_0] \to \mathbb{R}$ satisfying $g(0) = a$ and $g(t_0) = \delta_2$. Note that, by Young's relation (3.3), $\{t_1 \in [0,t_0]; \hat{g}_{[0,t_0]}(t_1) = 0\} = [s_1^L, t_0 - \theta]$, and also δ_2 is sufficiently small. Similarly, on the interval $[t_0, 1]$, we can show that

$$\Sigma_{[t_0,1]}(g) \geq \Sigma_{[t_0,1]}(\hat{g}_{[t_0,1]}) = (\delta_2 + b)\sqrt{2\xi^\varepsilon} - \xi^\varepsilon(1 - t_0).$$

Therefore, for g mentioned above, we have that

$$\Sigma^*(g) \geq \Sigma_{[0,t_0]}(\hat{g}_{[0,t_0]}) + \Sigma_{[t_0,1]}(\hat{g}_{[t_0,1]}) - \min \Sigma = 2\sqrt{2\xi^\varepsilon}\delta_2. \tag{3.4}$$

Next, consider the case where the discrepancy occurs at $t_0 \in [0, s_1^L]$. For such g, we have that

$$\Sigma_{[0,t_0]}(g) \geq \Sigma_{[t_0,1]}(g^{t_0}) = \frac{t_0}{2}\left(\frac{a}{s_1^L} + \frac{\delta_2}{t_0}\right)^2 \left(= \frac{t_0}{2}\left(\sqrt{2\xi^\varepsilon} + \frac{\delta_2}{t_0}\right)^2\right),$$
$$\Sigma_{[t_0,1]}(g) \geq \Sigma_{[t_0,1]}(\hat{g}_{[t_0,1]}) = \left((a - \sqrt{2\xi^\varepsilon}t_0 - \delta_2) + b\right)\sqrt{2\xi^\varepsilon} - \xi^\varepsilon(1 - t_0),$$

where $g^{t_0} : [0,t_0] \to \mathbb{R}$ is a linear function satisfying $g^{t_0}(t_0) = \hat{h}^{(1)}(t_0) - \delta_2 \left(= a - \frac{a}{s_1^L}t_0 - \delta_2\right)$, and $\hat{g}_{[t_0,1]} : [t_0, 1] \to \mathbb{R}$ is the minimizer of $\Sigma_{[t_0,1]}$ satisfying $\hat{g}_{[t_0,1]}(t_0) = \hat{h}^{(1)}(t_0) - \delta_2$. Therefore, for such g, we have that

$$\Sigma^*(g) \geq \Sigma_{[t_0,1]}(g^{t_0}) + \Sigma_{[t_0,1]}(\hat{g}_{[t_0,1]}) - \min \Sigma = \frac{\delta_2^2}{2t_0} > \delta_2^2,$$

since $t_0 < \frac{1}{2}$. The case where $t_0 \in [s_1^R, 1]$ is similar, and this together with (3.4) shows (3.2). The conclusion of the lemma follows from (3.1) and (3.2) if $\delta_1 \leq (2\sqrt{2\xi^\varepsilon})^2$. ■

We prepare another lemma.

Lemma 3.4 *Assume $d \geq 2$. Then, $\Sigma^*(h) \leq C_1$ implies $\|h\|_{L^q} \leq C_2$ for every $2 \leq q \leq \frac{2d}{d-2}$ (or $2 \leq q < \infty$ when $d = 2$) and some $C_2 = C_2(q, C_1) > 0$.*

Proof. The condition $\Sigma^*(h) \leq C_1$ shows

$$\frac{1}{2}\int_D |\nabla h(t)|^2 dt \leq C_1 + \xi^\varepsilon + \min \Sigma.$$

This, together with Poincaré inequality noting that $h = a$ on $\partial_L D$ and $h = b$ on $\partial_R D$, proves that $\|h\|_{W^{1,2}(D)} \leq C_2$. However, Sobolev's imbedding theorem (e.g., [1], p85) implies the continuity of the imbedding $W^{1,2}(D) \subset L^q(D)$ for $2 \leq q \leq \frac{2d}{d-2}$ and this concludes the proof of the lemma. ■

We are now at the position to give the proof of Proposition 3.1.

Proof of Proposition 3.1. Assume that h satisfies

$$\Sigma^*(h) = \int_{\mathbb{T}^{d-1}} \Sigma^{(1),*}(h(\cdot, \underline{t}))d\underline{t} + \frac{1}{2}\int_D |\nabla_{\underline{t}} h(t_1, \underline{t})|^2 dt < \delta_1, \tag{3.5}$$

where $\Sigma^{(1)}(g)$ is the energy of $g : [0,1] \to \mathbb{R}$, $\Sigma^{(1),*} = \Sigma^{(1)} - \min \Sigma^{(1)}$; recall (1.9). Note that $\min \Sigma^{(1)} = \min \Sigma$ so that we have the above expression for $\Sigma^*(h)$. We assume $\delta_1 > 0$ is sufficiently small. For $M \geq 2$ chosen later, set

$$S^{d-1}_{M\delta_1} := \{\underline{t} \in \mathbb{T}^{d-1}; \Sigma^{(1),*}(h(\cdot,\underline{t})) < M\delta_1\},$$
$$\hat{S}^{d-1}_{M\delta_1} := \mathbb{T}^{d-1} \setminus S^{d-1}_{M\delta_1} = \{\underline{t} \in \mathbb{T}^{d-1}; \Sigma^{(1),*}(h(\cdot,\underline{t})) \geq M\delta_1\}.$$

Then, by (3.5) and Chebyshev's inequality,

$$|\hat{S}^{d-1}_{M\delta_1}| \leq \frac{\delta_1}{M\delta_1} = \frac{1}{M},$$

and

$$|S^{d-1}_{M\delta_1}| \geq 1 - \frac{1}{M}.$$

We first estimate the contribution to $d_{L^1(D)}(h,\bar{h}) = \|h - \bar{h}\|_{L^1(D)}$ and $d_{L^1(D)}(h,\hat{h})$ from the region $\hat{S}^{d-1}_{M\delta_1}$, or more generally regions $S \subset \mathbb{T}^{d-1}$ such that $|S| \leq \frac{1}{M}$:

$$\begin{aligned} \text{(3.6)} \qquad & \int_S \|h(\cdot,\underline{t}) - \bar{h}^{(1)}\|_{L^1([0,1])} d\underline{t} \\ & \leq \int_S \{\|h(\cdot,\underline{t})\|_{L^1([0,1])} + \|\bar{h}^{(1)}\|_{L^1([0,1])}\} d\underline{t} \\ & = \int_D 1_{[0,1]\times S}(t)|h(t)|dt + C|S| \\ & \leq \sqrt{|[0,1]\times S|}\; \|h\|_{L^2(D)} + \frac{C}{M} \\ & \leq \frac{C_2}{\sqrt{M}} + \frac{C}{M} \leq \frac{C_3}{\sqrt{M}}, \end{aligned}$$

where $C = \|\bar{h}^{(1)}\|_{L^1([0,1])} < \infty$ and $C_3 = C_2 + C$. We have applied Schwarz's inequality for the fourth line and Lemma 3.4 for the fifth line with $C_2 = C_2(2,\delta_1)$. We similarly have

$$\int_S \|h(\cdot,\underline{t}) - \hat{h}^{(1)}\|_{L^1([0,1])} d\underline{t} \leq \frac{C_4}{\sqrt{M}}.$$

For $\underline{t} \in S^{d-1}_{M\delta_1}$, by Lemma 3.3, we see that

$$d_{L^\infty}(h(\cdot,\underline{t}), \{\bar{h}^{(1)}, \hat{h}^{(1)}\}) < \delta_2 (= \sqrt{M\delta_1}).$$

Set

$$S^{d-1,(1)}_{M\delta_1} := \{\underline{t} \in S^{d-1}_{M\delta_1}; d_{L^\infty}(h(\cdot,\underline{t}), \bar{h}^{(1)}) < \delta_2\},$$
$$S^{d-1,(2)}_{M\delta_1} := \{\underline{t} \in S^{d-1}_{M\delta_1}; d_{L^\infty}(h(\cdot,\underline{t}), \hat{h}^{(1)}) < \delta_2\}.$$

If $|S^{d-1,(2)}_{M\delta_1}| \leq \frac{1}{M}$, we have from (3.6) that

$$\begin{aligned} \text{(3.7)} \qquad d_{L^1(D)}(h,\bar{h}) = \|h - \bar{h}\|_{L^1(D)} &= \int_{\mathbb{T}^{d-1}} \|h(\cdot,\underline{t}) - \bar{h}^{(1)}\|_{L^1([0,1])} d\underline{t} \\ &\leq \sqrt{M\delta_1} + \frac{2C_3}{\sqrt{M}} = C_5 \delta_1^{1/4}, \end{aligned}$$

by dividing $\mathbb{T}^{d-1} = S^{d-1,(1)}_{M\delta_1} \cup (\hat{S}^{d-1}_{M\delta_1} \cup S^{d-1,(2)}_{M\delta_1})$ and choosing $M = 1/\sqrt{\delta_1}$, with $C_5 = 1+2C_3$. We have a similar bound:

$$d_{L^1(D)}(h, \hat{h}) = \|h - \hat{h}\|_{L^1(D)} \le C_6 \delta_1^{1/4}, \tag{3.8}$$

if $|S^{d-1,(1)}_{M\delta_1}| \le \frac{1}{M} = \sqrt{\delta_1}$.

Therefore, the case where both $|S^{d-1,(1)}_{M\delta_1}|$, $|S^{d-1,(2)}_{M\delta_1}| \ge \frac{1}{M} = \sqrt{\delta_1}$ is left. In this case, since $|S^{d-1}_{M\delta_1}| \ge 1 - \frac{1}{M} \ge \frac{1}{2}$ (since $M \ge 2$), the volume of $S^{d-1,(1)}_{M\delta_1}$ or $S^{d-1,(2)}_{M\delta_1}$ is larger than $\frac{1}{4}$. Let us assume $|S^{d-1,(1)}_{M\delta_1}| \ge \frac{1}{4}$ and $|S^{d-1,(2)}_{M\delta_1}| \ge \sqrt{\delta_1}$. The case $|S^{d-1,(2)}_{M\delta_1}| \ge \frac{1}{4}$ and $|S^{d-1,(1)}_{M\delta_1}| \ge \sqrt{\delta_1}$ can be treated similarly. Then, choosing a subset $S \subset S^{d-1,(2)}_{M\delta_1}$ such that $|S| = \sqrt{\delta_1}$, we have that

$$\begin{aligned} &\int_0^1 dt_1 \int_{S^{d-1,(1)}_{M\delta_1}} d\underline{t} \int_S d\underline{t}^* \, \mathrm{Av}\!\int_0^1 (\underline{t} - \underline{t}^*) \cdot \nabla_{\underline{t}} h(t_1, \alpha \underline{t} + (1-\alpha)\underline{t}^*) d\alpha \\ &\quad = \int_{S^{d-1,(1)}_{M\delta_1}} d\underline{t} \int_S d\underline{t}^* \int_0^1 \{h(t_1, \underline{t}) - h(t_1, \underline{t}^*)\} dt_1 \\ &\quad \ge \frac{c_{\delta_2}}{4} \sqrt{\delta_1} \ge \frac{C_7}{4} \sqrt{\delta_1}, \end{aligned} \tag{3.9}$$

by integrating in α first, where

$$\mathrm{Av}\!\int_0^1 f(\underline{t}, \underline{t}^*, \alpha) d\alpha := \frac{1}{2^{d-1}} \sum_{\underline{s}^* \in E} \int_0^1 f(\underline{s}, \underline{s}^*, \alpha) d\alpha,$$

by embedding $\underline{t}, \underline{t}^* \in \mathbb{T}^{d-1}$ into $\underline{s} \in [0,1)^{d-1}$ such that $\underline{s} = \underline{t}$ mod 1 componentwisely and $E = \{\underline{s}^* \in \mathbb{R}^{d-1}; \underline{s}^* = \underline{t}^* \text{ mod } 1 \text{ and } |\underline{s} - \underline{s}^*| < \sqrt{d-1}\}$, and

$$\begin{aligned} c_{\delta_2} &= \int_0^1 \{h(t_1, \underline{t}) - h(t_1, \underline{t}^*)\} dt_1 \\ &\ge \|\bar{h}^{(1)} - \hat{h}^{(1)}\|_{L^1([0,1])} - 2\delta_2 \ge C_8, \end{aligned}$$

for some $C_8 > 0$, if $\delta_2 = c\delta_1^{1/4}$ and therefore δ_1 are sufficiently small. Estimating $|\underline{t} - \underline{t}^*| \le \sqrt{d-1}$, the left hand side of (3.9) is bounded from above by

$$\begin{aligned} &\sqrt{d-1} \int_0^1 dt_1 \int_{\mathbb{T}^{d-1}} d\underline{t} \int_S d\underline{t}^* \, \mathrm{Av}\!\int_0^1 |\nabla_{\underline{t}} h(t_1, \alpha \underline{t} + (1-\alpha)\underline{t}^*)| d\alpha \\ &\quad = \sqrt{d-1} \int_0^1 E\left[1_S(\underline{t}^*) |\nabla_{\underline{t}} h(t_1, \alpha \underline{t} + (1-\alpha)\underline{t}^*)|\right] dt_1. \end{aligned}$$

Here, under the expectation, $\underline{t}$ and $\underline{t}^*$ are $\mathbb{T}^{d-1}$-valued uniformly distributed random variables, α is $[0,1]$-valued uniformly distributed random variable and $\{\underline{t}, \underline{t}^*, \alpha\}$ are mutually independent. Then, by Schwarz's inequality, we have that

$$\begin{aligned} &E\left[1_S(\underline{t}^*) |\nabla_{\underline{t}} h(t_1, \alpha \underline{t} + (1-\alpha)\underline{t}^*)|\right] \\ &\quad \le \sqrt{|S|} \sqrt{E[|\nabla_{\underline{t}} h(t_1, \alpha \underline{t} + (1-\alpha)\underline{t}^*)|^2]} \\ &\quad = \delta_1^{1/4} \left(\int_{\mathbb{T}^{d-1}} |\nabla_{\underline{t}} h(t_1, \underline{t})|^2 d\underline{t} \right)^{1/2}, \end{aligned}$$

since $\alpha\underline{t} + (1-\alpha)\underline{t}^*$ is also $\mathbb{T}^{d-1}$-valued uniformly distributed random variable. Thus, applying Schwarz's inequality again, the left hand side of (3.9) is bounded from above by

$$\sqrt{d-1}\delta_1^{1/4}\|\nabla_{\underline{t}}h\|_{L^2(D)} \le \sqrt{d-1}\delta_1^{1/4}\sqrt{2\delta_1},$$

by the condition (3.5). Combined with (3.9), this implies $\sqrt{2(d-1)}\delta_1^{1/4} \ge \frac{C_7}{4}$, which contradicts that we assume δ_1 is sufficiently small. Thus, (3.7) and (3.8) complete the proof of the proposition by taking $c = \max\{C_5, C_6\}$. ■

3.2 Stability at mesoscopic level

Given $0 < \beta < 1$, we divide D_N into $N^{d(1-\beta)}$ subboxes of sidelength N^β. For the sake of simplicity, we assume that N^βdivides N. We write $\mathcal{B}_{N,\beta}$ for the set of these subboxes, and $\hat{\mathcal{B}}_{N,\beta}$ for the set of unions of boxes in $\mathcal{B}_{N,\beta}$. The sets $B \in \hat{\mathcal{B}}_{N,\beta}$ are called mesoscopic regions.

For $B \in \hat{\mathcal{B}}_{N,\beta}$ (and actually for general $B \subset D_N$), set

$$\begin{aligned} E_N(B) &= E_{N,0}(B) - \xi^\varepsilon |B^c|, \\ E_{N,0}(B) &= \inf_{\phi \in \mathbb{R}^{D_N}:(3.11)} H_N(\phi), \\ E_N^*(B) &= E_N(B) - \min_{B \in \hat{\mathcal{B}}_{N,\beta}} E_N(B), \end{aligned} \tag{3.10}$$

where the infimum in (3.10) is taken over all $\phi \in \mathbb{R}^{D_N}$ satisfying the condition:

$$\phi_i = \begin{cases} aN & \text{if } i \in \partial_L D_N \\ bN & \text{if } i \in \partial_R D_N \\ 0 & \text{if } i \in D_N^\circ \setminus B \end{cases}. \tag{3.11}$$

Let $\bar\phi^B = (\bar\phi_i^B)_{i \in D_N}$ be the harmonic function on B subject to the condition (3.11). Then, $\bar\phi^B$ is the minimizer of the variational problem (3.10). The macroscopic profile $h^N = h_B^N (\equiv h_{B,\mathrm{PL}}^N) \in C(D)$ is defined from the microscopic profile $\bar\phi^B$ by polilinearly interpolating $\frac{1}{N}\bar\phi^B_{[Nt]}, t \in D$, where $[Nt]$ stands for the integer part of Nt taken componentwisely; see (1.5).

The stability at mesoscopic level is formulated as follows:

Proposition 3.5 *Assume $\alpha > 0$ is given and $\beta, \gamma > 4\alpha$. Then, if N is sufficiently large, $E_N^*(B) \le N^{d-\gamma}$ for $B \in \hat{\mathcal{B}}_{N,\beta}$ implies $d_{L^1}(h_B^N, \{\bar h, \hat h\}) \le N^{-\alpha}$.*

From (1.22) in [5], the polilinear interpolation has the property:

$$\frac{1}{2}\int_D |\nabla h^N(t)|^2 dt \le \frac{1}{2N^d}\sum_{i \in D_N} |\nabla^N \bar\phi_i^B|^2 = \frac{1}{N^d} H_N(\bar\phi^B) = \frac{1}{N^d} E_{N,0}(B).$$

We also see that $\{t \in D; h^N(t) = 0\} \supset \frac{1}{N}(B^c)^\circ$, which implies that

$$-|\{t \in D; h^N(t) = 0\}| + \frac{1}{N^d}|B^c| \le \frac{1}{N^d}|\partial B| \le \frac{1}{N^d} dN^{d-\beta} = dN^{-\beta}.$$

These two bounds show that

$$\Sigma(h^N) \le \frac{1}{N^d} E_N(B) + \xi^\varepsilon d N^{-\beta}. \tag{3.12}$$

We need the next lemma.

Lemma 3.6

$$\frac{1}{N^d} \min E_N(B) \le \min \Sigma \le \frac{1}{N^d} \min E_N(B) + \xi^\varepsilon d N^{-\beta}.$$

Proof. The upper bound follows from (3.12). To show the lower, recall $\min \Sigma = \frac{1}{2}(a-b)^2$. Define $\bar{\phi} \equiv \bar{\phi}^{D_N} = (\bar{\phi}_i)_{i \in D_N}$ by

$$\bar{\phi} = \psi_{i_1} := aN + (b-a)i_1, \quad i \in D_N,$$

where i_1 is the first component of i. Then, we see that

$$E_N(D_N) = H_N(\bar{\phi}) = \frac{1}{2} \sum_{(i_2,\dots,i_d) \in \mathbb{T}_N^{d-1}} \sum_{i_1=0}^{N-1} (\psi_{i_1+1} - \psi_{i_1})^2 = \frac{N^d}{2}(b-a)^2.$$

This proves the lower bound. ■

From the lower bound in this lemma and (3.12), we see that $E^*(B) \le N^{d-\gamma}$ implies $\Sigma^*(h^N) \le N^{-\gamma} + \xi^\varepsilon d N^{-\beta}$. Thus, Proposition 3.5 follows from Proposition 3.1.

We slightly extend Proposition 3.5 and this will be used in Section 6.3.

Proposition 3.7 *Let a mesoscopic region B and $A_2 \subset B$ such that $|B \setminus A_2| \le N^{d-\frac{1}{8}}$ be given, and assume that*

$$E_{N,0}(A_2) - \xi^\varepsilon |B^c| - \min E_N \le N^{d-\gamma}. \tag{3.13}$$

Then, we have that

$$d_{L^1}\left(h_{A_2}^N, \left\{\bar{h}, \hat{h}\right\}\right) \le N^{-\alpha}, \tag{3.14}$$

where $h_{A_2}^N$ is defined from $\bar{\phi}^{A_2}$, which is harmonic on A_2 subject to the condition (3.11) *with B replaced by A_2.*

Proof. As we saw above, we have that

$$\frac{1}{2} \int_D |\nabla h_{A_2}^N(t)|^2 dt \le \frac{1}{N^d} E_{N,0}(A_2)$$

and also, since $\{t \in D; h_{A_2}^N(t) = 0\} \supset \frac{1}{N}(B^c)^\circ$ (we don't need the condition on $|B \setminus A_2|$),

$$-|\{t \in D; h_{A_2}^N(t) = 0\}| + \frac{1}{N^d}|B^c| \le d N^{-\beta}.$$

Therefore, (3.13) together with the lower bound in Lemma 3.6 implies $\Sigma^*(h_{A_2}^N) \le N^{-\gamma} + \xi^\varepsilon d N^{-\beta}$, and we obtain (3.14) from Proposition 3.1. ■

4 Proof of the lower bound (1.11)

This section is concerned with the lower bound on

$$\Xi_N := \frac{Z_N^{aN,bN,\varepsilon}}{Z_N^{aN,bN}} \mu_N^{aN,bN,\varepsilon}(\|h^N - \hat{h}\|_{L^p(D)} \le \delta), \tag{4.1}$$

where we take $\delta = N^{-\alpha}$ with $\alpha < 1$; see Remark 4.1 below. We divide D_N° into five disjoint regions: $D_N^\circ = A_L \cup \gamma_L \cup B \cup \gamma_R \cup A_R$, where

$$\begin{aligned}
A_L &= \big([1, Ns_1^L - K - 1] \cap \mathbb{Z}\big) \times \mathbb{T}_N^{d-1}, \\
\gamma_L &= \big([Ns_1^L - K, Ns_1^L] \cap \mathbb{Z}\big) \times \mathbb{T}_N^{d-1}, \\
B &= \big([Ns_1^L + 1, Ns_1^R] \cap \mathbb{Z}\big) \times \mathbb{T}_N^{d-1}, \\
\gamma_R &= \big([Ns_1^R + 1, Ns_1^R + K] \cap \mathbb{Z}\big) \times \mathbb{T}_N^{d-1}, \\
A_R &= \big([Ns_1^R + K + 1, N - 1] \cap \mathbb{Z}\big) \times \mathbb{T}_N^{d-1},
\end{aligned}$$

for $K > 0$, where s_1^L and $s_1^R \in (0,1)$ are the first and the last s's such that $\hat{h}^{(1)}(s) = 0$ and we assume that $Ns_1^L, Ns_1^R \in \mathbb{Z}$ for simplicity. Note that the side lengths in i_1-direction of these five rectangles are $Ns_1^L - K - 1, K + 1, N(s_1^R - s_1^L), K$ and $N(1 - s_1^R) - K - 1$ for $A_L, \gamma_L, B, \gamma_R$ and A_R, respectively. Then, restricting the probability in (4.1) on the event:

$$\mathcal{A} := \{\phi; \phi_i \ne 0 \text{ for } i \in A_L \cup A_R \text{ and } \phi_i = 0 \text{ for } i \in \gamma_L \cup \gamma_R\},$$

we have

$$\begin{aligned}
\Xi_N \ge& \frac{Z_N^{aN,bN,\varepsilon}}{Z_N^{aN,bN}} \mu_N^{aN,bN,\varepsilon}(\|h^N - \hat{h}\|_{L^p(D)} \le \delta, \mathcal{A}) \\
=& \Xi_N^1 \times \mu_{A_L}^{aN,0}(\|h^N - \hat{h}\|_{L^p(D_L)} \le \delta) \\
&\times \mu_B^{0,\varepsilon}(\|h^N - \hat{h}\|_{L^p(D_M)} \le \delta) \mu_{A_R}^{0,bN}(\|h^N - \hat{h}\|_{L^p(D_R)} \le \delta),
\end{aligned} \tag{4.2}$$

by the Markov property of $\mu_N^{aN,bN,\varepsilon}$, where $\mu_{A_L}^{aN,0}$ is defined on A_L with boundary conditions aN and 0 at the left respectively right boundaries of A_L without pinning, $\mu_{A_R}^{0,bN}$ is similarly defined on A_R, $\mu_B^{0,\varepsilon}$ is defined on B with boundary condition 0 with pinning,

$$\Xi_N^1 = \frac{Z_{A_L}^{aN,0} Z_B^{0,\varepsilon} Z_{A_R}^{0,bN}}{Z_N^{aN,bN}} \varepsilon^{|\gamma_L| + |\gamma_R|},$$

and D_L, D_M and D_R are the macroscopic regions corresponding to A_L, B and A_R, respectively. Since γ_L and γ_R are macroscopically close to the hyperplanes $\{t_1 = s_1^L\}$ and $\{t_1 = s_1^R\}$ in D, respectively (i.e., γ_L/N is in a $c\delta$-neighborhood of $\{t_1 = s_1^L\}$ with suitable $c > 0$ etc.), by the LDP [2] for $\mu_{A_L}^{aN,0}$, $\mu_{A_R}^{0,bN}$ and the LDP for $\mu_B^{0,0}$ combined with the coupling argument (see Lemma 4.4 below) implying $-\tilde{\phi}_i^{(2)} \le \phi_i^{(1)} \le \phi_i^{(2)}, i \in B$ for $\phi^{(1)} \sim \mu_B^{0,\varepsilon}$, $\tilde{\phi}^{(2)}, \phi^{(2)} \sim \mu_B^{0,0,+} := \mu_B^{0,0}(\cdot | \phi \ge 0)$, three probabilities in the right hand side of (4.2) are close to 1 as $N \to \infty$. Therefore, for every $c > 0$, we have

$$\Xi_N \ge (1 - c)\Xi_N^1 \tag{4.3}$$

as $N \to \infty$.

Remark 4.1 (1) *If $d \ge 3$, the Gaussian property implies*

$$E^{\mu^{aN,0}_{A_L}}\left[\|h^N - \hat{h}\|^2_{L^2(D_L)}\right] \le \frac{C}{N^2},$$

and others. Therefore, (4.3) holds even for $\delta = N^{-\alpha}$ with $\alpha < 1$ at least for $p = 2$ (so that for every $1 \le p \le 2$). For $d = 2$, this statement is also true since the above expectation behaves as $C \log N/N^2$.
(2) *To show the weaker estimate (1.14), we can simply estimate $\Xi_N \ge \Xi^1_N$ so that the LDP and the coupling argument for the above three probabilities are unnecessary.*

We now give the lower bound on Ξ^1_N. Since $A_L = E_{Ns^L_1-K-1}$ and $A_R = E_{N(1-s^R_1)-K-1}$ (which is reversed), Lemma 2.1 shows that

$$\begin{aligned}
Z^{aN,bN}_N &= \exp\left\{-\frac{N^d}{2}(a-b)^2\right\} Z^{0,0}_N, \\
Z^{aN,0}_{A_L} &= \exp\left\{-\frac{a^2N^d}{2(s^L_1 - K/N)}\right\} Z^0_{A_L}, \\
Z^{0,bN}_{A_R} &= \exp\left\{-\frac{b^2N^d}{2(1-s^R_1 - K/N)}\right\} Z^0_{A_R}.
\end{aligned}$$

Therefore, from $1/(s^L_1 - K/N) = 1/s^L_1 + KN^{-1}/(s^L_1)^2 + O_\varepsilon(N^{-2})$ and a similar expansion for $1/(1-s^R_1 - K/N)$ as $N \to \infty$, we have

$$\Xi^1_N \ge \exp\left\{f(a,b)N^d - K\tilde{f}(a,b)N^{d-1} - O_\varepsilon(N^{d-2})\right\}\Xi^2_N,$$

where $O_\varepsilon(N^{d-2})$ means that the constant may depend on ε (since s^L_1 and s^R_1 depend on ε), and

$$\begin{aligned}
\Xi^2_N &= \frac{Z^0_{A_L} Z^{0,\varepsilon}_B Z^0_{A_R}}{Z^{0,0}_N}\varepsilon^{|\gamma_L|+|\gamma_R|}, \\
f(a,b) &= \frac{1}{2}(a-b)^2 - \frac{a^2}{2s^L_1} - \frac{b^2}{2(1-s^R_1)} \\
&= \Sigma(\bar{h}) - \Sigma(\hat{h}) - \xi^\varepsilon(s^R_1 - s^L_1), \\
\tilde{f}(a,b) &= \frac{a^2}{2(s^L_1)^2} + \frac{b^2}{2(1-s^R_1)^2}.
\end{aligned}$$

However, we have $\tilde{f}(a,b) = 2\xi^\varepsilon$ from Young's relation for the angles of $\hat{h}$ at $s = s^L_1$ and s^R_1: $a/s^L_1 = b/(1-s^R_1) = \sqrt{2\xi^\varepsilon}$, see Section 1.3 of [3] or Section 6 of [6] for example. Moreover, by Lemma 2.3 and Remark 2.8-(2)

$$\frac{Z^0_{A_L} Z^0_{A_R}}{Z^{0,0}_N} \ge \exp\left\{\hat{q}^0(|A_L| + |A_R| - |D^\circ_N|) - 4rN^{d-1} - C\right\},$$

and by the lower bound in Lemma 2.7

$$Z^{0,\varepsilon}_B \ge \exp\left\{\hat{q}^\varepsilon|B| - \frac{3}{2}\bar{c}N^{d-1}\right\}.$$

Thus, since $|A_L| + |A_R| + |B| + |\gamma_L| + |\gamma_R| = |D_N^\circ| \big(= N^{d-1}(N-1) \big)$ and $\hat{q}^\varepsilon - \hat{q}^0 = \xi^\varepsilon$, we obtain

$$\begin{aligned}\log \Xi_N^1 \geq & f(a,b)N^d + \hat{q}^0(|A_L| + |A_R| - |D_N^\circ|) + \hat{q}^\varepsilon |B| \\ & - \big(4r + 3\bar{c}/2 + 2K\xi^\varepsilon\big)N^{d-1} + (|\gamma_L| + |\gamma_R|)\log\varepsilon - O_\varepsilon(N^{d-2}) - C \\ \geq & f(a,b)N^d + \xi^\varepsilon|B| - (C_1 + 2K\xi^\varepsilon)N^{d-1} + (|\gamma_L| + |\gamma_R|)(\log\varepsilon - \hat{q}^0) - O_\varepsilon(N^{d-2}),\end{aligned}$$

with a constant $C_1 = 4r + 3\bar{c}/2 > 0$ independent of ε; the constant C is included in $O_\varepsilon(N^{d-2})$. However, the balance condition: $\Sigma(\bar{h}) = \Sigma(\hat{h})$ and $|B| = N^d(s_1^R - s_1^L)$ imply that $f(a,b)N^d + \xi^\varepsilon|B| = 0$, so that the volume order terms cancel. Therefore, from $|\gamma_L| + |\gamma_R| = (2K+1)N^{d-1}$, we have

$$\begin{aligned}\log \Xi_N^1 &\geq \big((2K+1)(\log\varepsilon - \hat{q}^0) - 2K\xi^\varepsilon - C_1\big)N^{d-1} - O_\varepsilon(N^{d-2}) \\ &\geq \big(\log\varepsilon - (2K+1)\hat{q}^0 - 2K\log 2 - C_1\big)N^{d-1} - O_\varepsilon(N^{d-2}),\end{aligned}$$

where the second line follows from the upper bound on ξ^ε given in Lemma 4.2 below. It is now clear that, for $\varepsilon > 0$ large enough, the coefficient of N^{d-1} in the right hand side is positive and thus the proof of the lower bound (1.11) is concluded.

Lemma 4.2 *For $\varepsilon \geq 1$, we have that*

$$\log\varepsilon - \hat{q}^0 \leq \xi^\varepsilon \leq \log 2\varepsilon.$$

Proof. We have an expansion:

$$Z_{\Lambda_\ell}^{0,\varepsilon} = \sum_{A \subset \Lambda_\ell} \varepsilon^{|\Lambda_\ell \setminus A|} Z_A^0.$$

To show the upper bound, we rudely estimate: $\varepsilon^{|\Lambda_\ell \setminus A|} \leq \varepsilon^{\ell^d}$ for $\varepsilon \geq 1$ and $Z_A^0 \leq e^{\hat{q}^0|A|} \leq e^{\hat{q}^0\ell^d}$ by Lemma 2.3-(1); note that its upper bound holds with q in place of q^N for $A \Subset \mathbb{Z}^d$. Then, since $\sharp\{A : A \subset \Lambda_\ell\} = 2^{\ell^d}$, we obtain

$$Z_{\Lambda_\ell}^{0,\varepsilon} \leq 2^{\ell^d} \varepsilon^{\ell^d} e^{\hat{q}^0 \ell^d}$$

and therefore

$$\hat{q}^\varepsilon \equiv \lim_{\ell\to\infty} \frac{1}{\ell^d} \log Z_{\Lambda_\ell}^{0,\varepsilon} \leq \log 2\varepsilon + \hat{q}^0,$$

from which the upper bound on $\xi^\varepsilon = \hat{q}^\varepsilon - \hat{q}^0$ follows (or, recall (1.7) for ξ^ε and note that Lemma 2.3 also shows $\lim_{\ell\to\infty} \ell^{-d} \log Z_{\Lambda_\ell}^0 = \hat{q}^0$). Taking only the term with $A = \emptyset$ in the expansion, we have $Z_{\Lambda_\ell}^{0,\varepsilon} \geq \varepsilon^{\ell^d}$ and this implies the lower bound. ■

Remark 4.3 (1) *To have the large factor $\log\varepsilon$, we need to allow some spaces for γ_L and γ_R. For this purpose, in the above proof, we have cut off the regions A_L and A_R by letting $K \geq 1$, while the volume of the region B are maintained. It is also possible to maintain the spaces for A_L and A_R by taking $K = 0$. Instead, we may cut off the region B, but the results are the same.*
(2) *In fact, one can take $K = 0$ for γ_L and $K = 1$ for γ_L so that the required condition for $\varepsilon > 0$ is: $\log\varepsilon > \log 2 + 2\hat{q}^0 + 4r + 3\bar{c}/2$.*

We finally give the coupling argument used above. Consider the Gibbs probability measure $\mu_A^{\psi,\varepsilon}$ on A under the boundary condition ψ given on $D_N \setminus A$. Assuming that $\psi \geq 0$ (i.e., $\psi_i \geq 0$ for all $i \in D_N \setminus A$), we compare it with

$$\mu_A^{\psi,0,+}(\cdot) := \mu_A^{\psi,0}(\cdot|\phi \geq 0),$$

by adding the effect of a wall located at the level of $\phi = 0$ to the Gaussian measure $\mu_A^{\psi,0}(\cdot)$ without the pinning effect. In fact, we have the following lemma from an FKG type argument.

Lemma 4.4 *We have the stochastic domination: $\mu_A^{\psi,\varepsilon} \leq \mu_A^{\psi,0,+}$. Namely, one can find a coupling of $\phi^\varepsilon = \{\phi_i^\varepsilon\}_{i\in D_N}$ and $\phi^{0,+} = \{\phi_i^{0,+}\}_{i\in D_N}$ on a common probability space such that $P(\phi_i^\varepsilon \leq \phi_i^{0,+}$ for all $i \in D_N) = 1$, and ϕ^ε and $\phi^{0,+}$ are distributed under $\mu_A^{\psi,\varepsilon}$ and $\mu_A^{\psi,0,+}$, respectively.*

Proof. For $\phi = (\phi_i)_{i\in D_N} \in \mathbb{R}^{D_N}$ satisfying the conditions $\phi_k = \psi_k$ on $D_N \backslash A$, we consider two Hamiltonians

$$H_N^{(\ell)}(\phi) = H_N^{\psi}(\phi) + \sum_{i\in D_N\backslash A} U^{(\ell)}(\phi_i), \ \ell = 1, 2,$$

by adding the self potentials $U^{(\ell)}$ defined by $U^{(1)}(r) = -\beta 1_{[0,\alpha]}(r)$ and $U^{(2)}(r) = K1_{(-\infty,0]}(r)$, $r > 0$, with $\alpha, \beta, K > 0$ to the original Hamiltonian H_N^ψ defined under the boundary condition ψ. The corresponding Gibbs probability measures $\mu_N^{(\ell)}$ are defined by

$$\mu_N^{(\ell)}(d\phi) = \frac{1}{Z_N^{(\ell)}} \mathrm{e}^{-H_N^{(\ell)}(\phi)} \prod_{i\in A} d\phi_i \prod_{k\in D_N\backslash A} \delta_{\psi_k}(d\phi_k), \ \ell = 1, 2.$$

It will be shown that the stochastic domination $\mu_N^{(1)} \leq \mu_N^{(2)}$ holds if $K \geq \beta$. Once this is shown, by taking the limits $\alpha \to 0$, $\beta \to \infty$ such that $\varepsilon = \alpha\left(\mathrm{e}^\beta - 1\right)$ (see e.g. (6.34) in [6], and $K \to \infty$, the lemma is concluded.

It is known that the stochastic domination $\mu_N^{(1)} \leq \mu_N^{(2)}$ holds if the two Hamiltonians satisfy Holley's condition:

$$H_N^{(2)}(\phi) + H_N^{(1)}(\bar\phi) \geq H_N^{(2)}(\phi \vee \bar\phi) + H_N^{(1)}(\phi \wedge \bar\phi), \tag{4.4}$$

for every $\phi, \bar\phi \in \mathbb{R}^{D_N}$, where $(\phi \vee \bar\phi)_i = \phi_i \vee \bar\phi_i$ and $(\phi \wedge \bar\phi)_i = \phi_i \wedge \bar\phi_i$, see Theorem 2.2 of [9]. Since (4.4) holds for H_N^ψ (i.e., if $U^{(1)} = U^{(2)} = 0$), it is enough to show that

$$U^{(2)}(x) + U^{(1)}(y) \geq U^{(2)}(x \vee y) + U^{(1)}(x \wedge y)$$

for all $x, y \in \mathbb{R}$. However, this is equivalent to

$$U^{(2)}(x) - U^{(2)}(y) \geq U^{(1)}(x) - U^{(1)}(y) \tag{4.5}$$

for every $x, y \in \mathbb{R}$. It is now easy to see that this is true under the condition $K \geq \beta$. ■

Remark 4.5 *If the self potentials $U^{(\ell)}$ are smooth, the condition* (4.5) *is equivalent to* $\{U^{(2)}\}' \le \{U^{(1)}\}'$ *on* $\mathbb{R}$.

Remark 4.6 *Lemma 4.4 was applied under the boundary condition $\psi \equiv 0$. In this case, by the symmetry $\phi \mapsto -\phi$ under $\mu_A^{0,\varepsilon}$, we also have the lower bound. When $\psi \equiv 0$, it might hold the stochastic domination: $|\phi_i^\varepsilon| \le |\phi_i|, i \in D_N$, $\phi^\varepsilon \sim \mu_A^{0,\varepsilon}$, $\phi \sim \mu_A^{0,0}$ (this was true at least when $d = 1$, see Section 4.2.3 [3]).*

5 Proof of the upper bound (1.12)

We write $Z_N^\varepsilon, Z_N, \mu_N^\varepsilon$ instead of $Z_N^{aN,bN,\varepsilon}, Z_N^{aN,bN}, \mu_N^{aN,bN,\varepsilon}$, respectively, and similar at other places. We expand as

$$\frac{Z_N^\varepsilon}{Z_N}\mu_N^\varepsilon\left(\left\|h^N - \overline{h}\right\|_{L^p(D)} \le \delta\right) = \sum_{A \subset D_N^\circ} \varepsilon^{|A^c|}\frac{Z_A}{Z_N}\mu_A\left(\left\|h^N - \overline{h}\right\|_{L^p(D)} \le \delta\right). \tag{5.1}$$

Here, Z_A refers to boundary conditions 0 on A^c, and the usual one on the cylinder (A^c stands for the complement of A in D_N°), and μ_A is defined with similar boundary conditions. We will consider the Gaussian field μ_N on D_N° with the above boundary conditions. Note that the Gaussian field $\{\phi_i\}_{i \in D_N^\circ}$ on $\mathbb{R}^{D_N^\circ}$ distributed under μ_N has covariance matrix

$$\Gamma \overset{\text{def}}{=} \frac{1}{2d}(I - P)^{-1},$$

where P is the random walk transition kernel with killing at the boundary ∂D_N. Furthermore, ϕ_i has mean $m(i) = m_{aN,bN}(i)$ which is given by linearly interpolating between the boundary condition aN on $\partial_L D_N$ and bN on $\partial_R D_N$.

We take $\delta = (\log N)^{-\alpha_0}$ with $\alpha_0 > d/p$ in (5.1). We show that

$$\sum_{A \subset D_N^\circ,\ |A^c| \le (N/\log N)^d} \varepsilon^{|A^c|}\frac{Z_A}{Z_N} \le 2 \tag{5.2}$$

if N is large enough. Note that, if $|A^c| \ge (N/\log N)^d$, then $h^N = 0$ on A^c so that

$$\left\|h^N - \overline{h}\right\|_{L^p(D)} \ge (a \wedge b)(\log N)^{-d/p}.$$

In particular, for such A, we have

$$\mu_A\left(\left\|h^N - \overline{h}\right\|_{L^p(D)} \le (\log N)^{-\alpha_0}\right) = 0$$

as $\alpha_0 > d/p$. Thus (5.2) proves (1.12).

Now we give the proof of (5.2). Recall that

$$\frac{Z_A}{Z_N} = \frac{1}{Z_N}\int_{\mathbb{R}^{D_N^\circ}} \exp\left[-H_N(\phi)\right]\prod_{i \in A} d\phi_i \prod_{i \in A^c} \delta_0(d\phi_i).$$

The function

$$f_{A^c}\left(\{\phi_i\}_{i\in A^c}\right) \stackrel{\text{def}}{=} \frac{1}{Z_N}\int \exp\left[-H_N\left(\phi\right)\right]\prod_{i\in A} d\phi_i$$

is the density function of the Gaussian distribution on $\mathbb{R}^{A^c}$ obtained as the marginal from the Gaussian distribution μ_N on $\mathbb{R}^{D_N^\circ}$. This marginal Gaussian field has the same mean as μ_N and the covariance matrix Γ_{A^c} which comes from restricting the covariance matrix Γ to $A^c \times A^c$. This covariance matrix has the representation $\Gamma_{A^c} = \left(I - P_{A^c}\right)^{-1}$, where $P_{A^c}\left(i,j\right)$ for $i,j \in A^c$ is the probability for a random walk to enter A^c at j after leaving i with absorption at ∂D_N. So

$$\sum_{j\in A^c} P_{A^c}\left(i,j\right) \leq 1.$$

We also write for the escape probability

$$e_{A^c}\left(i\right) \stackrel{\text{def}}{=} 1 - \sum_{y\in A^c} P_{A^c}\left(i,j\right),$$

and then the capacity of A^c with respect to the transient random walk on D_N° with killing at the boundary is

$$\operatorname{cap}_{D_N}\left(A^c\right) \stackrel{\text{def}}{=} \sum_{i\in A^c} e_{A^c}\left(j\right).$$

Then we have

$$f_{A^c}\left(\{\phi_i\}_{i\in A^c}\right) = \frac{1}{\sqrt{(2\pi)^{|A^c|}\det\Gamma_{A^c}}}\exp\left[-d\left\langle \phi - m, \left(I - P_{A^c}\right)\left(\phi - m\right)\right\rangle_{A^c}\right],$$

where $\langle \phi, \psi\rangle_{A^c} \stackrel{\text{def}}{=} \sum_{i\in A^c}\phi_i\psi_i$, and $m = m_{aN,bN}$. We therefore get

$$\frac{Z_A}{Z_N} = \frac{1}{\sqrt{(2\pi)^{|A^c|}\det\Gamma_{A^c}}}\exp\left[-d\left\langle m, \left(I - P_{A^c}\right)m\right\rangle_{A^c}\right]. \tag{5.3}$$

We first estimate the determinant from below

$$\begin{aligned}\sqrt{(2\pi)^{|A^c|}\det\Gamma_{A^c}} &= \int \exp\left[-d\left\langle \phi - m, \left(I - P_{A^c}\right)\left(\phi - m\right)\right\rangle_{A^c}\right]\prod_{i\in A^c} d\phi_i \\ &\geq \int_{\{|\phi_i - m_i|\leq 1/2,\ \forall i\in A^c\}} \exp\left[-d\left\langle \phi - m, \left(I - P_{A^c}\right)\left(\phi - m\right)\right\rangle_{A^c}\right]\prod_{i\in A^c} d\phi_i.\end{aligned}$$

On the other hand,

$$\sup_{\{|\phi_i - m_i|\leq 1/2,\ \forall i\in A^c\}} \left\langle \phi - m, \left(I - P_{A^c}\right)\left(\phi - m\right)\right\rangle_{A^c} \leq \left|A^c\right|,$$

and therefore

$$\sqrt{(2\pi)^{|A^c|}\det\Gamma_{A^c}} \geq \exp\left[-d\left|A^c\right|\right].$$

We write $p_L(i)$ for the probability that the random walk starting in $i \in A^c$ does not return to A^c and leaves D_N on the left side, and correspondingly $p_R(i)$ for the right exit. Clearly $p_L(i) + p_R(i) = e_{A^c}(i)$. Then

$$m(i) = \sum_j P_{A^c}(i,j)\, m(j) + p_L(i)\, aN + p_R(i)\, bN.$$

So

$$m(i) - \sum_j P_{A^c}(i,j)\, m(j) \geq \min(a,b)\, N e_{A^c}(i).$$

Of course, also $m(i) \geq \min(a,b)\, N$. Therefore from (5.3),

$$\frac{Z_A}{Z_N} \leq \exp\left[d\,|A^c|\right] \exp\left[-dN^2 \min(a,b)^2 \operatorname{cap}_{D_N}(A^c)\right].$$

Lemma 5.1 proved below implies that

$$\operatorname{cap}_{D_N}(A^c) \geq c\,|A^c|^{(d-2)/d}, \tag{5.4}$$

from which we conclude that for some $c > 0$, depending on d, a, b

$$\begin{aligned}
&\sum_{A \subset D_N^\circ,\ |A^c| \leq (N/\log N)^d} \varepsilon^{|A^c|} \frac{Z_A}{Z_N} \\
&\leq \sum_{m=0}^{(N/\log N)^d} \chi(m)\, \varepsilon^m \exp\left[dm - \bar{c} N^2 m^{(d-2)/d}\right],
\end{aligned}$$

where $\bar{c} > 0$ and $\chi(m)$ is the number of subset A in D_N° with $|A^c| = m$. Clearly,

$$\chi(m) \leq \exp\left[dm \log N\right].$$

So

$$\begin{aligned}
&\sum_{A \subset D_N^\circ,\ |A^c| \leq (N/\log N)^d} \varepsilon^{|A^c|} \frac{Z_A}{Z_N} \\
&\leq 1 + \left(\frac{N}{\log N}\right)^d \times \max_{1 \leq m \leq \left(\frac{N}{\log N}\right)^d} \\
&\qquad \exp\left[m\left(d \log N + \log \varepsilon + d\right) - \bar{c} N^2 m^{(d-2)/d}\right].
\end{aligned} \tag{5.5}$$

As the function of m in the exponent is convex, it takes its maximum either at $m = 1$, or at $m = \left(\frac{N}{\log N}\right)^d$ (assuming for simplicity that the latter is an integer). If it takes the maximum at $m = 1$, then we clearly for large N that the whole expression on the right hand side of (5.5) is ≤ 2. At $m = \left(\frac{N}{\log N}\right)^d$, one has the same situation. We get for the expression in the exponent

$$N^d \left[\frac{d}{\log^{d-1} N} + \frac{\log \varepsilon + d}{\log^d N} - \frac{\bar{c}}{\log^{d-2} N}\right]. \tag{5.6}$$

If N is sufficiently large, this is dominated by the third summand, and therefore the expression in the exponent is for $m = \left(\frac{N}{\log N}\right)^d$ bounded by

$$-\frac{CN^d}{\log^{d-2} N},$$

with some $C > 0$. This gives for the summand after 1 in (5.5) even something smaller, namely an expression of order

$$\frac{N^d}{\log^d N} \exp\left[-\frac{CN^d}{\log^{d-2} N}\right].$$

This completes the proof of (5.2) and therefore (1.12).

The rest of this section is devoted to the proof of the capacity estimate (5.4). Recall that, for $A \subset D_N^\circ$, the capacity with respect to D_N is defined by

$$\mathrm{cap}_{D_N}(A) := \sum_{x \in A} P_x^{RW_N^d}\left(T_{\partial D_N} < T_A\right)$$

where T_A denotes the first hitting time of A after time 0 for a random walk on the discrete cylinder.

Lemma 5.1 *For some constant $c > 0$, depending only on the dimension d, one has*

$$\mathrm{cap}_{D_N}(A) \geq c\,|A|^{(d-2)/d}. \tag{5.7}$$

Proof. We will use $c > 0$ as a notation for a generic positive (small) constant which depends only on the dimension and which may change from line to line. In the course of the proof, we need two other capacities. First the discrete capacity on $\mathbb{Z}^d$: For a finite subset $A \subset \mathbb{Z}^d$,

$$\mathrm{cap}_{\mathbb{Z}^d}(A) := \sum_{x \in A} P_x^{RW^d}\left(T_A = \infty\right),$$

where the random walk here is the standard random walk on $\mathbb{Z}^d$. We will compare cap_{D_N} with $\mathrm{cap}_{\mathbb{Z}^d}$ and then the latter with the usual Newtonian capacity.

We assume (for simplicity), that $N - 3$ is divisible by 6 : $N = 3(2M+1)$ and identify $\mathbb{T}_N$ with $\{3M-1, \dots, 3M+1\}$. Then, subdivide $\mathbb{T}_N$ into the 3 subintervals $J_{-1} := \{-3M-1, \dots, -M-1\}$, $J_0 := \{-M, \dots, M\}$, $J_1 := \{M+1, \dots, 3M+1\}$, and $\mathbb{T}_N^{d-1}$ into the 3^{d-1} subboxes $R_{\mathbf{i}} := J_{i_1} \times \cdots \times J_{i_{d-1}}$, $\mathbf{i} = (i_1, \dots, i_{d-1}) \in \{-1, 0, 1\}^{d-1}$, and for given $A \subset D_N^\circ$, we consider

$$A_{\mathbf{i}} := A \cap \left([1, N-1] \times R_{\mathbf{i}}\right),$$

where $[1, N-1] \stackrel{\text{def}}{=} \{1, \dots, N-1\}$. From the monotonicity of the capacity, we get

$$\mathrm{cap}_{D_N}(A) \geq \mathrm{cap}_{D_N}(A_{\mathbf{i}})$$

for every choice of $\mathbf{i}$. We choose $\mathbf{i}$ such that $|A_\mathbf{i}|$ is maximal. If we can prove

$$\operatorname{cap}_{D_N}(A_\mathbf{i}) \geq c\,|A_\mathbf{i}|^{(d-2)/d},$$

then we obtain (5.7) with an adjustment of c. We therefore can restrict to sets A which are contained in one of the sets $\{1,\ldots,N-1\}\times R_\mathbf{i}$, and we may assume that $\mathbf{i}=(0,\ldots,0)$ i.e. A is contained in the middle subbox. As we have periodic boundary conditions on $\mathbb{T}^{d-1}$, this is no loss of generality.

We can then view A also as a subset of $\mathbb{Z}^d$ by the identification $\mathbb{T}_N^{d-1} = [3M-1, 3M+1]^{d-1} \subset \mathbb{Z}^{d-1}$. We now claim that for such an A one has

$$\operatorname{cap}_{D_N}(A) \geq c\operatorname{cap}_{\mathbb{Z}^d}(A). \tag{5.8}$$

We denote by $\|\cdot\|_{d-1,\infty}$ the subnorm in $\mathbb{Z}^{d-1}$. We also write for $0\leq k\leq l$

$$\begin{aligned}
S_{k,l} &\stackrel{\text{def}}{=} [1,N-1]\times\left\{x\in\mathbb{Z}^{d-1}: k\leq\|x\|_{d-1,\infty}\leq l\right\},\\
\hat{S}_{k,l} &\stackrel{\text{def}}{=} \{0,N\}\times\left\{x\in\mathbb{Z}^{d-1}: k\leq\|x\|_{d-1,\infty}\leq l\right\}.
\end{aligned}$$

For $k=l$, we write S_k instead of $S_{k,k}$. So $A\subset S_{0,M}$. The boundary of $S_{M+1,3M}$, regarded as a subset of $\mathbb{Z}^d$ consists of the three parts $\hat{S}_{M+1,3M},\ S_M,\ S_{3M+1}$. An evident fact is

$$P_x^{RW^d}\left(X_{\tau_{S_{M+1,3M}}}\in\hat{S}_{M+1,3M}\right)\geq c>0,\ x\in S_{2M}, \tag{5.9}$$

where τ_S is the first exit time from S of a random walk $\{X_n\}$, starting in x. This follows for instance from the weak convergence of the random walk path to Brownian motion, and the elementary fact that for a d-dimensional Brownian motion starting in 0, the first exit from a cylinder $[-\gamma,\gamma]\times\left\{x\in\mathbb{R}^{d-1}:|x|\leq 1\right\}$ through $\{-\gamma,\gamma\}\times\{|x|\leq 1\}$ has probability $p(d,\gamma)>0$.

Consider now a random walk on $\mathbb{Z}^d$ starting at $x\in A$. The escape probability $e_A(x)$ to ∞ can be bounded as follows

$$\begin{aligned}
e_A(x) &= P_x^{RW^d}(T_A=\infty)\leq P_x^{RW^d}\left(\tau_{S_{0,2M-1}}<T_A\right)\\
&= P_x^{RW^d}\left(\tau_{S_{0,2M-1}}<T_A, X\left(\tau_{S_{0,2M-1}}\right)\in\hat{S}_{0,2M-1}\right)\\
&\quad + P_x^{RW^d}\left(\tau_{S_{0,2M-1}}<T_A, X\left(\tau_{S_{0,2M-1}}\right)\in S_{2M}\right)\\
&\leq P_x^{RW^d}\left(\tau_{S_{0,3M}}<T_A, X\left(\tau_{S_{0,3M}}\right)\in\hat{S}_{0,3M}\right)\\
&\quad + P_x^{RW^d}\left(\tau_{S_{0,2M-1}}<T_A, X\left(\tau_{S_{0,2M-1}}\right)\in S_{2M}\right).
\end{aligned}\tag{5.10}$$

The inequality is coming from the fact that on $\left\{\tau_{S_{0,2M-1}}<T_A, X\left(\tau_{S_{0,2M-1}}\right)\in\hat{S}_{0,2M-1}\right\}$ one has $\left\{\tau_{S_{0,2M-1}}=\tau_{S_{0,3M}}\right\}$. We estimate the second summand on the right hand side by (5.9). For abbreviation, we set $\tau_1\stackrel{\text{def}}{=}\tau_{S_{0,2M-1}}$ and $\tau_2=\tau_{S_{M+1,3M}}$. Then, denoting by θ_{τ_1} the shift operator by τ_1, we have

$$\left\{\tau_1<T_A,\ X_{\tau_1}\in S_{2M},\ X_{\tau_2}\circ\theta_{\tau_1}\in\hat{S}_{M+1,3M}\right\}\subset\left\{\tau_{S_{0,3M}}<T_A,\ X_{S_{0,3M}}\in\hat{S}_{0,3M}\right\},$$

and therefore, by the strong Markov property, and (5.9)

$$\begin{aligned}
&P_x\left(\tau_{S_{0,3M}} < T_A, X_{S_{0,3M}} \in \hat{S}_{0,3M}\right) \\
&\geq P_x\left(\tau_1 < T_A,\ X_{\tau_1} \in S_{2M},\ X_{\tau_2} \circ \theta_{\tau_1} \in \hat{S}_{M+1,3M}\right) \\
&= E_x\left(1_{\left\{\tau_1 < T_A,\ X_{\tau_1} \in S_{2M}\right\}} E_{X_{\tau_1}}\left(X_{\tau_2} \in \hat{S}_{M+1,3M}\right)\right) \\
&\geq c P_x\left(\tau_1 < T_A,\ X_{\tau_1} \in S_{2M}\right).
\end{aligned}$$

Combining this with (5.10) gives

$$e_A(x) \leq \left(1 + c^{-1}\right) P_x\left(\tau_{S_{0,3M}} < T_A, X_{S_{0,3M}} \in \hat{S}_{0,3M}\right).$$

If for the random walk on $\mathbb{Z}^d$, one has $\tau_{S_{0,3M}} < T_A, X_{S_{0,3M}} \in \hat{S}_{0,3M}$, then the random walk on $D_N^\circ = [1, N-1] \times \mathbb{T}^{d-1}$ obtained through periodizing the torus part reaches ∂D_N before returning to A. Therefore

$$P_x\left(\tau_{S_{0,3M}} < T_A, X_{S_{0,3M}} \in \hat{S}_{0,3M}\right) \leq e_A^{\mathbb{T}^{d-1}}(x).$$

Summing over $x \in A$, this implies (5.8) (with a changed c).

In order to prove the lemma, it therefore remains to prove that for a finite subset $A \subset \mathbb{Z}^d$, we have

$$\mathrm{cap}_{\mathbb{Z}^d}(A) \geq c\,|A|^{(d-2)/d}.$$

We denote by $(k_1, \ldots, k_d)$, $k_i \in \{0,1\}$ the 2^d corner points of a unit box in $\mathbb{Z}^d$ spanned by the unit vectors $e_1, \ldots, e_d$, and we write $Q \subset \mathbb{R}^d$ for the closed unit box itself. The discrete translations are $Q_y := y + Q$, $y \in \mathbb{Z}^d$. Set

$$\bar{A} := \bigcup_{k \in \{0,1\}^d} (A + k) \subset \mathbb{Z}^d,\ \hat{A} := \bigcup_{x \in A} Q_x \subset \mathbb{R}^d.$$

By the subadditivity and shift invariance of the discrete capacity, we have

$$\mathrm{cap}_{\mathbb{Z}^d}(A) \geq 2^{-d}\,\mathrm{cap}_{\mathbb{Z}^d}\left(\bar{A}\right).$$

Define ϕ to be the discrete harmonic extension of $1_{\bar{A}}$, i.e.

$$\phi(x) = P_x\left(S_{\bar{A}} < \infty\right),$$

where

$$S_{\bar{A}} := \inf\left\{n \geq 0 : X_n \in \bar{A}\right\},$$

$\{X_n\}_{n \geq 0}$ being the symmetric nearest neighbor random walk on $\mathbb{Z}^d$. ϕ is discrete harmonic outside $\bar{A}$ and satisfies $\lim_{|x| \to \infty} \phi(x) = 0$ as $d \geq 3$. We write

$$\delta\phi(x) := \left(\delta_i \phi(x)\right)_{i=1,\ldots,d},\ \delta_i \phi(x) := \phi(x + e_i) - \phi(x).$$

It is well known that the discrete lattice capacity satisfies

$$\mathrm{cap}_{\mathbb{Z}^d}\left(\bar{A}\right) = \frac{1}{2d} \sum_x |\delta\phi(x)|^2 = \frac{1}{2d} \inf\left\{\sum\nolimits_x |\delta\psi(x)|^2 : \psi \geq 1_{\bar{A}}\right\}.$$

We interpolate ϕ on each of the boxes $Q_y,\ y \in \mathbb{Z}^d$, to a continuous function $\hat{\phi} : \mathbb{R}^d \to [0,1]$ by defining for $y + x \in \mathbb{R}^d,\ y \in \mathbb{Z}^d,\ x \in Q$:

$$\hat{\phi}\,(y+x) := \sum_{k \in \{0,1\}^d} \prod_{i=1}^{k} x_i^{(k_i)} \phi\,(y + (k_1, \ldots, k_d)),$$

where

$$x_i^{(k_i)} = \begin{cases} 1 - x_i & \text{for } k_i = 0 \\ x_i & \text{for } k_i = 1 \end{cases}.$$

By the construction, $\hat{\phi}$ is uniquely defined also on the intersections of different boxes. It is evident that

$$\hat{\phi} \geq 1_{\hat{A}} \tag{5.11}$$

because for $x \in A$, all corner points of Q_x belong to $\bar{A}$ on which ϕ is 1. The partial derivatives inside of box Q_y are

$$\begin{aligned} \frac{\partial \hat{\phi}}{\partial x_i}\,(y+x) = \sum_{k_1,\ldots,k_{i-1},k_{i+1},\ldots,k_d \in \{0,1\}} \prod_{j:j \neq i} x_j^{(k_j)} \Big[\\ - \phi\,(y + (k_1, \ldots, k_{i-1}, 0, k_{i+1}, \ldots, k_d)) \Big] \end{aligned}$$

From this representation, it follows that with some constant $C\,(d) > 0$

$$\frac{1}{2} \int_{\mathbb{R}^d} \left| \nabla \hat{\phi}\,(x) \right|^2 dx \leq C\,(d) \sum_{x \in \mathbb{Z}^d} |\delta \phi\,(x)|^2 = C\,(d)\, \mathrm{cap}_{\mathbb{Z}^d}\left(\bar{A}\right).$$

The Newtonian capacity of a compact subset $K \subset \mathbb{R}^d$ is defined by

$$\mathrm{cap}_d\,(K) \overset{\mathrm{def}}{=} \inf \left\{ \frac{1}{2} \int |\nabla \psi\,(x)|^2\, dx : \psi \in H_1\left(\mathbb{R}^d\right),\ \psi \geq 1_{\hat{A}} \right\},$$

where H_1 is the Sobolev space of weakly once differentiable functions on $\mathbb{R}^d$ with square integrable derivative. Using (5.11), we get

$$\begin{aligned} \mathrm{cap}_d\left(\hat{A}\right) &\leq C\,(d)\, \mathrm{cap}_{\mathbb{Z}^d}\left(\bar{A}\right) \\ &\leq 2^d C\,(d)\, \mathrm{cap}_{\mathbb{Z}^d}\,(A). \end{aligned} \tag{5.12}$$

By the Poincaré-Faber-Szegö inequality for the Newtonian capacity (see [13] for $d = 3$, and [10], Appendix A for general $d \geq 3$), one has with some new constant $c > 0$

$$\mathrm{cap}_d\left(\hat{A}\right) \geq c \left(\mathrm{vol}\left(\hat{A}\right) \right)^{(d-2)/d} = c\,|A|^{(d-2)/d},$$

which, together with (5.12) proves the claim. ■

6 The large deviation estimate: Proof of (1.13)

6.1 Preliminaries

Convention: All statements we make are only claimed to be true for large enough N without special mentioning.

Markov property: Let μ_Λ be the probability measure of the free field, that is the Gaussian field without pinning, on a finite subset Λ of the cylinder $\mathbb{Z} \times \mathbb{T}_N^d$, with arbitrary boundary conditions on $\partial\Lambda$, and let $B \subset \Lambda$. We write $\mathcal{F}_A$ for $\sigma(\phi_i : i \in A)$. Then for any $X \in \mathcal{F}_B$ we have

$$\mu_\Lambda\left(X \mid \mathcal{F}_{B^c}\right) = \mu_\Lambda\left(X \mid \mathcal{F}_{\partial B \cap \Lambda}\right). \tag{6.1}$$

FKG-inequality: Let $G : \mathbb{R}^\Lambda \to \mathbb{R}$ be a measurable function which is non-decreasing in all arguments, and let $\mu_{\Lambda,\mathbf{x}}$ be the free field on Λ with boundary condition $\mathbf{x} \in \mathbb{R}^{\partial\Lambda}$. The FKG-property states that $\int G \, d\mu_{\Lambda,\mathbf{x}}$ is nondecreasing as a function of $\mathbf{x} \in \mathbb{R}^{\partial\Lambda}$ in all coordinates.

We will use the expansion

$$\mu_N^{aN,bN,\varepsilon} = \sum_{A \subset D_N^\circ} p_N^\varepsilon(A)\, \mu_A^{aN,bN}, \tag{6.2}$$

where μ_A is the standard free field with boundary condition 0 on $\partial A \cap D_N^\circ$ and aN, respectively bN on ∂D_N, extended by the Dirac measure at 0 on $A^c \overset{\text{def}}{=} D_N^\circ \backslash A$, and where

$$p_N^\varepsilon(A) = \frac{\varepsilon^{|A^c|} Z_A^{aN,bN}}{Z_A^{aN,bN,\varepsilon}}$$

$\{p_N^\varepsilon(A)\}_{A \subset D_N^\circ}$ is a probability distribution on the set of subsets of D_N°.

We write

$$A_{N,\alpha} \overset{\text{def}}{=} \left\{ \operatorname{dist}_{L^1}(h^N, \{\hat{h}, \bar{h}\}) \geq N^{-\alpha} \right\},$$

so, in order to prove (1.13), we have to prove $\mu_N^\varepsilon(A_{N,\alpha}) \to 0$ for small enough α. Let

$$\Omega_N^+ \overset{\text{def}}{=} \{\phi_i \geq -\log N, \ \forall i \in D_N^\circ\}.$$

Lemma 6.1

$$\lim_{N\to\infty} \mu_N^{aN,bN,\varepsilon}\left(\Omega_N^+\right) = 1.$$

Proof. We use

$$\begin{aligned}
\mu_A^{aN,bN}\left((\Omega_N^+)^c\right) &\leq N^d \sup_{i \in A} \mu_A^{aN,bN}(\phi_i \leq -\log N) \\
&\leq N^d \sup_{i \in A} \mu_A^0(\phi_i \leq -\log N) \\
&\leq N^d \sup_{i \in A} \exp\left[-\frac{(\log N)^2}{2G_A(i,i)}\right] \leq N^d \exp\left[-\frac{(\log N)^2}{2C}\right],
\end{aligned}$$

where μ_A^0 has boundary conditions 0 on A^c (and not just on $A^c \cap D_N^\circ$). In the last inequality, we have used $G_A(i,i) \leq G_{\mathbb{Z}^d}(i,i) = C < \infty$ as we assume $d \geq 3$. For the second inequality, we use FKG and $a, b \geq 0$. Combining with (6.2) shows the conclusion. ■

Using this lemma, it suffices to prove

$$\lim_{N\to\infty} \mu_N^{aN,bN,\varepsilon}\left(A_{N,\alpha} \cap \Omega_N^+\right) = 0 \tag{6.3}$$

for α chosen sufficiently small.

We will consider the random fields on an extended set

$$D_{N,\mathrm{ext}} \stackrel{\mathrm{def}}{=} \{-N, -N+1, \ldots, 2N\} \times \mathbb{T}_N^{d-1},$$

with

$$D_{N,\mathrm{ext}}^\circ \stackrel{\mathrm{def}}{=} \{-N+1, \ldots, 2N-1\} \times \mathbb{T}_N^{d-1},$$

$$\partial D_{N,\mathrm{ext}} \stackrel{\mathrm{def}}{=} \{-N, 2N\} \times \mathbb{T}_N^{d-1},\ \check{D}_{N,\mathrm{ext}} \stackrel{\mathrm{def}}{=} D_{N,\mathrm{ext}}^\circ \backslash D_N^\circ.$$

We define the measure $\mu_{N,\mathrm{ext}}^\varepsilon$ on $\mathbb{R}^{D_{N,\mathrm{ext}}}$ with 0 boundary conditions on $\partial D_{N,\mathrm{ext}}$ and ε-pinning on D_N°, i.e.

$$\begin{aligned}\mu_{N,\mathrm{ext}}^\varepsilon(d\phi) =& \frac{1}{Z_{N,\mathrm{ext}}^\varepsilon}\exp\left[-\frac{1}{2}\sum_{\langle i,j\rangle \subset D_{N,\mathrm{ext}}}(\phi_i - \phi_j)^2\right] \\ &\times \prod_{i\in D_N^\circ}(d\phi_i + \varepsilon\delta_0(d\phi_i)) \prod_{i\in D_{N,\mathrm{ext}}^\circ \backslash D_N^\circ} d\phi_i, \quad \phi \equiv 0 \text{ on } \partial D_{N,\mathrm{ext}}.\end{aligned}$$

$\mu_{N,\mathrm{ext}}$ is the usual Gaussian field corresponding to $\varepsilon = 0$. The reader should pay attention to the fact that pinning for $\mu_{N,\mathrm{ext}}^\varepsilon$ is *only* on D_N°.

We write $\mathbb{F}$ for the set of subsets of $D_{N,\mathrm{ext}}^\circ$ satisfying $\check{D}_{N,\mathrm{ext}} \subset F$. For $F \in \mathbb{F}$ we write μ_F^0 for the Gaussian field on $\mathbb{R}^F$ with 0 boundary condition on ∂F. It is sometimes convenient to extend μ_F to $\mathbb{R}^{D_{N,\mathrm{ext}}}$ by multiplying it with $\prod_{i\notin F}\delta_0(d\phi_i)$. Remark that $\partial D_N \subset F$.

We need the following lemma for the proof of Lemma 6.5 below.

Lemma 6.2 *Let $F \in \mathbb{F}$, and $s, t > 0$ satisfy $s > t/2, t > s/2$. Let $\psi_F : F \cup \partial F \to \mathbb{R}$ be a function which minimizes $H(\psi)$ subject to the boundary conditions 0 at $\partial F, \psi_F \geq s$ on $\partial_L D_N$, $\psi_F \geq t$ on $\partial_R D_N$. Then ψ_F is unique, and is the harmonic function on $F\backslash\partial D_N$ with boundary condition 0 on ∂F, s on $\partial_L D_N$ and t on $\partial_R D_N$.*

Furthermore, one has

$$\Delta\psi_F(i) = \sum_{j:|i-j|=1}(\psi_F(j) - \psi_F(i)) \leq 0, \quad i \in \partial D_N. \tag{6.4}$$

Remark 6.3 *The condition $s > t/2, t > s/2$ is needed to ensure that piecewise linear function on $[-1,2]$ which is s at 0, t at 1, and 0 at $\{-1,2\}$ is concave. We will later apply* *the lemma with* $s = aN + o(N)$, $t = bN + o(N)$, *so that we should have* $a > b/2$, $b > a/2$ *(and* N *large). If this is not satisfied, we can take instead of* $D_{N,\mathrm{ext}}$ *the smaller extensions* $\{-cN, -cN+1, \dots, N+cN\} \times \mathbb{T}_N^{d-1}$ *with* c *satisfying*

$$\frac{bc}{1+c} < a, \ \frac{ac}{1+c} < b,$$

in which case the corresponding piecewise linear function on $[-c, 1+c]$ *is concave. After this modification, all the arguments below go through. For the sake of notational simplicity, we stay with our choice for* $D_{N,\mathrm{ext}}$ *and the conditions on* s, t.

To prove this lemma, we prepare another lemma, which reduces the variational problem to that on superharmonic functions and gives a comparison for such functions.

Lemma 6.4 (1) *The minimizer* ψ_F *of* $H(\psi)$ *subject to the conditions*

$$\psi_F = 0 \text{ at } \partial F, \quad \psi_F \ge s \text{ on } \partial_L D_N, \quad \psi_F \ge t \text{ on } \partial_R D_N, \tag{6.5}$$

is characterized as the unique solution satisfying this condition and

$$\begin{cases} \Delta\psi_F = 0 & \text{on } F \cup (\partial D_N \backslash I) \\ \Delta\psi_F \le 0 & \text{on } I \end{cases}, \tag{6.6}$$

where $I = I(\psi_F)$ *is a region in* ∂D_N *given by* $I \equiv I_L \cup I_R := \{i \in \partial_L D_N; \psi_F(i) = s\} \cup \{i \in \partial_R D_N; \psi_F(i) = t\}$.
(2) *Assume that* $\psi^{(1)}$ *and* $\psi^{(2)}$ *are two solutions of the problem* (6.6) *satisfying* $\psi^{(1)} \ge \psi^{(2)}$ *on* F^c *instead of* $\psi^{(1)} = \psi^{(2)} = 0$ *on* F^c *in* (6.5). *Then, we have that* $\psi^{(1)} \ge \psi^{(2)}$ *on* F.

Proof. (1) Let ψ_F be the minimizer of $H(\psi)$ subject to the conditions (6.5). Then, ψ_F is harmonic on $F \cup (\partial D_N \setminus I)$, since

$$\begin{aligned} 0 = \frac{d}{da} H(\psi_F + a\delta_i)\big|_{a=0} &= \frac{d}{da} \sum_{j:|j-i|=1} (\psi_F(j) - \psi_F(i) + a)^2\big|_{a=0} \\ &= -2 \sum_{j:|j-i|=1} (\psi_F(j) - \psi_F(i)) = -2(\Delta\psi_F)(i), \end{aligned}$$

for every $i \in F \cup (\partial D_N \setminus I)$, where $\delta_i \in \mathbb{R}^{D_{N,\mathrm{ext}}}$ is defined by $\delta_i(j) = \delta_{ij}$. For $i \in I$, since

$$\frac{d}{da} H(\psi_F + a\delta_i)\big|_{a=0+} \ge 0,$$

we have $\Delta\psi_F \le 0$. Thus the minimizer ψ_F satisfies (6.6).

To show the uniqueness of the solution ψ_F of (6.6), let $\psi^{(1)}$ and $\psi^{(2)}$ be two solutions of the problem (6.6). Then, we have that

$$\left(\psi^{(1)}(i) - \psi^{(2)}(i)\right) \left(\Delta\psi^{(1)}(i) - \Delta\psi^{(2)}(i)\right) \ge 0, \tag{6.7}$$

for all $i \in F$. In fact, denoting $I^{(k)} = I(\psi_F^{(k)}), I_L^{(k)} = I_L(\psi_F^{(k)}), I_R^{(k)} = I_R(\psi_F^{(k)})$ for $k = 1, 2$, if $i \in F \cup (\partial D_N \setminus (I^{(1)} \cup I^{(2)}))$, then $\Delta\psi^{(1)}(i) = \Delta\psi^{(2)}(i) = 0$. If $i \in I_L^{(1)} \setminus I^{(2)}$, then

$\psi^{(1)}(i) - \psi^{(2)}(i) = s - \psi^{(2)}(i) < 0$ and $\Delta\psi^{(1)}(i) - \Delta\psi^{(2)}(i) = \Delta\psi^{(1)}(i) \le 0$. The case $i \in I_L^{(2)} \setminus I^{(1)}$ and the cases with $I_R^{(1)}, I_R^{(2)}$ are similar. If $i \in I^{(1)} \cap I^{(2)}$, then $\psi^{(1)}(i) = \psi^{(2)}(i)$. In all cases, (6.7) holds.

From (6.7), setting $\psi = \psi^{(1)} - \psi^{(2)}$, since $\psi(i) = 0$ on ∂F, we have that

$$0 \le \sum_{i \in F} \psi(i)\Delta\psi(i) = - \sum_{i,j \in \bar{F}:|i-j|=1} (\psi(i) - \psi(j))^2,$$

see (2.19) in [6] for this summation by parts formula. This shows $\psi(i) = \psi(j)$ for all $i, j \in \bar{F} = F \cup \partial F : |i - j| = 1$. Since $\psi(i) = 0$ at ∂F, this proves $\psi = 0$ on F, and therefore the uniqueness.

(2) Set $\psi = \psi^{(1)} - \psi^{(2)}$ and assume that $-m = \min_{i \in F} \psi(i) < 0$. Let $i_0 \in F$ be the point such that $\psi(i_0) = -m$. Then, since $\psi^{(2)}(i_0) = \psi^{(1)}(i_0) + m > \psi^{(1)}(i_0)$, from the first condition in (6.6), we see $\Delta\psi^{(2)}(i_0) = 0$. Thus, $\Delta\psi(i_0) = \Delta\psi^{(1)}(i_0) - \Delta\psi^{(2)}(i_0) = \Delta\psi^{(1)}(i_0) \le 0$. Since we have shown

$$0 \ge \Delta\psi(i_0) = \sum_{j:|i_0-j|=1} (\psi(j) - \psi(i_0))$$

and $\psi(j) - \psi(i_0) \ge 0$, we obtain that $\psi(j) = \psi(i_0) (= -m)$ for all $j : |i_0 - j| = 1$. Continuing this procedure, we see that $\psi \equiv -m < 0$ on the connected component of $F \cup \partial F$ containing i_0, but this contradicts with the boundary condition: $\psi \ge 0$ on F^c. ■

Proof of Lemma 6.2. The harmonic property of ψ_F on F and the property (6.4) are immediate from Lemma 6.4. What are left are to show that $\psi_F = s$ on $\partial_L D_N$, $\psi_F = t$ on $\partial_R D_N$ and to give the explicit form of ψ_F on $D^\circ_{N,\mathrm{ext}} \setminus D_N$ stated in the lemma. Indeed, define $\psi^{(1)}$ by

$$\psi^{(1)}(i) = \begin{cases} \left(\frac{i_1}{N} + 1\right) s & \text{on } \{-N, \dots, 0\} \times \mathbb{T}_N^{d-1} \\ \frac{N-i_1}{N} s + \frac{i_1}{N} t & \text{on } \{1, \dots, N-1\} \times \mathbb{T}_N^{d-1} \\ \left(2 - \frac{i_1}{N}\right) t & \text{on } \{N, \dots, 2N\} \times \mathbb{T}_N^{d-1} \end{cases}.$$

Then, by the concavity condition on the segments mentioned in the lemma, $\psi^{(1)}$ satisfies the condition (6.6) and $\psi^{(1)} \ge \psi^{(2)} := \psi_F$ on F^c. Thus, Lemma 6.4-(2) proves $\psi^{(1)} \ge \psi_F$ on F. This implies that $\psi_F = s$ on $\partial_L D_N$, $\psi_F = t$ on $\partial_R D_N$. Once this is shown, the rest is easy, since ψ_F is harmonic on $D^\circ_{N,\mathrm{ext}} \setminus D_N$. ■

With F still as above, and $\mathbf{x}_L \in \mathbb{R}^{\partial_L D_N}$, $\mathbf{x}_R \in \mathbb{R}^{\partial_R D_N}$, let $\phi_{F,\mathbf{x}_L,\mathbf{x}_R} : F \cap D_N^\circ \to \mathbb{R}$ be the harmonic function with 0 boundary condition on $\partial F \cap D_N^\circ$, $\mathbf{x}_L$ on $\partial_L D_N$, and $\mathbf{x}_R$ on $\partial_R D_N$. We set $\Xi(F, \mathbf{x}_L, \mathbf{x}_R) \stackrel{\text{def}}{=} H(\phi_{F,\mathbf{x}_L,\mathbf{x}_R})$.

Lemma 6.5 *Let $F \in \mathbb{F}$. Then, we have the followings.*
(1) *Let $s, t \ge 0$. Then*

$$\begin{aligned} &\mu_{F,\mathrm{ext}}\left(\phi|_{\partial_L D_N} \ge s,\ \phi|_{\partial_R D_N} \ge t\right) \\ &\le \exp\left[-\Xi(F, s, t) - \tfrac{s^2}{2} N^{d-2} - \tfrac{t^2}{2} N^{d-2}\right]. \end{aligned}$$

(2) *Let $\delta > 0$ and $\mathbf{x}_L,\ \mathbf{x}_R$ satisfy $aN - N^{1-\delta} \leq \mathbf{x}_L \leq aN$, $bN - N^{1-\delta} \leq \mathbf{x}_R \leq bN$. Then*

$$\Xi\left(F, aN, bN\right)\left(1 - \frac{2N^{-\delta}}{\min\left(a, b\right)}\right) \leq \Xi\left(F, \mathbf{x}_L, \mathbf{x}_R\right) \leq \Xi\left(F, aN, bN\right).$$

Proof. (1) We consider ψ_F as in the previous lemmas. With the transformation of variables $\phi_i = \bar{\phi}_i + \psi_F\left(i\right)$, we obtain

$$\begin{aligned}
&\mu_{F,\mathrm{ext}}\left(\phi|_{\partial_L D_N} \geq s, \phi|_{\partial_R D_N} \geq t\right) \\
&= \left[-\Xi\left(F, s, t\right) - \tfrac{s^2}{2}N^{d-2} - \tfrac{t^2}{2}N^{d-2}\right] \\
&\times \int_{\bar{\phi}_i \geq 0,\ i \in \partial D_N} \exp\left[-2\sum\nolimits_{i \in \partial D_N} \bar{\phi}_i \sum\nolimits_j \left(\psi_F\left(j\right) - \psi_F\left(i\right)\right)\right] \mu_{F,\mathrm{ext}}\left(d\bar{\phi}\right).
\end{aligned}$$

By Lemma 6.2, the integrand is ≤ 1 in the domain of integration, which proves the claim.

(2) It evidently suffices to prove

$$\Xi\left(F, aN - N^{1-\delta}, bN - N^{1-\delta}\right) \geq \Xi\left(F, aN, bN\right)\left(1 - \frac{2N^{-\delta}}{\min\left(a, b\right)}\right).$$

Without loss of generality, we assume $b \geq a$. Then

$$\frac{bN}{bN - N^{1-\delta}} \leq \frac{aN}{aN - N^{1-\delta}}.$$

Let ψ be the harmonic function on F which is 0 on $\partial F \cap D_N^{\circ}$, $aN - N^{1-\delta}$ on $\partial_L D_N$ and $bN - N^{1-\delta}$ on $\partial_R D_N$. Define

$$\psi' \stackrel{\mathrm{def}}{=} \frac{aN}{aN - N^{1-\delta}}\psi$$

which is harmonic on F, 0 on $\partial F \cap D_N^{\circ}$, aN on $\partial_L D_N$ and $\geq bN$ on $\partial_R D_N$. If we define ψ'' to be the harmonic function on F which has boundary conditions aN, bN on $\partial_L D_N$, $\partial_R D_N$, respectively, and 0 on $\partial F \cap D_N^{\circ}$, we get

$$\begin{aligned}
H\left(\psi\right) &= \left(1 - \frac{N^{-\delta}}{a}\right)^2 H\left(\psi'\right) \geq \left(1 - \frac{N^{-\delta}}{a}\right)^2 H\left(\psi''\right) \\
&= \left(1 - \frac{N^{-\delta}}{a}\right)^2 \Xi\left(F, aN, bN\right) \geq \left(1 - \frac{2N^{-\delta}}{a}\right) \Xi\left(F, aN, bN\right).
\end{aligned}$$

■

6.2 Superexponential estimate

Given $0 < \beta < 1$, we consider the following coarse graining: We divide D_N into $N^{d(1-\beta)}$ subboxes of sidelength N^{β}. For the sake of simplicity, we assume that N^{β} divides N as before. We write $\mathcal{B}_N \equiv \mathcal{B}_{N,\beta}$ for the set of these subboxes, and $\hat{\mathcal{B}}_N \equiv \hat{\mathcal{B}}_{N,\beta}$ for the set of unions of boxes in $\mathcal{B}_N$. We attach to every subbox $C \in \mathcal{B}_N$ the arithmetic mean

$$\phi_C^{\mathrm{cg},\beta,N} \stackrel{\mathrm{def}}{=} N^{-d\beta} \sum_{j \in C} \phi_j.$$

Then define

$$\phi^{\mathrm{cg},\beta,N}(i) = \phi_C^{\mathrm{cg},\beta,N},\ i \in C,$$
$$h^{\mathrm{cg},\beta,N}(x) = \frac{1}{N}\phi^{\mathrm{cg},\beta,N}([xN]),\ x \in D = [0,1]\times \mathbb{T}^{d-1}.$$

Proposition 6.6 *For every $\eta > 0$ satisfying $2\eta+\beta < 1$ and for large enough N (as stated at the beginning of Section 6.1),*

$$\mu_N^{aN,bN,\varepsilon}\left(\left\|h^{\mathrm{cg},\beta,N} - h^N\right\|_{L^1(D)} \geq N^{-\eta}\right) \leq C\exp\left[-\frac{1}{C}N^{d+1-2\eta-\beta}\right].$$

Proof. We first consider the $\mu_{N,\mathrm{ext}}^{\varepsilon}$ which is defined as the free field with 0 boundary conditions (and no boundary conditions on ∂D_N). We use the extension as explained in Section 6.1. Expanding the product in the usual way, we get

$$\mu_{N,\mathrm{ext}}^{\varepsilon} = \sum_{A\in\mathbb{F}} \frac{Z_A}{Z_{N,\mathrm{ext}}^{\varepsilon}}\varepsilon^{|A^c|}\mu_A, \tag{6.8}$$

where $A^c \stackrel{\mathrm{def}}{=} D_{N,\mathrm{ext}}^{\circ}\backslash A$, and μ_A is the centered Gaussian field on $D_{N,\mathrm{ext}}^{\circ}$ with zero boundary conditions outside on ∂A. The covariance function of μ_A is denoted by G_A. It is convenient to extend $G_A(i,j)$ to i or $j \notin A$ by putting it 0. It is the Green's function for a random walk on A with Dirichlet boundary condition.

We can define $h^N, h^{\mathrm{cg},\beta,N}$ in the same way as before, but on the extended space. The coarse graining is done here on the full $D_{N,\mathrm{ext}}$. We first prove that

$$\mu_{N,\mathrm{ext}}^{\varepsilon}\left(\left\|h^N - h^{\mathrm{cg},\beta,N}\right\|_{L^1(D)} \geq N^{-\eta}\right) \leq C\exp\left[-\frac{1}{C}N^{d+1-2\eta-\beta}\right] \tag{6.9}$$

provided $2\eta+\beta<1$.

Using the expansion (6.8), it suffices to prove the inequality for μ_A, uniformly in A. So we have to estimate

$$\mu_A\left(\sum\nolimits_{i\in D_N^{\circ}}\left|N^{-d\beta}\sum\nolimits_{j\in C_i}(\phi_j-\phi_i)\right| \geq N^{1+d-\eta}\right)$$

where $C_i \in \mathcal{B}_{N,\beta,\mathrm{ext}}$ denotes the box in which i lies. The sum over the extended region $D_{N,\mathrm{ext}}^{\circ}$ of the absolute values is

$$\sup_{\sigma}\sum_{i\in D_{N,\mathrm{ext}}^{\circ}}\sigma_i\left(N^{-d\beta}\sum\nolimits_{j\in C_i}(\phi_j-\phi_i)\right),$$

where $\sigma = (\sigma_i) \in \{-1,1\}^{D_{N,\mathrm{ext}}^{\circ}}$. Therefore, with

$$X(\sigma) \stackrel{\mathrm{def}}{=} \sum\nolimits_{i\in D_{N,\mathrm{ext}}^{\circ}}\sigma_i\left(N^{-d\beta}\sum\nolimits_{j\in C_i}(\phi_j-\phi_i)\right),$$

we have

$$\begin{aligned}
&\mu_A\left(\sum\nolimits_{i\in D^\circ_{N,\mathrm{ext}}}\left|N^{-d\beta}\sum\nolimits_{j\in C_i}\phi_j-\phi_i\right|\geq N^{1+d-\eta}\right)\\
&\leq 2^{\left|D^\circ_{N,\mathrm{ext}}\right|}\sup_\sigma \mu_A\left(X(\sigma)\geq N^{1+d-\eta}\right),
\end{aligned}$$

where $\mu_A=\mu_{A,\mathrm{ext}}$. The $X(\sigma)$ are centered Gaussian variables, so we just have to estimate the variances, uniformly in σ and A.

$$\begin{aligned}
\mathrm{var}_{\mu_A}(X(\sigma)) &\leq \sum_{i,k\in D^\circ_{N,\mathrm{ext}}}\left|E_{\mu_A}\left(N^{-2d\beta}\sum\nolimits_{j'\in C_i}(\phi_{j'}-\phi_i)\sum\nolimits_{j\in C_k}(\phi_j-\phi_k)\right)\right|\\
&\leq 2\sum_{i,k\in D^\circ_{N,\mathrm{ext}}}\left|E_{\mu_A}\left(N^{-d\beta}\phi_i\sum\nolimits_{j\in C_k}(\phi_j-\phi_k)\right)\right|\\
&\leq 2\sum_{i\in D^\circ_{N,\mathrm{ext}}}N^{-d\beta}\sum_{k\in D^\circ_{N,\mathrm{ext}}}\sum_{j\in C_k}\left|G_A(i,j)-G_A(i,k)\right|\\
&\leq 2\sum_{i\in D^\circ_{N,\mathrm{ext}}}N^{-d\beta}\sum_{k\in D^\circ_{N,\mathrm{ext}}}\sum_{j:d(j,k)\leq\rho(d,\beta)}\left|G_A(i,j)-G_A(i,k)\right|,
\end{aligned}$$

where G_A is the Green's function of ordinary random walk with killing at exiting A or reaching $\partial D_{N,\mathrm{ext}}$. $d(j,k)$ is any reasonable distance on the discrete torus, for instance the length of the shortest path from j to k. $\rho(d,\beta)$ is the diameter of the boxes in $\mathcal{B}_{N,\beta}$. If we define $K(d,\beta)$ to be the ball of radius $\rho(d,\beta)$ around $0\in D_{N,\mathrm{ext}}$, we can also write the above expression as

$$2\sum_{j\in K}N^{-d\beta}\sum_{i\in D^\circ_{N,\mathrm{ext}}}\sum_{k\in D^\circ_{N,\mathrm{ext}}}\left|G_A(i,k+j)-G_A(i,k)\right|.$$

For $i\in A$, let $\pi_A(i,\cdot)$ be the first exit distribution from A of a random walk starting in i. It is well known that

$$G_A(i,k)=G_{N,\mathrm{ext}}(i,k)-\sum_s\pi_A(i,s)\,G_{N,\mathrm{ext}}(s,k)$$

where $G_{N,\mathrm{ext}}$ is the the Green's function on $D_{N,\mathrm{ext}}$ with Dirichlet boundary condition on $\partial D_{N,\mathrm{ext}}$. Therefore

$$\begin{aligned}
\left|G_A(i,k+j)-G_A(i,k)\right| &\leq \left|G_{N,\mathrm{ext}}(i,k+j)-G_{N,\mathrm{ext}}(i,k)\right|\\
&\quad+\sum_s\pi_A(i,s)\left|G_{N,\mathrm{ext}}(s,k+j)-G_{N,\mathrm{ext}}(s,k)\right|.
\end{aligned}$$

Let

$$\mu(j)\stackrel{\mathrm{def}}{=}\sup_{i\in A}\sum_{k\in D_{N,\mathrm{ext}}}\left|G_{N,\mathrm{ext}}(i,k+j)-G_{N,\mathrm{ext}}(i,k)\right|.$$

Then we obtain

$$\begin{aligned}
\sum_{i\in D^\circ_{N,\mathrm{ext}}}\sum_{k\in D^\circ_{N,\mathrm{ext}}}\left|G_A(i,k+j)-G_A(i,k)\right| &\leq \mu(j)\,|A|+\mu(j)\sum_{i\in A}\sum_s\pi_A(i,s)\\
&=2\mu(j)\,|A|.
\end{aligned}$$

We prove further down that

$$\mu(j) \leq Cd(j,0)\, N \tag{6.10}$$

From that, we obtain

$$\operatorname{var}_{\mu_A}(X(\sigma)) \leq CN^{1+\beta}|A| \leq CN^{1+d+\beta},$$

and therefore

$$\begin{aligned}&\mu_A\left(\sum\nolimits_{i\in D^\circ_{N,\text{ext}}}\left|N^{-d\beta}\sum\nolimits_{j\in C_i}\phi_j-\phi_i\right|\geq N^{1+d-\eta}\right)\\ &\leq 2^{3N^d}\exp\left[-N^{2+2d-2\eta}N^{-1-d-\beta}\right]\leq\exp\left[-\frac{1}{C}N^{1+d-2\eta-\beta}\right]\end{aligned}$$

provided $2\eta+\beta<1$, and N is large enough. This proves (6.9), but we still have to prove (6.10).

For a fixed $j\in K(d,\beta)$ we can find a nearest neighbor path of length $d(j,0)$ connecting 0 with j. In order to prove (6.10), we therefore only have to prove that for any e with $|e|=1$, we have

$$\sum_k|G_N(0,k)-G_N(0,k+e)|=O(N).$$

This was shown in Lemma 2.5.

Next, we discuss how to transfer the result to the one we are interested in, namely the corresponding approximation result on D_N with boundary conditions aN and bN, respectively. For $a,b>0$ consider the event

$$\begin{aligned}\Lambda_{N,a,b}\stackrel{\text{def}}{=}\Big\{\phi:\phi_i\in\left[aN,aN+N^{-2d}\right],\ i\in\partial_L D_N,\\ \phi_i\in\left[bN,bN+N^{-2d}\right],\ i\in\partial_R D_N\Big\}.\end{aligned} \tag{6.11}$$

Applying Lemma 6.5 with $F=D^\circ_{N,\text{ext}},\ s=aN,\ t=bN$, we get

$$\mu_{N,\text{ext}}(\Lambda_{N,a,b})=\exp\left[-N^d\frac{a^2+(b-a)^2+b^2}{2}+O\left(N^{d-1}\right)\right]\mu_{N,\text{ext}}(\Lambda_{N,0,0}). \tag{6.12}$$

Furthermore

$$\mu_{N,\text{ext}}(\Lambda_{N,0,0})\geq\left(CN^{-2d}\right)^{2N^{d-1}}. \tag{6.13}$$

To prove this, we enumerate the points in ∂D_N as $k_1,\ldots,k_{2N^{d-1}}$, and prove

$$\mu_{N,\text{ext}}\left(\phi_{k_1}\in\left[0,N^{-2d}\right]\right)\geq CN^{-2d}, \tag{6.14}$$

$$\mu_{N,\text{ext}}\left(\phi_{k_{j+1}}\in\left[0,N^{-2d}\right]\Big|\,\phi_{k_i}=x_i,\ \forall i\leq j\right)\geq CN^{-2d}, \tag{6.15}$$

uniformly in $x_i \in \left[0, N^{-2d}\right]$, and $j \leq 2N^{d-1}$. (6.14) follows from the fact that ϕ_{k_1} is centered under $\mu_{N,\text{ext}}$ and $\operatorname{var}(\phi_{k_1})$ is bounded and bounded away from 0, uniformly in N, as we assume $d \geq 3$. Under the conditional distribution $\mu_{N,\text{ext}}\left(\cdot \mid \phi_{k_i} = x_i,\ \forall i \leq j\right)$, $\phi_{k_{j+1}}$ is not centered, but has an expectation in $\left[0, N^{-2d}\right]$. Furthermore, the conditional variance is bounded and bounded away from 0, uniformly in N, the choice of the enumeration, and j. So (6.15) follows, too. This implies (6.13).

From that, we get

$$\mu^{\varepsilon}_{N,\text{ext}}\left(\Lambda_{N,a,b}\right) \geq \frac{Z_N}{Z_{N,\text{ext}}} \mu_{N,\text{ext}}\left(\Lambda_{N,a,b}\right) \geq \exp\left[-CN^d\right]. \tag{6.16}$$

Some more notations: If $\mathbf{x} = (x_i)_{i \in \partial_L D_N}$, $\mathbf{y} = (y_i)_{i \in \partial_R D_N}$, we write $\mu_N^{\mathbf{x},\mathbf{y},\varepsilon}$ for the field on D_N with boundary conditions $\mathbf{x}$ and $\mathbf{y}$ on ∂D_N, and ε-pinning. If we have an event Q which depends on the field variables only inside D_N°, then

$$\mu^{\varepsilon}_{N,\text{ext}}\left(Q \mid \phi_L = \mathbf{x}, \phi_R = \mathbf{y}\right) = \mu_N^{\mathbf{x},\mathbf{y},\varepsilon}(Q),$$

where $\phi_L = \{\phi_i\}_{i \in \partial_L D_N}$, and ϕ_R similarly. This follows from the Markov property and the fact that the pinning is only inside D_N°.

If ϕ is an element in $\mathbb{R}^{D_N^\circ}$, we write $\phi \vee \{\mathbf{x}, \mathbf{y}\}$ for the configuration which is extended by $\mathbf{x}$ on $\partial_L D_N$, and $\mathbf{y}$ on $\partial_R D_N$. We set

$$U_{N,a,b} \overset{\text{def}}{=} \left\{(\mathbf{x}, \mathbf{y}) : x_i \in \left[aN, aN + N^{-2d}\right],\ y_i \in \left[bN, bN + N^{-2d}\right],\right\}$$

If ϕ is a configuration which satisfies $|\phi_i| \leq N^d$ for all $i \in D_N^\circ$, and $(\mathbf{x}, \mathbf{y}) \in U_{N,a,b}$, then

$$H_N\left(\phi \vee \{\mathbf{x}, \mathbf{y}\}\right) = H_N\left(\phi \vee \{aN, bN\}\right) + O\left(N^{d-1} N^{-d}\right).$$

Therefore, it follows that for any $Q \subset \left\{\phi : |\phi_i| \leq N^d,\ \forall i \in D_N^\circ\right\}$, one has

$$\mu_N^{\varepsilon,\mathbf{x},\mathbf{y}}(Q) = \mu_N^{\varepsilon,aN,bN}(Q)\left(1 + O\left(N^{-1}\right)\right).$$

We therefore have

$$\begin{aligned}
&\mu_N^{aN,bN,\varepsilon}(Q)\, \mu^{\varepsilon}_{N,\text{ext}}\left(\Lambda_{N,a,b}\right) \\
&= \int_{U_{N,a,b}} \mu_N^{\mathbf{x},\mathbf{y},\varepsilon}(Q)\, \mu^{\varepsilon}_{N,\text{ext}}\left(\phi_L \in d\mathbf{x}, \phi_R \in d\mathbf{y}\right)\left(1 + O\left(N^{-1}\right)\right) \\
&= \int_{U_{N,a,b}} \mu^{\varepsilon}_{N,\text{ext}}\left(Q \mid \phi_L = \mathbf{x}, \phi_R = \mathbf{y}\right) \mu^{\varepsilon}_{N,\text{ext}}\left(\phi_L \in d\mathbf{x}, \phi_R \in d\mathbf{y}\right)(1 + O(N^{-1})) \\
&= \mu^{\varepsilon}_{N,\text{ext}}\left(Q \cap \Lambda_{N,a,b}\right)(1 + O(N^{-1})) \leq \mu^{\varepsilon}_{N,\text{ext}}(Q)\,(1 + O(N^{-1})),
\end{aligned} \tag{6.17}$$

i.e., with (6.16)

$$\mu_N^{aN,bN,\varepsilon}(Q) \leq \mu^{\varepsilon}_{N,\text{ext}}(Q) \exp\left[CN^d\right]. \tag{6.18}$$

We apply this to

$$Q \overset{\text{def}}{=} \left\{\left\|h^{\text{cg},\beta,N} - h^N\right\|_{L^1(D)} \geq N^{-\eta}\right\} \cap \left\{|\phi_i| \leq N^d,\ \forall i \in D_N\right\}.$$

Evidently, the restriction to $|\phi_i| \le N^d$ is harmless, as

$$\mu_N^{aN,bN,\varepsilon}\left(|\phi_i| > N^d, \text{ some } i\right) \le CN^d \exp\left[-\frac{1}{C}N^{2d}\right], \tag{6.19}$$

and therefore, from (6.9) and (6.18),

$$\begin{aligned}\mu_N^{aN,bN,\varepsilon}\left(\left\|h^{\mathrm{cg},\beta,N} - h^N\right\|_{L^1(D)} \ge N^{-\eta}\right) &\le C\exp\left[-\frac{1}{C}N^{d+1-2\eta-\beta} + CN^d\right] + CN^d\exp\left[-\frac{1}{C}N^{2d}\right] \\ &\le C\exp\left[-\frac{1}{C}N^{d+1-2\eta-\beta}\right],\end{aligned}$$

for large enough N, provided $0 < 2\eta + \beta < 1$. This proves Proposition 6.6. ∎

One simple consequence of this proposition is the following lemma; recall (1.5) for h_{PL}^N.

Lemma 6.7 *For every $\eta > 0$, we have that*

$$\mu_N^{aN,bN,\varepsilon}\left(\|h^N - h_{\mathrm{PL}}^N\|_{L^1(D)} \ge N^{-\eta}\right) \le \exp\{-CN^{d+1-2\eta}\}.$$

Proof. First, noting that $\sum_{v\in\{0,1\}^d}\left[\prod_{\alpha=1}^d\left(v_\alpha\{Nt_\alpha\} + (1-v_\alpha)(1-\{Nt_\alpha\})\right)\right] = 1$, we see that

$$\begin{aligned}\|h^N - h_{\mathrm{PL}}^N\|_{L^1(D)} &\le \frac{1}{N^{d+1}}\sum_{i\in D_N}\sum_{v\in\{0,1\}^d}|\phi(i) - \phi(i+v)| \\ &\le \frac{C_d}{N^{d+1}}\sum_{i,j\in D_N:|i-j|=1}|\phi(i) - \phi(j)|.\end{aligned}$$

Therefore, from (6.18) in the proof of Proposition 6.6 and the expansion (6.8), it suffices to prove

$$\mu_{A,\mathrm{ext}}\left(\sum_{i,j\in D_N:|i-j|=1}|\phi(i) - \phi(j)| \ge N^{d+1-\eta}\right) \le \exp\{-CN^{d+1-2\eta}\},$$

uniformly in $A \subset D_{N,\mathrm{ext}}^\circ$. As we discussed in the proof of Proposition 6.6, setting

$$X(\sigma) = \sum_{i,j\in D_N:|i-j|=1}\sigma_{ij}(\phi(i) - \phi(j))$$

for $\sigma = (\sigma_{ij}) \in \{-1,1\}^{\mathbb{B}_N}$, $\mathbb{B}_N = \{(i,j); i,j \in D_N, |i-j| = 1\}$, it suffices to show that

$$\mu_{A,\mathrm{ext}}\left(X(\sigma) \ge N^{d+1-\eta}\right) \le \exp\{-CN^{d+1-2\eta}\}, \tag{6.20}$$

uniformly in A and σ. However, $X(\sigma)$ are centered Gaussian variables and

$$\begin{aligned}
\mathrm{var}_{\mu_{A,\mathrm{ext}}}(X(\sigma)) &= \sum_{\substack{i,j\in D_N:|i-j|=1\\ i',j'\in D_N:|i'-j'|=1}} \sigma_{ij}\sigma_{i'j'}\big(G_A(i,i')-G_A(i,j')-G_A(j,i')+G_A(j,j')\big)\\
&\le C_1 \sum_{i,j\in D_N,|e|=1} |G_A(i,j)-G_A(i,j+e)|\\
&\le C_2 \sum_{i,j\in D_N,|e|=1} |G_{N,\mathrm{ext}}(i,j)-G_{N,\mathrm{ext}}(i,j+e)|\\
&\le C_3 N^{d+1},
\end{aligned}$$

by the estimate shown in the proof of Proposition 6.6. This combined with the Gaussian property of $X(\sigma)$ immediately implies (6.20). ■

We draw some other easy consequences from the coarse graining estimate: Given $\gamma > 0$ we define the **mesoscopic wetted region** by

$$\mathcal{M}_N \equiv \mathcal{M}_N(\phi) \stackrel{\mathrm{def}}{=} \bigcup \left\{ C \in \mathcal{B}_N : \phi_C^{\mathrm{cg},\beta,N} \ge N^\gamma \right\}.$$

We write

$$\begin{aligned}
\mu_N^{aN,bN,\varepsilon}\left(A_{N,\alpha}\cap\Omega_N^+\right) &= \sum_{B\in\hat{\mathcal{B}}} \mu_N^{aN,bN,\varepsilon}\left(A_{N,\alpha}\cap\Omega_N^+\cap\{\mathcal{M}_N=B\}\right)\\
&\le \left|\hat{\mathcal{B}}\right| \max_{B\in\hat{\mathcal{B}}} \mu_N^{aN,bN,\varepsilon}\left(A_{N,\alpha}\cap\Omega_N^+\cap\{\mathcal{M}_N=B\}\right)\\
&= \exp\left[N^{d(1-\beta)}\log 2\right] \max_{B\in\hat{\mathcal{B}}} \mu_N^{aN,bN,\varepsilon}\left(A_{N,\alpha}\cap\Omega_N^+\cap\{\mathcal{M}_N=B\}\right).
\end{aligned}$$

In order to prove (6.3), it therefore suffices to prove that there exists $\delta_1 < d\beta$ and $\alpha > 0$ such that

$$\max_{B\in\hat{\mathcal{B}}} \mu_N^{aN,bN,\varepsilon}\left(A_{N,\alpha}\cap\Omega_N^+\cap\{\mathcal{M}_N=B\}\right) \le \mathrm{e}^{-N^{d-\delta_1}}, \tag{6.21}$$

N large, uniformly in B.

Let $\partial^* B \stackrel{\mathrm{def}}{=} \partial B \cap D_N^\circ$. Any point $i \in \partial^*\mathcal{M}_N$ is in block C with $\phi_C \le N^\gamma$. If also $\phi \in \Omega_N^+$, we conclude that

$$\phi(i) \le N^{d\beta+\gamma}\log N.$$

We will choose γ, β such that $d\beta + \gamma < 1$, and then choose

$$\kappa_1 \stackrel{\mathrm{def}}{=} \frac{1-d\beta-\gamma}{2}, \tag{6.22}$$

so that if $i \in \partial^*\mathcal{M}_N$ we have

$$\phi(i) \le N^{1-\kappa_1}. \tag{6.23}$$

Lemma 6.8 (Volume filling lemma) *Assume $\gamma + \eta > 1$, and $2\eta + \beta < 1$. Then*

$$\mu_N^{aN,bN,\varepsilon}\left(|\mathcal{M}_N\cap\{i:\phi(i)=0\}| \ge N^{d+1-\gamma-\eta}\right) \le C\exp\left[-\frac{1}{C}N^{d+1-2\eta-\beta}\right].$$

Proof. Remark that

$$\sum_i \left|\phi(i) - \phi^{\mathrm{cg},\beta,N}(i)\right|$$
$$\geq \sum_{i\in\mathcal{M}_N\cap\{i:\phi(i)=0\}} \left|\phi(i) - \phi^{\mathrm{cg},\beta,N}(i)\right| \geq \left|\mathcal{M}_N\cap\{i:\phi(i)=0\}\right| N^{\gamma}.$$

Therefore, from Proposition 6.6 we get

$$\mu_N^{aN,bN,\varepsilon}\left(\left|\mathcal{M}_N\cap\{i:\phi(i)=0\}\right| \geq N^{d+1-\gamma-\eta}\right)$$
$$\leq \mu_N^{aN,bN,\varepsilon}\left(N^{-d-1}\sum_i \left|\phi(i) - \phi^{\mathrm{cg},\beta,N}(i)\right| \geq N^{-\eta}\right)$$
$$\leq C\exp\left[-\frac{1}{C}N^{d+1-2\eta-\beta}\right]$$

which proves the claim. ■

The different requirements on $\beta,\eta,\gamma>0$ are

$$2\eta+\beta<1,$$
$$d\beta+\gamma<1,$$
$$\eta+\gamma>1.$$

We can fulfill them by taking for instance

$$\beta=\frac{1}{10d},\ \gamma=\frac{4}{5},\ \eta=\frac{1}{4}.$$

From now on, we keep these constants fixed under the above restrictions, for instance with the above values. We put

$$\kappa_2 \overset{\mathrm{def}}{=} \gamma+\eta-1,\ \kappa_3 \overset{\mathrm{def}}{=} \frac{1-(2\eta+\beta)}{2},$$

so that, by the volume filling lemma, we have

$$\mu_N^{\varepsilon}\left(\left|\mathcal{M}_N\cap\{i:\phi(i)=0\}\right| \geq N^{d-\kappa_2}\right) \leq \exp\left[-N^{d+\kappa_3}\right]. \tag{6.24}$$

6.3 Proof of (6.21)

If $A\subset D_N^{\circ}$, we write $A_{\mathrm{ext}} \overset{\mathrm{def}}{=} A\cup(D_{N,\mathrm{ext}}\backslash D_N^{\circ})$. Using Lemma 2.6 (patching at ∂D_N), we have

$$Z_{A_{\mathrm{ext}}} = Z_A Z_{D_{N,\mathrm{ext}}\backslash D_N^{\circ}} \exp\left[O\left(N^{d-1}\right)\right],$$

and using Lemma 2.3, one has

$$Z_{D_{N,\mathrm{ext}}\backslash D_N^{\circ}} = \exp\left[2\hat{q}^0 N^d + O\left(N^{d-1}\right)\right].$$

Note that these partition functions are defined without pinning. Therefore

$$
\begin{aligned}
Z^{\varepsilon}_{N,\text{ext}} &:= \sum_{A \subset D^{\circ}_N} \varepsilon^{\left|D^{\circ}_N \setminus A\right|} Z_{A_{\text{ext}}} \\
&= \exp\left[2N^d \hat{q}^0 + O\left(N^{d-1}\right)\right] \sum_{A \subset D^{\circ}_N} \varepsilon^{\left|D^{\circ}_N \setminus A\right|} Z_A \\
&= \exp\left[2N^d \hat{q}^0 + N^d \hat{q}^{\varepsilon} + O\left(N^{d-1}\right)\right],
\end{aligned}
$$

where we have used a version of (2.3.4) of [4]. Therefore,

$$
\mu^{\varepsilon}_{N,\text{ext}} = \exp\left[-N^d \hat{q}^{\varepsilon} - 2N^d \hat{q}^0 + O\left(N^{d-1}\right)\right] \sum_{A \subset D^{\circ}_N} \varepsilon^{\left|D^{\circ}_N \setminus A\right|} Z_{A_{\text{ext}}} \mu_{A,\text{ext}}.
$$

However, we can estimate

$$
\begin{aligned}
&\sum_{A \subset D^{\circ}_N} \varepsilon^{\left|D^{\circ}_N \setminus A\right|} Z_{A,\text{ext}} \mu_{A,\text{ext}}\left(\Lambda_{N,a,b}\right) \geq Z_{D^{\circ}_{N,\text{ext}}} \mu_{N,\text{ext}}\left(\Lambda_{N,a,b}\right) \\
&= Z_{D^{\circ}_{N,\text{ext}}} \exp\left[-\frac{N^d}{2}\left(a^2 + b^2 + (b-a)^2\right) + O\left(N^{d-1}\log N\right)\right]
\end{aligned}
$$

by (6.12) and (6.13). Using

$$
Z_{D^{\circ}_{N,\text{ext}}} = \exp\left[3N^d \hat{q}^0 + O\left(N^{d-1}\right)\right],
$$

and recalling $\xi^{\varepsilon} = \hat{q}^{\varepsilon} - \hat{q}^0$ as in Remark 2.8, we obtain

$$
\mu^{\varepsilon}_{N,\text{ext}}\left(\Lambda_{N,a,b}\right) \geq \exp\left[-N^d\left\{\frac{a^2+b^2+(b-a)^2}{2} + \xi^{\varepsilon}\right\} + O\left(N^{d-1}\log N\right)\right]. \tag{6.25}
$$

We use now $\mu^{\varepsilon,\mathbf{x},\mathbf{y}}_N$ as defined after (6.16). Arguing in the same way as in (6.17), we obtain with the abbreviation $\mathcal{B}_{N,\alpha} \stackrel{\text{def}}{=} \{\mathcal{M}_N = B\} \cap \Omega^+_N \cap A_{N,\alpha}$,

$$
\begin{aligned}
&\mu^{\varepsilon,aN,bN}_N\left(\mathcal{B}_{N,\alpha}\right) \mu^{\varepsilon}_{N,\text{ext}}\left(\Lambda_{N,a,b}\right) \\
&= \mu^{\varepsilon}_{N,\text{ext}}\Big(\mathcal{B}_{N,\alpha} \cap \{\phi|_{\partial D_N} \in U_{N,a,b}\}\Big)\left(1 + O\left(N^{-1}\right)\right).
\end{aligned}
$$

Combining this with (6.25) gives

$$
\begin{aligned}
\mu^{\varepsilon,aN,bN}_N\left(\mathcal{B}_{N,\alpha}\right) &\leq \mu^{\varepsilon}_{N,\text{ext}}\left(\mathcal{B}_{N,\alpha} \cap \{\phi|_{\partial D_N} \in U_{N,a,b}\}\right) \\
&\quad \times \exp\left[N^d\left\{\frac{a^2+b^2+(b-a)^2}{2} + \xi^{\varepsilon}\right\} + O\left(N^{d-1}\log N\right)\right].
\end{aligned} \tag{6.26}
$$

For the expression on the right hand side, we use the usual splitting

$$
\mu^{\varepsilon}_{N,\text{ext}}(\cdot) = \sum_{A \subset D^{\circ}_{N,\text{ext}},\ A^c \subset D^{\circ}_N} \frac{\varepsilon^{\left|D^{\circ}_N \setminus A\right|} Z_{A_{\text{ext}}}}{Z^{\varepsilon}_{N,\text{ext}}} \mu_{A,\text{ext}}(\cdot).
$$

From (6.24), we know that we can restrict the summation to A with $|B \setminus A| \leq N^{d-\kappa_2}$, up to a contribution of order $\exp\left[-N^{d+\kappa_3}\right]$, which we can neglect. Splitting A into $A_1 \cup A_2$ with $A_2 \overset{\text{def}}{=} A \cap B$, and using (2.3.4) of [4] and Lemma 2.3,

$$\begin{aligned} Z_{A_1 \cup A_2,\text{ext}} &\leq Z_{A_2,\text{ext}} Z_{A_1} \exp\left[O\left(N^{d-\beta}\right)\right] \\ &\leq Z_{B,\text{ext}} Z_{A_1} \exp\left[O\left(N^{d-\beta}\right)\right] \\ &\leq Z_{A_1} \exp\left[(2N^d + |B|)\hat{q}^0 + O\left(N^{d-\beta}\right)\right], \end{aligned}$$

it suffices to estimate

$$J_N(B, A_2) = \sum_{A_1 : A_1 \cap B = \emptyset} \varepsilon^{|B^c \cap A_1^c|} Z_{A_1} \mu_{A_1 \cup A_2,\text{ext}} \left(\mathcal{B}_{N,\alpha} \cap \{\phi|_{\partial D_N} \in U_{N,a,b}\}\right)$$

uniformly in B, A_2. If we prove that for all $\delta > 0$ sufficiently small, there exists $\alpha < 1$ such that for all mesoscopic B and all $A_2 \subset B$ with $|B \backslash A_2| \leq N^{d-\kappa_2}$ we have

$$J_N(B, A_2) \exp\left[N^d \frac{a^2 + b^2 + (b-a)^2}{2} - |B^c|\, \hat{q}^0\right] \leq \exp\left[-N^{d-\delta}\right] \tag{6.27}$$

for large enough N (uniformly in B, A_2), we have proved (6.21).

Note that

$$\mathcal{B}_{N,\alpha} \subset \left\{-\log N \leq \phi|_{\partial^* B} \leq N^{d-\kappa_1}\right\} \cap \{\mathcal{M}_N = B\} \cap A_{N,\alpha}.$$

On $\partial^* B \cap (A_1 \cup A_2)^c$, ϕ is of course 0 under $\mu_{A_1 \cup A_2,\text{ext}}$. We define $\hat{\mu}_{B,A_1,A_2,\mathbf{x}}$ to be the free field on $\mathbb{R}^{A_2 \cup \left(D_{N,\text{ext}} \backslash D_N^\circ\right)}$ with boundary condition 0 on $\partial D_{N,\text{ext}} \cap (A_1 \cup A_2)^c$ and boundary condition $\mathbf{x}$ on $\partial^* B \cap (A_1 \cup A_2)$. Then

$$\begin{aligned} &\mu_{A_1 \cup A_2,\text{ext}} \left(\mathcal{B}_{N,\alpha} \cap \{\phi|_{\partial D_N} \in U_{N,a,b}\} \cap A_{N,\alpha}\right) \\ &\leq \mu_{A_1 \cup A_2,\text{ext}} \left(\left\{-\log N \leq \phi|_{\partial^* B} \leq N^{d-\kappa_1}\right\}, \mathcal{M}_N = B, \phi|_{\partial D_N} \in U_{N,a,b}, A_{N,\alpha}\right) \\ &\leq \int_{-\log N \leq \mathbf{x} \leq N^{1-\kappa_1}} \hat{\mu}_{B,A_1,A_2,\mathbf{x}} \left(\phi|_{\partial D_N} \in U_{N,a,b}, \mathcal{M}_N = B, A_{N,\alpha}\right) \mu_{A_1 \cup A_2,\text{ext}} \left(\phi|_{\partial^* B} \in d\mathbf{x}\right) \\ &\leq \mu_{A_1 \cup A_2,\text{ext}} \left(-\log N \leq \phi|_{\partial^* B} \leq N^{1-\kappa_1}\right) \\ &\times \sup_{\mathbf{x} \leq N^{1-\kappa_1}} \hat{\mu}_{B,A_1,A_2,\mathbf{x}} \left(\phi|_{\partial D_N} \in U_{N,a,b}, \mathcal{M}_N = B, A_{N,\alpha}\right) \\ &\leq \sup_{\mathbf{x} \leq N^{1-\kappa_1}} \hat{\mu}_{B,A_1,A_2,\mathbf{x}} \left(\phi|_{\partial D_N} \in U_{N,a,b}, \mathcal{M}_N = B, A_{N,\alpha}\right). \end{aligned}$$

There is a slightly awkward dependence of the right hand side on A_1: If a point $i \in \partial^* B$ is in $\partial^* A_2$ but not in A_1, then the boundary condition there is 0. However, if it is in A_1, then the boundary condition can be arbitrary $\leq N^{1-\kappa_1}$. If we allow for arbitrary boundary condition $\mathbf{x}$ on $\partial^* A_2$, of course with $\mathbf{x} \leq N^{1-\kappa_1}$ and denote the corresponding measure on $\mathbb{R}^{A_2}$ by $\bar{\mu}_{A_2,\mathbf{x}}$, then

$$\begin{aligned}&\sup_{\mathbf{x}\leq N^{1-\kappa_1}} \hat{\mu}_{B,A_1,A_2,\mathbf{x}}\left(\phi|_{\partial D_N}\in U_{N,a,b}, \mathcal{M}_N=B, A_{N,\alpha}\right)\\ &\leq \sup_{\mathbf{x}\leq N^{1-\kappa_1}} \bar{\mu}_{A_2,\mathbf{x}}\left(\phi|_{\partial D_N}\in U_{N,a,b}, \mathcal{M}_N=B,\ A_{N,\alpha}\right),\end{aligned}$$

and the right hand side has no longer a dependence on A_1. Therefore, we just get

$$\begin{aligned}J_N\left(B,A_2\right)&=\sum_{A_1:A_1\cap B=\emptyset}\varepsilon^{\left|B^c\cap A_1^c\right|}Z_{A_1}\mu_{A_1\cup A_2,\mathrm{ext}}\left(\mathcal{B}_{N,\alpha},\ \phi|_{\partial D_N}\in U_{N,a,b}\right)\\ &\leq\left(\sum_{A_1:A_1\cap B=\emptyset}\varepsilon^{\left|B^c\cap A_1^c\right|}Z_{A_1}\right)\sup_{\mathbf{x}\leq N^{1-\kappa_1}}\bar{\mu}_{A_2,\mathbf{x}}\left(\phi|_{\partial D_N}\in U_{N,a,b},\mathcal{M}_N=B,\ A_{N,\alpha}\right)\\ &=\exp\left[\left|B^c\right|\hat{q}^{\varepsilon}+O\left(N^{d-\beta}\right)\right]\sup_{\mathbf{x}\leq N^{1-\kappa_1}}\bar{\mu}_{A_2,\mathbf{x}}\left(\phi|_{\partial D_N}\in U_{N,a,b},\mathcal{M}_N=B,\ A_{N,\alpha}\right).\end{aligned}$$

Therefore, we are left with estimating the above supremum. We distinguish two cases:
First case:

$$E_{N,0}\left(A_2\right)-\xi^{\varepsilon}\left|B^c\right|\geq N^d\inf_h\Sigma\left(h\right)+N^{d-\chi} \tag{6.28}$$

with $\chi>0$ to be chosen later. In this case, we drop $\mathcal{M}_N=B,\ A_{N,\alpha}$ and obtain

$$\begin{aligned}&\sup_{\mathbf{x}\leq N^{1-\kappa_1}}\bar{\mu}_{A_2,\mathbf{x}}\left(\phi|_{\partial D_N}\in U_{N,a,b},\mathcal{M}_N=B,\ A_{N,\alpha}\right)\\ &\leq\sup_{\mathbf{x}\leq N^{1-\kappa_1}}\bar{\mu}_{A_2,\mathbf{x}}\left(\phi|_{\partial_L D_N}\geq aN,\phi|_{\partial_R D_N}\geq bN\right).\end{aligned}$$

By the FKG inequality, the last expression can be estimated from above by putting all boundary conditions (including at $\partial D_{N,\mathrm{ext}}$) at $N^{1-\kappa_1}$. By shifting the field and the boundary conditions down by $N^{1-\kappa_1}$, we obtain from Lemma 6.5 that the right hand side is

$$\begin{aligned}&\leq\exp\left[-\Xi\left(A_2,aN-N^{1-\kappa_1},bN-N^{1-\kappa_1}\right)-N^d\frac{a^2+b^2}{2}+O\left(N^{d-\kappa_4}\right)\right]\\ &=\exp\left[-\Xi\left(A_2,aN,bN\right)-N^d\frac{a^2+b^2}{2}+O\left(N^{d-\kappa_5}\right)\right]\\ &=\exp\left[-E_{N,0}\left(A_2\right)-N^d\frac{a^2+b^2}{2}+O\left(N^{d-\kappa_5}\right)\right],\end{aligned}$$

with some constant $\kappa_4,\kappa_5>0$, which depend only on the fixed values β,γ,η. Summarizing, we get

$$\begin{aligned}&\exp\left[N^d\frac{a^2+b^2+(b-a)^2}{2}-\left|B^c\right|\hat{q}^0\right]J_N\left(B,A_2\right)\\ &\leq\exp\left[N^d\frac{(b-a)^2}{2}+\left|B^c\right|\xi^{\varepsilon}-E_{N,0}\left(A_2\right)+O\left(N^{d-\min(\beta,\kappa_5)}\right)\right].\end{aligned}$$

Remember now, that we have

$$\frac{(b-a)^2}{2} = \inf_h \Sigma(h).$$

Therefore, from (6.28), if we choose $\chi > 0$ small enough, but smaller than $\min(\beta, \kappa_5)$, we have proved the bound (6.27) in this case. (Here actually, α plays no role). This χ will be fixed from now on.

Second case:

$$E_{N,0}(A_2) - \xi^{\varepsilon}|B^c| \le N^d \inf_h \Sigma(h) + N^{d-\chi}. \tag{6.29}$$

Given $\mathbf{x} \in \mathbb{R}_{\partial^* A_2}$, $-\log N \le \mathbf{x} \le N^{1-\kappa_3}$, $\mathbf{y}_L \in \mathbb{R}^{\partial_L D_N}$, and $\mathbf{y}_R \in \mathbb{R}^{\partial_R D_N}$ with $aN \le \mathbf{y}_L \le aN + N^{-2d}$, $bN \le \mathbf{y}_R \le bN + N^{-2d}$, we write $\phi_{\mathbf{x},\mathbf{y}_L,\mathbf{y}_R}$ for the harmonic function with these boundary conditions. If the boundary conditions are 0 and aN, bN respectively, we write ϕ_{A_2} (or $\bar{\phi}^{A_2}$ in Section 3.2). From the maximum principle, we know that

$$\sup_{i \in A_2} |\phi_{\mathbf{x},\mathbf{y}_L,\mathbf{y}_R}(i) - \phi_{A_2}(i)| \le N^{1-\kappa_3},$$

and therefore

$$\sum_{i \in A_2} |\phi_{\mathbf{x},\mathbf{y}_L,\mathbf{y}_R}(i) - \phi_{A_2}(i)| \le N^{d+1-\kappa_3}.$$

By the stability (rigidity) results obtained in Proposition 3.7, we have that either

$$\sum_i \left| \phi_{A_2}(i) - N\bar{h}\left(\frac{i}{N}\right) \right| \le N^{d+1-\kappa_6}$$

or

$$\sum_i \left| \phi_{A_2}(i) - N\hat{h}\left(\frac{i}{N}\right) \right| \le N^{d+1-\kappa_6},$$

where $\kappa_6 > 0$ depends on χ. Therefore, putting $\kappa_7 \stackrel{\text{def}}{=} \min(\kappa_6, \kappa_3)$, we have, uniformly in $\mathbf{x}, \mathbf{y}_L, \mathbf{y}_R$ satisfying the above conditions that either

$$\sup_{\mathbf{x},\mathbf{y}_L,\mathbf{y}_R} \sum_i \left| \phi_{\mathbf{x},\mathbf{y}_L,\mathbf{y}_R}(i) - N\bar{h}\left(\frac{i}{N}\right) \right| \le N^{d+1-\kappa_7}$$

or

$$\sup_{\mathbf{x},\mathbf{y}_L,\mathbf{y}_R} \sum_i \left| \phi_{\mathbf{x},\mathbf{y}_L,\mathbf{y}_R}(i) - N\hat{h}\left(\frac{i}{N}\right) \right| \le N^{d+1-\kappa_7}.$$

Therefore, if we choose $\alpha > 0$ smaller than κ_7 we have that

$$\operatorname{dist}_{L_1}\left(h_N, \left\{\hat{h}, \bar{h}\right\}\right) \ge N^{-\alpha}$$

implies

$$\sum_i |\phi_{\mathbf{x},\mathbf{y}_L,\mathbf{y}_R}(i) - \phi(i)| \ge \frac{1}{2} N^{1+d-\alpha}$$

for all $\mathbf{x}, \mathbf{y}_L, \mathbf{y}_R$ under the above restrictions. Therefore,

$$\begin{aligned}&\bar{\mu}_{A_2,\mathbf{x}}\left(\phi|_{\partial D_N} \in U_{N,a,b}, \mathcal{M}_N = B,\ A_{N,\alpha}\right)\\ &\leq \bar{\mu}_{A_2,\mathbf{x}}\left(\phi|_{\partial D_N} \in U_{N,a,b}, \sum_i \left|\phi_{\mathbf{x},\mathbf{y}_L,\mathbf{y}_R}(i) - \phi(i)\right| \geq \frac{1}{2}N^{1+d-\alpha}\right).\end{aligned}$$

Applying the Markov property at ∂D_N, we can bound that by

$$\bar{\mu}_{A_2,\mathbf{x}}\left(\phi|_{\partial_L D_N} \geq aN,\ \phi|_{\partial_R D_N} \geq bN\right) \sup_{\mathbf{x},\mathbf{y}_L,\mathbf{y}_R} \widetilde{\mu}_{A_2,\mathbf{x},\mathbf{y}_L,\mathbf{y}_R}\left(\sum_i \left|\phi_{\mathbf{x},\mathbf{y}_L,\mathbf{y}_R}(i) - \phi(i)\right| \geq \frac{1}{2}N^{1+d-\alpha}\right),$$

where $\widetilde{\mu}_{A_2,\mathbf{x},\mathbf{y}_L,\mathbf{y}_R}$ is the free field on $\mathbb{R}^{A_2}$ with boundary conditions $\mathbf{x}, \mathbf{y}_L, \mathbf{y}_R$. Remark that $\phi_{\mathbf{x},\mathbf{y}_L,\mathbf{y}_R}(i)$ is the expectation of $\phi(i)$ under $\widetilde{\mu}_{A_2,\mathbf{x},\mathbf{y}_L,\mathbf{y}_R}$. We write $\tilde{E}$ for the expectation under $\tilde{\mu} := \widetilde{\mu}_{A_2,\mathbf{x},\mathbf{y}_L,\mathbf{y}_R}$. Then,

$$\begin{aligned}m :=& \tilde{E}\left[\sum_i \left|\tilde{E}[\phi(i)] - \phi(i)\right|\right]\\ \leq& \sum_i \sqrt{\operatorname{var}_{\tilde{\mu}}(\phi(i))} = O(N^d),\end{aligned}$$

uniformly in $A_2, \mathbf{x}, \mathbf{y}_L, \mathbf{y}_R$. Therefore, if $\alpha < 1$, by (4.4) of [12]

$$\begin{aligned}\tilde{\mu}\left(\sum_i \left|\tilde{E}[\phi(i)] - \phi(i)\right| \geq \tfrac{1}{2}N^{1+d-\alpha}\right) &\leq \tilde{\mu}\left(\sum_i \left|\tilde{E}[\phi(i)] - \phi(i)\right| \geq m + \tfrac{1}{4}N^{1+d-\alpha}\right)\\ &\leq \exp\left(-\frac{N^{2+2d-2\alpha}}{32\sigma^2}\right),\end{aligned}$$

where

$$\sigma^2 = \sup\left\{\operatorname{var}_{\tilde{\mu}}\left(\sum_i g(i)\phi(i)\right); \sup_i |g(i)| \leq 1\right\}.$$

However, one can estimate

$$\sigma^2 \leq \sum_{i,j\in A_2} G_{A_2}(i,j) \leq CN^{d+2}.$$

Therefore, if $0 < 2\alpha < \delta$, we get

$$\widetilde{\mu}_{A_2,\mathbf{x},\mathbf{y}_L,\mathbf{y}_R}\left(\sum_i \left|\phi_{\mathbf{x},\mathbf{y}_L,\mathbf{y}_R}(i) - \phi(i)\right| \geq \frac{1}{2}N^{1+d-\alpha}\right) \leq \exp\left[-N^{d-\delta}\right],$$

uniformly in $A_2, \mathbf{x}, \mathbf{y}_L, \mathbf{y}_R$. Estimating $\bar{\mu}_{A_2,\mathbf{x}}\left(\phi|_{\partial_L D_N} \geq aN,\ \phi|_{\partial_R D_N} \geq bN\right)$ in the same way as in the first case, we arrive at (6.27) also in this case. .

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