MA231 Vector Analysis

2009, term 1
Stefan Adams

Example Sheet 4: Hints and partial solutions

A1 (b) If $f = u(x,y) + iv(x,y)$ then $u =$ so that $v_y = u_x = 0$ and $v_x = -u_y = 0$. Hence v is a constant. (c) $u_{xx} = (v_y)_x = v_{xy}$ and $u_{yy} = (-v_x)_y = -v_{xy}$.

A2 (a) $2i/3$. (b) 1.

A3 (a) (i) $2\pi i$e. (ii) π. (iii) 0. (b) $\frac{2\pi i e}{(n-1)!}$.

A4 (i) 0, (ii) π, (iii) $-\pi$, (iv) 0, (v) π.

A5 Integrate around the semicircle $\gamma = \gamma_1 \cup \gamma_2$ where $\gamma_1(t) = Re^{it}$ for $t \in [0,\pi]$ and $\gamma_2(t) = t$ for $t \in [-R,R]$. (a) $|z^2 - 2z + 2| \geq |z^2| - |2z| - 2 = R^2 - 2R - 2$ for $z \in \gamma_1$ by the triangle inequality. Use this to show $|f(z)| \leq 1/(R^2 - 2R - 2)$ for $z \in \gamma_1$ and hence, using the estimation lemma, that $\int_{\gamma_1} f \, \mathrm{d}z \to 0$ as $R \to \infty$. $\int_{\gamma_2} f(z) \, \mathrm{d}z \to \int_{-\infty}^{\infty} \frac{\cos(\pi x)}{x^2-2x+2} \, \mathrm{d}x + i\int_{-\infty}^{\infty} \frac{\sin(\pi x)}{x^2-2x+2} \, \mathrm{d}x$. $z^2 - 2z + 2 = (z-(1+i))(z-(i-i))$ so that $f(z)$ is holomorphic on and inside γ except at $z = 1+i$. Hence by Cauchy's integral formula $\int_\gamma = 2\pi i \left.\frac{e^{\pi} iz}{z-(1-i)}\right\|_{z=1+i} = -\pi e^{-\pi}$. Conclude that $\int_{-\infty}^{\infty} \frac{\cos(\pi x)}{x^2-2x+2} \, \mathrm{d}x = -\pi e^{-\pi}$ and $\int_{-\infty}^{\infty} \frac{\sin(\pi x)}{x^2-2x+2} \, \mathrm{d}x = 0$.

B1 (a) (a) f only differentiable at 0 and g only at the points $x+iy$ where $3xy = -1$. Neither are holomorphic anywhere. h is differentiable at the origin and nowhere else. (b) applying the CR equations to $f(x+iy) = u(y) + iv(x)$ gives $v'(x) = -u'(y)$ for all $x, y \in \mathbb{R}$. Thus u' and v' are constant and thus f is of the form $f(z) = ay + b + i(cx + d)$ for some $a, b, c, d \in \mathbb{R}$. Using the condition from the CR equations again gives the claim.

B2 (a) parametrise the three sides of the triangle separately. From 1 to i: $\int_{\gamma_1} f = i$, from i to -1: $\int_{\gamma_2} f = i$, from -1 to 1: $\int_{\gamma_3} f = 0$. Thus the result is $2i$. This can be also derived from example 15.6 in the lecture where it was shown that $\int_\gamma \overline{z} \, \mathrm{d}z = 2i$(**area enclosed**). (b) Imitate the proof for $\int_{\partial B(0,\epsilon)} f(z)/z = 2\pi i f(0)$ from the lecture. (c) see part (a) and g is diff. on the circline around the origin with radius 1. (d) see lecture notes.

B3 (a) the integrals are zero ... (i) by the Cauchy integral formula, (ii) by Cauchy's theorem, and (iii) by the fundamental theorem of calculus ($-(z-2)^{-2}/2$ is a primitive). (b) (i) $2\pi i(i^3) = -2\pi + i6\pi$ by Cauchy's integral formula. (ii) Use Cauchy's representation for the coefficient $c_2 = f^{(2)}(1)/2$ in the power series of $f(z) = e^{z^2}$ about $z = 1$. (iii) poles are at i and $-i$, so we can split the integral around the circle line into two integral around the upper and lower half-circle which each only contain one pole. Result: $2\pi i(i \cos i) + 2\pi i(-i \cos(i)) = 0$.

B4 Imitate the calculation from the lecture. Result: $\int_{-\infty}^{\infty} \sin^2(x)/x^2 \, \mathrm{d}x = \pi$.

C1 $v(x,y) = \sqrt{|xy|}$ vanishes along the axes so has zero partial derivatives at the origin. The Cauchy-Riemann equations do hold at the origin. However $\lim_{r\to 0} \frac{f(re^{i\theta})}{re^{i\theta}} = \frac{i\sqrt{|\cos\theta \sin\theta}}{\cos\theta + i \sin\theta}$ which varies as θ varies so that $\lim_{z\to 0} \frac{f(z)-f(0)}{z}$ does not exist.

C2 If $f = u(x,y) + iv(x,y)$ then $u^2 + v^2 = 0$. Differentiate to find $uu_x + vv_x = 0$ and $uu_y + vv_y = 0$. Combine these with the Cauchy-Riemann equations to show that u and v are constants.

C2 $f'(z)$ is also holomorphic on C and so from question A4 we know that $f'(z)$ is a linear function and hence that f is a quadratic, namely $f(z) = f(0) + zf'(0) + z^2(f''(0)/2)$. The hypothesis implies that $f'(0) = 0$. Also by applying Cauchy's integral formula to $f'(z)$ we have that $|f''(0)| \leq \left\| \frac{1}{2\pi i} \int_{\partial B(0,1)} \frac{f'(z)}{z^2} dz \right\| \leq 1$.

C3 (a) Use the quotient rule for differentiating. (b) By the uniqueness theorem $(1+z)^k$ must have the same power series inside $|z| < 1$ as the real power series known via the binomial theorem on $\mathbb{R}$.

C4 (a) $|c_k| = \left\| \frac{1}{2\pi i} \int_{\partial B(0,R)} \frac{f(z)}{z^{k+1}} \,\mathrm{d}z \right\| \leq \frac{1}{2\pi} \frac{M}{R^{k+1}} 2\pi R = \frac{M}{R^k}$. Apply this with $M = 1 + R$ and let $R \to \infty$ to see that $c_k = 0$ whenever $k > 1$. (b) Apply the bound from part (a) with $M = A + BR^L$ and let $R \to \infty$ to see that $c_k = 0$ whenever $k > L$.

C5 Zeros at $\pm i$, $\pm 2i$. The usual semicircle contour therefore has 2 singularities inside it. Bound the integral around the top of the semicircle using $\frac{z^2}{(z^2+1)(z^2+4)} \leq \frac{R^2}{(R^2-1)(R^2-4)}$ when $|z| = R$ is larger than 2. The final integral has value $\pi/3$.

C6 $\int_0^\infty \cos^2 t \,\mathrm{d}t = \sqrt{\pi/8}$. The hard part is to bound $\int_{\gamma_2} f(z) \,\mathrm{d}z$. By the estimation lemma this is bounded by $\int_0^R \mathrm{e}^{-R^2 \cos(2t)} R \,\mathrm{d}t$. Now use the fact that $\cos(2t) \geq 1 - (4t/\pi)$ for $t \in [0, \pi/4]$ (draw the two functions on this interval). This implies that $\int_{\gamma_2} f(z) \,\mathrm{d}z \leq R \int_0^R \mathrm{e}^{-R^2} \mathrm{e}^{-4R^2 t/\pi} \,\mathrm{d}t$ which can be exactly calculated.