MA231 Vector Analysis
Example Sheet 0

2009, term 1
Stefan Adams

Parameterisations and multiple integrals

This sheet is **not** for credit. Parameterisations and multiple integrals, as discussed in 3D geometry and motion (or the equivalent set of notes for maths/physics students), will be used repeatedly and without much comment during the course. Hints and partial solutions are available on mathstuff. Please use your first supervision to ask about questions that you find hard.

1 **Sketching parameterised curves in the plane**

(a) Sketch the image of the following parameterised curves in the plane. Use arrows to indicate the direction along the curve in which the parameter increases.

$$(i)\ \ (\cos t, \sin t) \quad t \in [0, 4\pi] \qquad (ii)\ \ (\cos t, \sin t) \quad t \in [0, 2\pi]$$

$$(iii)\ \ (\sin t, \cos t) \quad t \in [0, 2\pi] \qquad (iv)\ \ (\cos t, -\sin t) \quad t \in [0, 2\pi]$$

$$(v)\ \ (\cos t, \frac{1}{3}\sin t) \quad t \in [0, 2\pi] \qquad (vi)\ \ (t\cos t, t\sin t) \quad t \in [0, 4\pi]$$

Show that the curve in part (v) is the ellipse $x^2 + 9y^2 = 1$.

(b) Sketch the curve parameterised by $(\sin t, \sin 2t)$ over $t \in [0, 2\pi]$.

2 **Sketching parameterised curves in $\mathbb{R}^3$**

(a) Sketch the image of the following parameterised curves in three space.

$$(i)\ \ (\cos t, \sin t, 1) \quad t \in [0, 2\pi] \qquad (ii)\ \ (\cos t, \sin t, t) \quad t \in [0, 6\pi]$$

$$(iii)\ \ (\sin t, \cos t, t^2) \quad t \in [0, 6\pi]$$

(b) Show that the curve $(t\cos t, t\sin t, t^2+1)$ lies on surface $x^2 + y^2 = z - 1$ (called a paraboloid). Make a rough sketch.

(c) Show that the curve $(\cos t \cos 4t, \cos t \sin 4t, \sin t)$ lies on the sphere $x^2 + y^2 + z^2 = 1$. Make a rough sketch.

3 **Finding parameterisations for curves**

Find a parameterisation for the following curves:

(a) (i) A line segment in $\mathbb{R}^3$ with ends $(0, 0, 0)$ and $(1, 1, 1)$. (ii) A line segment in $\mathbb{R}^3$ with ends $(-1, -1, -1)$ and $(1, 1, 1)$. (iii) A line segment in $\mathbb{R}^2$ with ends $(1, 3)$ and $(3, 1)$.

(b) (i) A circle in the plane centred at $(0, 0)$ with radius 2. (ii) A circle centred at $(1, 3)$ with radius 2. (iii) The ellipse $4x^2 + 9y^2 = 1$.

(c) Find a parameterisation of the circle of radius one formed by intersecting the plane $x + y + z = 0$ and the sphere $x^2 + y^2 + z^2 = 1$. (Hint: start by finding two orthogonal unit vectors in the plane).

4 **Sketching parameterised surfaces**

(a) Sketch the image of the following parameterised surfaces.

$$(i)\ \ (\cos t, \sin t, s) \quad t \in [0, 2\pi],\ s \in [-1, 1], \qquad (ii)\ \ (s\cos t, s\sin t, s) \quad t \in [0, 2\pi],\ s \in [0, 1],$$

$(iii)\ (s,t,s^2+t^2)\quad s\in[-1,1], t\in[-1,1], \qquad (iv)\ (s,t,s^2+t^2)\quad 0\le s^2+t^2\le 1.$

(b) What surface is parameterised by $((2+\cos t)\cos s, (2+\cos t)\sin s, \sin t)$ over $s,t\in[0,2\pi]$. (June 2003 exam question).

(c) Consider the surface parameterised by $((1-s\sin t)\cos 2t, (1-s\sin t)\sin 2t, s\cos t)$, for $t\in[0,\pi]$ and $s\in[-1,1]$. Sketch the curve formed when $s=0$ and only t varies. Check that when t is fixed and s varies you get a straight line. Can you see how these straight lines together make up a Möbius strip?

5 Finding parameterisations for surfaces

(a) Over what region must s,t vary for $(s,t,\sqrt{1-s^2-t^2})$ to be a parameterisation for the spherical cap $x^2+y^2+z^2=1$, $\frac{1}{2}\le z\le 1$.

(b) Over what region must s,t vary for $(s,t,1-s-t)$ to be a parameterisation for the triangular plane sheet $x+y+z=1$, $x\ge 0$, $y\ge 0$, $z\ge 0$.

6 Iterated integrals and volumes

(a) Calculate the iterated integrals

$$(i)\quad \int_0^1\int_1^2 \frac{1}{(x+y)^2}\,\mathrm{d}y\,\mathrm{d}x \qquad (ii)\quad \int_0^1\int_y^{2y} e^{x-y}\,\mathrm{d}x\,\mathrm{d}y.$$

(b) Show, by sketching the region of integration, that (for reasonable f) the following two integrals will have the same value:

$$\int_0^8\int_0^{4-\frac{x}{2}}\int_0^{2-\frac{y}{2}-\frac{x}{4}} f(x,y,z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \qquad \int_0^2\int_0^{4-2z}\int_0^{8-2y-4z} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

(c) Let B be the unit ball $\{x^2+y^2+z^2\le 1\}$. Explain, without calculating, why $\int_B \sin(x)\mathrm{d}V=0$.

(d) Calculate the volume of the region in $\mathbb{R}^3$ that lies under the paraboloid $z=x^2+y^2$ and above the unit square $x,y\in[0,1]$.

7 Polar coordinates in the plane

Recall the change of co-ordinates $F(r,\theta)=(r\cos\theta, r\sin\theta)$ for $r\ge 0$, $\theta\in[0,2\pi]$.

(a) Fix $r>0$. What curve is parameterised by $\theta\mapsto F(r,\theta)$?

(b) Fix θ. What curve is parameterised by $r\mapsto F(r,\theta)$?

(c) Use the change of co-ordinates $x=r\cos\theta$, $y=r\sin\theta$ to calculate the integral

$$\int\int_{\{x^2+y^2\le 1\}} (1+x^2+y^2)^{1/3}\mathrm{d}x\,\mathrm{d}y.$$

(d) Sketch and find the volume of the region in $\mathbb{R}^3$ that is bounded by the paraboloid $z=x^2+y^2$, the cylinder $x^2+y^2=1$ and the plane $z=0$.