MA231 Vector Analysis
Example Sheet 1

2009, term 1
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Students should hand in solutions to questions B1, B2, B3 and B4 by 3pm Monday of week 4 to the maths pigeonloft. Maths students hand in solutions to their supervisors and maths/physics students hand solutions into the slots marked *Vector Analysis Maths+Physics*.

A1 **Level sets of scalar fields**

Sketch level sets $f^{-1}(c)$, for $c = 0$ and for some values $c > 0$ and $c < 0$, of the following functions:

$$(a) \quad f(x, y) = y^2 + x, \qquad (b) \quad f(x, y) = xy.$$

A2 **Visualizing planar vector fields**

(a) Sketch the vector field $v(x, y) = (-1, 2y)$. Compare with your sketch of the level sets of $f = y^2 - x$ to confirm it looks like the gradient vector field of f.

(b) Sketch the vector fields (i) $v(x, y) = (-x, y)$ and (ii) $v(x, y) = \big(\frac{-x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\big)$.

A3 **Gradients of scalar fields**

(a) Find the gradient vector field ∇f for each of the following scalar fields:

$$\text{(i)} \quad f(x, y) = 2xy + y^2, \qquad \text{(ii)} \quad f(x, y) = xy \cos(\pi y).$$

(b) What is the directional derivative of the function $f(x, y) = 2xy+y^2$ at the point $(2, 3)$ in the direction $(-1, 5)$?

A4 **Line integrals**

(a) Find the arclength of the curve parameterized by (t^2, t^3) for $t \in [0, 1]$.

(b) Let v be the vector field $v(x, y) = (x + y^2, y - 1)$. Let $\mathcal{C}$ be the curve consisting of the line along the x-axis in the plane joining the points $(-2, 0)$ and $(2, 0)$ together with the upper semicircle of radius 2, centered at the origin. Find a parameterization for each part of $\mathcal{C}$. Then evaluate the tangential line integral $\int_{\mathcal{C}} v \cdot \hat{T} ds$, where $\mathcal{C}$ is traversed in the anticlockwise direction.

A5 **Gradient vector fields**

For the following vector fields v, find a scalar field f so that $v = \nabla f$.

$$(a) \quad v(x, y) = (2xy + 3x^2, x^2) \qquad (b) \quad v(x, y, z) = (2xyz + z, x^2z + 1, x^2y + x).$$

(c) Show that the vector field $v(x, y) = (3y, x + y)$ is not of gradient type.

A6 **Finding unit normals to surfaces**

(a) Find a unit normal to the surface $z = xy + 1$ at the point $(2, 2, 5)$.

(b) Find a unit normal to the surface parameterized by $x(s, t) = (st, s^2 + t^2, t^2s)$.

A7 **Surface integrals**

(a) The surface $\mathcal{S}$ is parameterized by $(s, t, s^2 + t)$ over $s \in [0, 1]$, $t \in [-1, 1]$. Calculate the integral $\int_{\mathcal{S}} x \, \mathrm{d}S$.

(b) Compute the surface area of the part of the paraboloid $z = x^2 + y^2$ that lies between the planes $z = 0$ and $z = L$.

B1 **Visualization of functions**

(a) Sketch level sets $f^{-1}(c)$, for $c = 0$ and some $c > 0$ and $c < 0$, of the following functions:

$$\text{(i)} \quad f(x,y) = x^2 - 4y^2, \qquad \text{(ii)} \quad f(x,y,z) = \sqrt{x^2 + y^2 + 3} - z.$$

(b) Sketch the following vector fields in the plane

$$\text{(i)} \quad u(x,y) = (1, -\frac{x}{2}), \qquad \text{(ii)} \quad v(x,y) = (y, -\sin x)$$

B2 **Gradients and Directional Derivatives**

(a) Find the gradient vector field ∇f for each of the following scalar fields $f : \mathbb{R}^3 \to \mathbb{R}$; recall $\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$:

$$\text{(i)} \quad f(x) = \log \|x\| \text{ for } x \neq 0, \qquad \text{(ii)} \quad f(x) = \frac{1}{\|x\|} \quad \text{for } x \neq 0,$$

$$\text{(iii)} \quad f(x,y,z) = \frac{x^3}{3(y^2+1)} \sin(3xz).$$

(b) What is the directional derivative of the function $f(x,y,z) = x^2yz + 4xz^2$ at the point $(1, -2, -1)$ in direction $(2, -1, -2)$?

B3 **Gradient Vector Fields**

(a) Find a potential $v : \mathbb{R}^3 \to \mathbb{R}$ for the following vector fields $f : \mathbb{R}^3 \to \mathbb{R}^3$:

$$\text{(i)} \quad f(x,y,z) = (0, y, 0), \qquad \text{(ii)} \quad f(x_1, x_2, x_3) = 2\|x\|^4 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, x = (x_1, x_2, x_3) \in \mathbb{R}^3$$

(b) Show that $f : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $f(x,y,z) = (z, 0, 0)$ is no gradient vector field.

B4 **Line integrals**

(a) Evaluate the line integral

$$\int_{\mathcal{C}} (x + y^2),$$

where $\mathcal{C}$ is the parabola $y = x^2$ in the plane $z = 0$ connecting the points $(0,0,0)$ and $(2,4,0)$.

(b) Calculate the tangent line integral of the vector field

$$v(x,y,z) = \big((x-1)(z-3), xyz, x+z\big)$$

along the straight line from $(1,1,1)$ to $(1,3,9)$.

(c) Consider the half circle $\mathcal{C} = \{y^2 + z^2 = 1, z \geq 0, x = 0\} \subseteq \mathbb{R}^3$ and the vector field $f(x,y,z) = (0, y, 0)$. Use the fundamental theorem of calculus for gradient vector fields to calculate the tangent line integral of f along $\mathcal{C}$ from $(0,-1,0)$ to $(0,1,0)$.

C1 **Directional derivatives** The temperature at the point (x, y, z) is given by $T(x, y, z) = z + x^2 + y^2$. Starting at the point $(1, 1, 0)$ you decide to move in the direction $(\cos(\theta), \sin(\theta), 1)$ for some $\theta \in [0, 2\pi]$. Which choice of θ will lead to the greatest rate of increase in the temperature?

C2 **Uniqueness of the potential for a gradient vector fields** Suppose a vector field v defined on all of $\mathbb{R}^n$ satisfies $v = \nabla f$ and $v = \nabla g$. Show that the scalar functions f and g differ by a constant. (Hint: What is the value of the line integral $\int_{\mathcal{C}} v \cdot \hat{T} \, \mathrm{d}s$ where $\mathcal{C}$ is the straight line starting at the origin and ending at the point $x_0 \in \mathbb{R}^n$?)

C3 **Gradients of compositions**

(a) Suppose $f: \mathbb{R}^n \to \mathbb{R}$ is a scalar field and $\varphi: \mathbb{R} \to \mathbb{R}$. Show that

$$\nabla(\varphi \circ g)(x) = \varphi'(g(x)) \, \nabla g(x).$$

(b) Show that $u = \frac{g}{g^2+1} \nabla g$ is a gradient vector field.

(c) What is the gradient of the scalar field $\varphi(r^2)$ where $r = \|x\|$? Show that $r^m x$ is a gradient vector field.

C4 **Lagrangian derivatives**

Suppose $u(t, x_1, x_2, x_3)$ is a fluid velocity at time t at position (x_1, x_2, x_3). A tracer particle is moving within the fluid and so its position $x(t) = (x_1(t), x_2(t), x_3(t))$ solves the differential equation $\frac{dx}{dt} = u(t, x(t))$. Show that the acceleration of the tracer particle is given by

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{3} u_i(t, x(t)) \frac{\partial u}{\partial x_i}(t, x(t)).$$

This is usually written in shorthand notation as $\frac{\partial u}{\partial t} + (u \cdot \nabla)u$ — can you see why?

C5 **The dipole vector field**

(a) Suppose $f: \mathbb{R}^n \to \mathbb{R}$ and $g: \mathbb{R}^n \to \mathbb{R}$ are scalar fields. Show that $\nabla(fg) = f\nabla g + g\nabla f$.

(b) For fixed $m \in \mathbb{R}^n$ calculate the gradient of the linear function $f(x) = \langle x, m \rangle$.

(c) Calculate the gradient of the radial function $g(x) = \frac{1}{|x|^3} = \frac{1}{r^3}$.

(d) Combine parts (a),(b) and (c) to find the gradient of the dipole potential function $\frac{\langle x, m \rangle}{\|x\|^3}$.

C6 **Length and area of graphs**

(a) The graph of the function $g: [a, b] \to \mathbb{R}$ is the curve $\mathcal{C}$ in $\mathbb{R}^2$ parameterized by $(t, g(t))$ for $t \in [a, b]$. Find the length of the tangent vector for this parameterization and hence show the arclength of $\mathcal{C}$ is given by $\int_a^b \sqrt{1 + g'(t)^2} \, \mathrm{d}t$.

(b) The graph of the function $g: \mathbb{R}^2 \to \mathbb{R}$ is the surface $\mathcal{S}$ in $\mathbb{R}^3$ parameterized by $x(s, t) = (s, t, g(s, t))$. Find an expression in terms of g for a unit normal to $\mathcal{S}$ at $x(s, t)$. Hence find an expression for the area of the part of the surface parameterized by $s, t \in [0, 1]$.