MA231 Vector Analysis
Example Sheet 3

2009, term 1
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This sheet contains some questions on the tail end of the vector analysis part of the course and some on the warm up sections of the complex variable part of the course. Hand in solutions to questions B1, B2, B3 and B4 by 3pm Monday of week 8.

A1 **Calculations in spherical coordinates**

At the back of this example sheet are the formulae for spherical coordinates.

Let $v = r^2 \sin\vartheta\, \hat{e}_r + \frac{1}{r\cos\vartheta}\, \hat{e}_\varphi + \frac{1}{r}\, \hat{e}_\vartheta$. Calculate $\operatorname{div}(v)$, giving the answer in in spherical coordinates (r, φ, ϑ). Calculate $\nabla \times v$ giving the answer in the curvilinear basis $\hat{e}_r, \hat{e}_\varphi, \hat{e}_\vartheta$.

A2 **Algebra on $\mathbb{C}$**

Algebraic manipulation of complex numbers is used in all the remaining parts of the course and you should practice if you feel at all rusty.

(a) Express the following numbers in polar form $r(\cos\vartheta + i\sin\vartheta)$: $i,\ -i,\ 1+i,\ -\frac{1}{2}+\frac{\sqrt{3}}{2}i$.

(b) Express the following in the form $x+iy$: $\frac{1}{6+2i}$, i^7, $\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^4$.

(c) Find both square roots of $-8+6i$. Then solve the quadratic equation $z^2 + i\sqrt{32}z - 6i = 0$.

(d) Find all four solutions to $z^4 = -1$. Plot them on the complex plane.

A3 **Complex conjugates**

(a) Give the complex conjugate $\bar{z}$ for a complex number z both in cartesian form $x+iy$ and in polar form $r(\cos\vartheta + i\sin\vartheta)$.

(b) Show $\overline{vw} = \bar{v}\bar{w}$ for complex numbers v, w.

A4 **Curves in $\mathbb{C}$**

Sketch the set $\{z \in \mathbb{C} \mid z\bar{z} = 1\}$ in the complex plane. Find a parametrisation $\varphi\colon [0,1] \to \mathbb{C}$ for this set.

A5 **Convergence of complex series**

(a) Calculate the radius of convergence of the following power series: (i) $\sum_{n=0}^{\infty} 2^{-n^2} z^n$, (ii) $\sum_{n=0}^{\infty} \frac{n!}{n^n} z^n$.

(b) Show that the series $\sum (3+4i)^n (z-4i)^n$ converges inside a certain ball in the complex plane.

(c) Show that $\sum_{n=0}^{\infty} \frac{i^n}{i+n^2}$ is convergent and that $\sum_{n=0}^{\infty} \frac{1}{i+n}$ is divergent.

(d) Find the power series $\sum c_n z^n$ for: (i) $\frac{1}{i+z}$, (ii) $\frac{\sin z}{z}$, (iii) $\cos(z+i)$.

(e) Find the power series $\sum c_n (z-i)^n$ for: (i) $\frac{1}{z}$, (ii) e^z.

A6 **Complex special functions**

(a) Show that $\sin z = \frac{\mathrm{e}^{iz} - \mathrm{e}^{-iz}}{2i}$.

(b) Show that $\sin(2z) = 2\sin z\cos z$.

(c) Show that $\sin(x+iy) = \sin x\cosh y + i\cos x\sinh y$.

(d) Find z so that $\sin z = 2$. Can you find all such z?

B1 **Stokes's theorem, Spherical coordinates**

(a) Let $\mathcal{C}$ be the curve formed by the intersection of the cylinder $x^2 + y^2 = 1$ in $\mathbb{R}^3$ and the plane $x + y + z = 1$. Orient the curve $\mathcal{C}$ so that the tangent vector at $(1, 0, 0)$ has a negative z-component. Use Stokes' theorem to calculate, for the vector field $v(x, y, z) = (-y^3, x^3, z^3)$, the tangential line integral $\int_{\mathcal{C}} \langle v, \hat{T} \rangle$.

(b) Check that the vectors $\hat{e}_r$, $\hat{e}_\varphi$ and $\hat{e}_\vartheta$ are orthonormal with $\hat{e}_r \times \hat{e}_\varphi = \hat{e}_\vartheta$.

(c) Calculate ∇T and ΔT for a radial symmetric scalar field $T(r, \varphi, \vartheta) = T(r)$.

Hint: You may use the formulae from the back of the assignment sheet for your solution.

B2 **Complex numbers**

(a) Determine the set of all complex numbers $z \in \mathbb{C}$ for which $|z + 1| < 2|z - i|$ holds, and draw a figure of that set.

(b) Write the number $z = 1 + i$ in polar form and determine z^7 with Moivre's formula.

(c) Determine all solutions of $z^4 = 1 + i\sqrt{3}$ in polar form.

(d) Determine $\sqrt[3]{1 - \sqrt{3}i}$.

B3 **Complex series**

(a) Show that the series $\sum_{n=1}^{\infty} \frac{(1+i)^{2n}}{n^4} (3z + i)^n$ converges inside a certain ball in the complex plane. At what points on the boundary of the ball does the series converge?

(b) Show that $\sum_{n=0}^{\infty} \frac{1}{z+n^2}$ defines a continuous function on the set $\{ z = x + iy \mid x > 0 \}$.

(c) Find the power series $\sum c_n (z - a)^n$, and their radius of convergence, for each of the following functions around the given points:

  (i) z^2 with $a = -i$

  (ii) $\frac{1}{z-1}$ with $a = i$

  (iii) $\frac{1}{1-z}$ with $a = 0$.

B4 **Special functions**

(a) Show that $\cos(2z) = 1 - 2\sin^2 z$.

(b) Show that $\cosh z$ is surjective.

(c) Define f by $f(z) = z\cos(1/z)$ when $z \neq 0$ and $f(0) = 0$. Decide whether f is continuous at 0 or not and explain your answer.

C1 **Gradient, divergence and curl in cylindrical coordinates**

(a) Consider the value of the tangential line integral $\int_{\mathcal{C}} \langle \nabla f, \hat{T} \rangle \, \mathrm{d}s$ for $f(r, \vartheta, z)$, written in cylindrical coordinates, along the line segment $\mathcal{C}$ joining the points (r, ϑ, z) to $(r, \vartheta + \Delta\vartheta, z)$. Apply the fundamental theorem of calculus for gradient vector fields to find $\nabla f \cdot \hat{e}_\vartheta$. (The other components of ∇f can be found in a similar way.)

(b) Consider the value of the surface integral $\int_{\mathcal{S}} \langle \nabla \times v, \hat{N} \rangle \, \mathrm{d}S$ for $v = v_r \, \hat{e}_r + v_\vartheta \, \hat{e}_\vartheta + v_z \, \hat{e}_r$, written in the cylindrical curvilinear basis, over the quadrilateral joining the points (r, ϑ, z), $(r, \vartheta + \Delta\vartheta, z)$, $(r + \Delta r, \vartheta, z)$ and $(r + \Delta r, \vartheta + \Delta\vartheta, z)$. Apply Stokes theorem and by considering the circulation $\int_{\partial\mathcal{S}} \langle v, \hat{T} \rangle \, \mathrm{d}s$ find the value of $\nabla \times v \cdot \hat{e}_z$. (The other components of $\nabla \times v$ can be found in a similar way.)

(c) Fix r, ϑ, z and small $\Delta r, \Delta\vartheta, \Delta z > 0$. Consider the region given by

$$\Omega = \big\{ (r', \vartheta', z') \bigm| r \leq r' \leq r + \Delta r, \; \vartheta \leq \vartheta' \leq \vartheta + \Delta\vartheta, \; z \leq z' \leq z + \Delta z \big\}.$$

  i. Sketch the region Ω and mark the unit outward normals to its six sides, given approximately by $\pm\hat{e}_r$, $\pm\hat{e}_\vartheta$, $\pm\hat{e}_z$.

  ii. Show that Volume(Ω) is approximately $r\Delta r \, \Delta\vartheta \, \Delta z$.

iii. Apply the divergence theorem to the region Ω to show that, ignoring higher order terms in Δr, $\Delta\vartheta, \Delta z$,

$$\begin{aligned}\nabla \cdot v\, r\Delta r\Delta\vartheta\Delta z \quad &\approx \quad (v_z(r,\vartheta,z+\Delta z) - v_z(r,\vartheta,z))\, r\Delta r\,\Delta\vartheta \\ &\quad + (v_\vartheta(r,\vartheta+\Delta\vartheta,z) - v_z(r,\vartheta,z))\,\Delta r\,\Delta z \\ &\quad + (v_r(r+\Delta r,\vartheta,z) - v_z(r,\vartheta,z))\,\Delta\vartheta\,\Delta z.\end{aligned}$$

(Hint: consider the flux across opposite faces of Ω).

iv. Let Δr, $\Delta\vartheta, \Delta z \to 0$ to deduce the formula for $\nabla \cdot v$ in cylindrical coordinates.

C2 Vector potential for the dipole vector field

The dipole vector field, in the direction m, was introduced in question C5 of example sheet 1 as the gradient vector field $v = \nabla\varphi$ arising from the potential $\varphi(x) = \frac{1}{r^3}\, x \cdot m$.

Use the identity

$$\nabla \times (\nabla \times u) = \nabla(\nabla \cdot u) - \Delta u$$

from question C2 on example sheet 2 to show that $A = \nabla \times u$, when $u(x) = -\frac{m}{r}$, is a vector potential for v in $\mathbb{R}^3$, i.e. that $v = \nabla \times A$.

C3 Newton potential

The gravitational vector field of a solid ball $B(0,R)$ (e.g. a planet) with constant mass density is the gradient of the Newtonian potential defined by

$$f_B(x) = -C \int_{B(0,R)} \frac{1}{\|x-y\|}\, \mathrm{d}V(y).$$

It is a calculation in Newton's Principia from 1687 that $f_B(x) = -\frac{C}{\|x\|}\,\mathrm{Volume}(B(0,R))$. This exercise shows one way to derive this result.

(a) Use symmetry to argue that $f_B(x)$ should be radial, that is of the form $F(r)$ for $r = \|x\|$.

(b) Check that $\Delta\frac{1}{\|x-x_0\|} = 0$. Passing the derivatives under the integral sign show that $\Delta f_B(x) = 0$ when $\|x\| > R$.

(c) In B2 we showed that the only radial functions $F(r)$ for which $\Delta F(r) = 0$ are $F(r) = a + \frac{b}{r}$ for constants a and b. Show that $|F_B(x)| \to 0$ as $\|x\| \to \infty$ and deduce that $b = 0$.

(d) Using the limit $\frac{\|x\|}{\|x-y\|} \to 1$ as $x \to \infty$ uniformly over $y \in B(0,R)$, deduce that $a = C\,\mathrm{Volume}(B(0,R))$.

spherical coordinates
coordinate map:

$$(r,\varphi,\vartheta) \mapsto \begin{pmatrix} r\cos(\varphi)\cos(\vartheta) \\ r\sin(\varphi)\cos(\vartheta) \\ r\sin(\vartheta) \end{pmatrix}$$

basis vectors:

$$\hat{\mathrm{e}}_r = \begin{pmatrix} \cos(\varphi)\cos(\vartheta) \\ \sin(\varphi)\cos(\vartheta) \\ \sin(\vartheta) \end{pmatrix}, \quad \hat{\mathrm{e}}_\varphi = \begin{pmatrix} -\sin(\varphi) \\ \cos(\varphi) \\ 0 \end{pmatrix}, \quad \hat{\mathrm{e}}_\vartheta = \begin{pmatrix} -\cos(\varphi)\sin(\vartheta) \\ -\sin(\varphi)\sin(\vartheta) \\ \cos(\vartheta) \end{pmatrix}$$

differential operators:

$$\mathrm{grad}(f) = \frac{\partial f}{\partial r}\hat{\mathrm{e}}_r + \frac{1}{r\cos\vartheta}\frac{\partial f}{\partial \varphi}\hat{\mathrm{e}}_\varphi + \frac{1}{r}\frac{\partial f}{\partial \vartheta}\hat{\mathrm{e}}_\vartheta$$

$$\mathrm{div}(v) = \frac{1}{r^2}\frac{\partial}{\partial r}\big(r^2 v_r\big) + \frac{1}{r\cos\vartheta}\frac{\partial v_\varphi}{\partial \varphi} + \frac{1}{r\cos\vartheta}\frac{\partial}{\partial \vartheta}\big(\cos(\vartheta)v_\vartheta\big)$$

$$\begin{aligned}\mathrm{curl}(v) = \frac{1}{r\cos\vartheta}\Big(\frac{\partial v_\vartheta}{\partial \varphi} - \frac{\partial}{\partial \vartheta}\big(\cos(\vartheta)v_\varphi\big)\Big)\hat{\mathrm{e}}_r + \frac{1}{r}\Big(\frac{\partial v_r}{\partial \vartheta} - \frac{\partial}{\partial r}\big(r v_\vartheta\big)\Big)\hat{\mathrm{e}}_\varphi \\ + \frac{1}{r}\Big(\frac{\partial}{\partial r}\big(r v_\varphi\big) - \frac{1}{\cos\vartheta}\frac{\partial v_r}{\partial \varphi}\Big)\hat{\mathrm{e}}_\vartheta\end{aligned}$$

cylindrical coordinates
coordinate map:

$$(r,\varphi,z) \mapsto \begin{pmatrix} r\cos(\varphi) \\ r\sin(\varphi) \\ z \end{pmatrix}$$

basis vectors:

$$\hat{\mathrm{e}}_r = \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \\ 0 \end{pmatrix}, \quad \hat{\mathrm{e}}_\varphi = \begin{pmatrix} -\sin(\varphi) \\ \cos(\varphi) \\ 0 \end{pmatrix}, \quad \hat{\mathrm{e}}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

differential operators:

$$\mathrm{grad}(f) = \frac{\partial f}{\partial r}\hat{\mathrm{e}}_r + \frac{1}{r}\frac{\partial f}{\partial \varphi}\hat{\mathrm{e}}_\varphi + \frac{\partial f}{\partial z}\hat{\mathrm{e}}_z$$

$$\mathrm{div}(v) = \frac{1}{r}\frac{\partial}{\partial r}\big(r v_r\big) + \frac{1}{r}\frac{\partial v_\varphi}{\partial \varphi} + \frac{\partial v_z}{\partial z}$$

$$\mathrm{curl}(v) = \Big(\frac{1}{r}\frac{\partial v_z}{\partial \varphi} - \frac{\partial v_\varphi}{\partial z}\Big)\hat{\mathrm{e}}_r + \Big(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\Big)\hat{\mathrm{e}}_\varphi + \frac{1}{r}\Big(\frac{\partial}{\partial r}\big(r v_\varphi\big) - \frac{\partial v_r}{\partial \varphi}\Big)\hat{\mathrm{e}}_z$$