MA231 Vector Analysis
Example Sheet 4

2009, term 1
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Hand in solutions to questions B1, B2, B3 and B4 by 3pm Monday of week 10.

On this example sheet, all contour integrals around simple closed curves are taken in anticlockwise direction.

A1 **Cauchy-Riemann equations**

(a) Find the real and imaginary parts of $f(z) = \mathrm{e}^z$ and show they satisfy the Cauchy Riemann equations.

(b) Suppose that a holomorphic function $f\colon \mathbb{C} \to \mathbb{C}$ satisfies $\mathrm{Re}(f(z)) = 0$ for all $z \in \mathbb{C}$. Show that f is a constant.

(c) Suppose that $f(x+iy) = u(x,y) + iv(x,y)$ is holomorphic on $\mathbb{C}$. Show that $\Delta u = \Delta v = 0$ on $\mathbb{R}^2$ (you may assume that the second partial derivatives exist and are continuous).

A2 **Complex contour integrals**

(a) Calculate $\int_\gamma f\,\mathrm{d}z$ when $f(x+iy) = xy$ and $\gamma(t) = \mathrm{e}^{it}$ for $t \in [0,\pi]$.

(b) Calculate $\int_{\mathcal{C}} f\,\mathrm{d}z$ where $f(z) = \overline{z}$ and $\mathcal{C}$ is the straight line from 0 to $1+i$.

A3 **Contour integrals via Cauchy's integral representation**

(a) Using Cauchy's integral formula calculate the following contour integrals:

$$(i)\ \int_{\partial B(0,2)} \frac{\mathrm{e}^z}{z-1}\,\mathrm{d}z, \quad (ii)\ \int_{\partial B(0,2)} \frac{\mathrm{e}^z}{\pi i - 2z}\,\mathrm{d}z, \quad (iii)\ \int_{\partial B(0,2)} \frac{\mathrm{e}^z}{6\pi i - 2z}\,\mathrm{d}z.$$

(b) Using Cauchy's integral formula for the coefficients c_k of a Taylor series, evaluate the contour integral

$$\int_{\partial B(i,2)} \frac{\mathrm{e}^z}{(z-1)^n}\,\mathrm{d}z \quad \text{when } n \text{ is a positive integer.}$$

A4 **More contour integrals**

Evaluate $\int_{\partial B} \frac{1}{1+z^2}\,\mathrm{d}z$ for each of the following balls:

$(i)\ B(1,1), \quad (ii)\ B(i,1), \quad (iii)\ B(-i,1), \quad (iv)\ B(0,2), \quad (v)\ B(3i,\pi).$

A5 **Evaluation of a real integral**

Using a contour integral of the function

$$f(z) = \frac{\mathrm{e}^{\pi i z}}{z^2 - 2z + 2}$$

around a semicircular contour, evaluate the real integrals

$$\int_{-\infty}^{\infty} \frac{\cos(\pi x)}{x^2 - 2x + 2}\,\mathrm{d}x \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{\sin(\pi x)}{x^2 - 2x + 2}\,\mathrm{d}x.$$

B1 Cauchy-Riemann equations

(a) At what points $z \in \mathbb{C}$ are the functions $f(z) = |z|^4$ and $g(z) = g(x+iy) = 6xy^2 + i(4x + 2y^3)$ and $h(z) = z\bar{z}^2$ differentiable? At what points are f and g and h holomorphic?

(b) Suppose that a holomorphic function f is of the special form $f(x+iy) = u(y) + iv(x)$. Show that $f(z) = aiz + b$ for some $a \in \mathbb{R}$ and $b \in \mathbb{C}$.

(c) At what points $z \in \mathbb{C}$ are the functions $f(z) = z\bar{z}^2$ and $g(z) = g(x+iy) = (x - xy^2) + ix^2y$ differentiable? At what points are f and g holomorphic?

(d) Let $f\colon B(0,R) \to \mathbb{C}$ be holomorphic (in $B(0,R)$). Prove that f is constant in $B(0,R)$ if $|f|$ is constant in $B(0,R)$.

B2 Contour integrals

(a) Calculate the integral $\int_{\mathcal{C}} f \,\mathrm{d}z$ where $f(z) = \bar{z}$ and $\mathcal{C}$ is the boundary of the triangle with vertices $1, i, -1$, traversed in an anticlockwise direction.

(b) For $\varepsilon > 0$ let γ_ε be the piece of circular arc of radius ε parameterised by $\gamma_\varepsilon(t) = \varepsilon \mathrm{e}^{it}$ for $t \in [\alpha, \beta] \subseteq [0, 2\pi]$. Show, for continuous $f\colon \mathbb{C} \to \mathbb{C}$, that

$$\int_{\gamma_\varepsilon} \frac{f(z)}{z} \,\mathrm{d}z \to f(0)(\beta - \alpha)i \quad \text{as } \varepsilon \to 0.$$

B3 More contour integrals

(a) Each of the following integrals is zero. Give a brief reason for each example.

$$(i) \int_{\partial B(1,2)} \frac{\sin z}{z} \,\mathrm{d}z, \quad (ii) \int_{\partial B(1,2)} \frac{1}{z+2} \,\mathrm{d}z, \quad (iii) \int_{\partial B(1,2)} \frac{1}{(z-2)^3} \,\mathrm{d}z.$$

(b) Evaluate the following integrals anticlockwise around the boundary of the ball $B(0,2)$:

$$(i) \int \frac{z^5 + 3}{z - i} \,\mathrm{d}z, \qquad (ii) \int \frac{\mathrm{e}^{z^2}}{(z-1)^3} \,\mathrm{d}z, \qquad (iii) \int \frac{z \cos z}{z^2 + 1} \,\mathrm{d}z$$

B4 Evaluating a real integral

The aim is to evaluate $\int \frac{\sin^2 x}{x^2} \,\mathrm{d}x$. The strategy is to do a contour integral of the function $f(z) = \frac{\mathrm{e}^{2iz} - 1}{z^2 + a^2}$ for real $a > 0$ around the contour γ consisting of two parts: γ_1 the semi-circle $\{z \in \mathbb{C} \mid |z| = R, \ \mathrm{Im}(z) > 0\}$, and γ_2 the straight line segment from $-R$ to R.

(i) Explain why $\int_{\gamma_1} f(z) \,\mathrm{d}z \to 0$ as $R \to \infty$.

(ii) Use Cauchy's integral formula to show that $\int_\gamma f(z) \,\mathrm{d}z = \frac{\pi}{a} \left(\mathrm{e}^{-2a} - 1\right)$.

(iii) Show that $\mathrm{e}^{2ix} - 1 = -2\sin^2 x + i \sin 2x$ and hence that $\int_{\gamma_2} f(z) \,\mathrm{d}z$ is real.

(iv) Evaluate $\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2 + a^2} \,\mathrm{d}x$.

(v) Let $a \to 0$ to find $\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} \,\mathrm{d}x$.

C1 Cauchy-Riemann equations

Let $f\colon \mathbb{C} \to \mathbb{C}$ be analytic and suppose that $|f(z)|$ is a constant. Show that f is a constant. (Hint: If $f = u + iv$ then $u^2 + v^2$ is constant. Differentiate in x and y and solve the resulting equations for u_x, u_y, v_x, v_y.)

C2 Differentiability

Consider the complex function $f(x+iy) = x^2 - y^2 + i\sqrt{|xy|}$. Are the Cauchy-Riemann equations satisfied at the origin? Is f holomorphic at the origin? Hint: examine

$$\lim_{r\to 0} \frac{f(r\mathrm{e}^{i\theta})}{r\mathrm{e}^{i\theta}}.$$

C3 Identities via the uniqueness theorem

Fix an integer $k > 0$.

(a) Explain why the function $f(z) = \frac{1}{(1+z)^k}$ is holomorphic on $|z| < 1$.

(b) Use the uniqueness theorem for power series to show, for $|z| < 1$, that

$$\frac{1}{(1+z)^k} = \sum_{n=0}^{\infty} \frac{k(k-1)\dots(k-n+1)}{n!} z^n.$$

C4 An improved version of Liouville's theorem

(a) Suppose that $f(z)$ is holomorphic on the whole of $\mathbb{C}$ and that $|f(z)| \le M$ along the circle $|z| = R$. Use Cauchy's integral formula for the coefficients c_k in the Taylor expansion of f about the origin to show that $|c_k| \le \frac{M}{R^k}$.

(b) Suppose that f is holomorphic on $\mathbb{C}$ and satisfies $|f(z)| \le A + B|z|^L$ at all points z, for some real $A, B > 0$ and positive integer $L \ge 0$. Show that $f(z)$ is a polynomial of order at most L.

C5 Another real integral

Use a semicircular contour to evaluate the integral $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,\mathrm{d}x$.

C6 Evaluation of $\int \cos(x^2)\,\mathrm{d}x$

It is, perhaps, initially surprising that $\int_0^\infty \cos(x^2)\,\mathrm{d}x$ has a well defined finite integral, since the function $\cos(x^2)$ continues to oscillate between 0 and 1 along the whole real line.

The idea is to use the contour integral of $f(z) = \mathrm{e}^{-z^2}$ along the contour γ made up of three parts: γ_1 the line segment from 0 to R; γ_2 the circular arc $R\mathrm{e}^{it}$ for $t \in [0, \pi/4]$; γ_3 the line segment from $R(1+i)/\sqrt{2}$ back to the origin.

(a) Show that $\int_\gamma f\,\mathrm{d}z = 0$.

(b) Show that $\int_{\gamma_2} f\,\mathrm{d}z \to 0$ as $R \to \infty$.

(c) Recall that $\int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x = \sqrt{\pi}$. Show that $\int_{\gamma_3} f\,\mathrm{d}z = \mathrm{e}^{(i\pi/4)} \int_0^R \mathrm{e}^{-it^2}\,\mathrm{d}t$.

(d) Combine the parts to evaluate $\int_0^\infty \cos(x^2)\,\mathrm{d}x$.